FINITE GROUP ACTIONS IN SEIBERG–WITTEN THEORY

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Abstract. Let $X$ be a closed and oriented Riemannian four-manifold with $b_3 (X) > 1$. We discuss the Seiberg–Witten invariants of $X$ and finite group actions on spin$^c$ structures of $X$. We introduce and comment some of our results on the subject.

1. Introduction

In the past twenty years, the symbiosis between mathematics and theoretical physics has always been a source of unexpected and profound results.

Even if we do not make attempt to relate it chronologically, the story begun with the Donaldson’s gauge theory aiming a nonabelian generalization of the classical electromagnetic theory.

As results of it the nonsmoothability of certain topological four-manifolds, exotic smooth structures on $\mathbb{R}^4$, and nondecomposability of some four-manifolds have been established.

The computation of Donaldson invariants however is highly nontrivial.

In 1994, the monopole theory in four-manifolds gave a rise to the Seiberg–Witten invariant which is much simpler than the Donaldson theory, also had almost the same effects on the Donaldson theory, and was used for a proof of the Thom conjecture.

At almost the same time the Gromov–Witten invariant of symplectic manifolds was introduced. Using it we may compute the number of algebraic curves, representing a two-dimensional homology class in a symplectic manifold.

In 1995 Taubes [26] proved that for symplectic four-manifolds the Seiberg–Witten invariant and the Gromov–Witten invariant are the same.

In 1983 Donaldson [12] proved the nonsmoothability of certain topological four-manifolds.
In 1994, Witten [29] had introduced the Seiberg–Witten theory which simplifies the
Donaldson’s gauge theory.
In 1994 Kontsevich and Manin [20] introduced the Gromov–Witten invariant and
applied it to enumerative problems of algebraic geometry.
In this article we want to introduce the Seiberg–Witten invariant and its fundamen-
tal consequences. We would like to survey finite group actions in the Seiberg–
Witten theory, and some of our results which were already published (see the re-
ferences at the end of the paper).

2. Review of Seiberg–Witten Invariant

Let $X$ be a closed, oriented four-manifold with $b_2^+(X) > 1$. Let $L \rightarrow X$ be a
complex line bundle with $c_1(L) = c_1(X) \text{ mod } 2$.
Let $W^\pm$ be the twisted spinor bundles on $X$ associated with the line bundle $L$. Let
$\sigma : W^+ \otimes T^*X \rightarrow W^-$ be the Clifford multiplication. There is a correspondence
$\tau : W^+ \times W^+ \rightarrow \text{End}(W^+)_c$ given by $\tau(\phi, \phi) = (\phi \otimes \phi)'_c$ which is a traceless
endomorphism of $W^+$.
The Levi-Civita connection on $X$ combined with a connection $A$ on $L$ induces a
Dirac operator

$$D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$$

along with the Seiberg–Witten (SW) equations

$$D_A \phi = 0, \quad F_A^+ = -\tau(\phi, \phi)$$

(1)
whose solutions correspond to the absolute minima of some functional.
The gauge group $C^\infty(X, U(1))$ of $L$ acts on the space of solutions $(A, \phi)$ of the
SW-equations and quotient $\mathcal{M}(L)$ of the space of solutions of the equations modulo
the gauge group is called the moduli space associated to the spin$^c$ structure $L$ on
$X$.
If we perturb the SW-equations or find a generic metric on $X$, the moduli space
$\mathcal{M}(L)$ is a compact orientable $d$-manifold where $d = \frac{1}{2}[c_1(L)^2 - (2\chi + 3\sigma)]$.
If $x_0$ is some fixed base point in $X$, the evaluation at $x_0$ gives a representation
$\rho : C^\infty(X, U(1)) \rightarrow U(1)$, which induces a $U(1)$-bundle $E \rightarrow \mathcal{M}(L)$.
If the dimension of $\mathcal{M}(L)$ is even, i.e., $d = 2s$, then the Seiberg–Witten invariant
of $L$ is given by

$$\text{SW}(L) = \langle c_1(E)^s, \mathcal{M}(L) \rangle.$$ 

The fundamental properties of the SW-invariants are as follows.

**Theorem 1.** Let $X$ be a closed smooth oriented four-manifold with $b_2^+(X) > 1$. 

1) There is only a finite number of spin$^c$ structures $L$ for which $SW(L) \neq 0$.
2) If $X = X_1 \times X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$, then $SW(L) = 0$ for all spin$^c$ structures $L$ on $X$.
3) For a spin$^c$ structures $L$ on $X$, the $SW(L)$ is independent of the metrics on $X$, and depends only on the class $c_1(L)$.
4) If $f$ is a diffeomorphism of $X$, then $SW(L) = \pm SW(f^*L)$.
5) If $X$ admits a metric of positive scalar curvature, then $SW(L) = 0$ for all spin$^c$ structures $L$ on $X$.
6) If $X$ is a closed symplectic four-manifold with canonical complex line bundle $K_X$, then $SW(K_X) = 1$.
7) A complex curve in a Kähler surface has minimum genus in its homology class.

3. Finite Group Actions on spin$^c$ Structures

Let $X$ be a closed, oriented, Riemannian four-manifold. Let $P$ be the principal bundle of oriented orthonormal frames associated to the tangent bundle $TX$ of $X$. Let $L \to X$ be a complex line bundle on $X$ satisfying $c_1(L) \equiv w_2(TX)$ mod 2. There is a one-to-one correspondence between the set of spin$^c$ structure on $X$ and the set of elements of $H^2(X, \mathbb{Z}) \otimes 2H^2(X, \mathbb{Z})$.

Let $P_L$ be the principal $U(1)$ bundle associated to the bundle $L$. Let $G$ be a finite group. Let $G$ act on $X$ by orientation preserving isometries. The induced action of $G$ on the frame bundle $\tilde{P}$ commutes with the right action of $SO(4)$ on $P$. Choose an action of $G$ over the principal $U(1)$ bundle $P_L \to X$ which is compatible with the action of $G$ on $X$, and commutes with the canonical right action of $U(1)$ on $P_L$. If the induced action of $G$ on the product $P \times P_L$ lifts to an action of $G$ on the associated principal spin$^c$ bundle $\tilde{P}$ which commutes with the right action of $Spin^c(4)$ on $P$, then the action of $G$ on $\tilde{P}$ is called a spin$^c$ action on the spin$^c$ structure $\tilde{P}$. Thus the spin$^c$ action of $G$ on $\tilde{P} \to X$ induces a diffeomorphism on $X$, $P$, $P_L$ and $\tilde{P}$ for each element in $G$ and induces bundle automorphisms on $P$, $P_L$ and $\tilde{P}$ which cover the action of $G$ on $X$.

**Proposition 1.** If the action of a finite group $G$ on a spin$^c$ structure $\tilde{P}$ associated to a line bundle $L$ over $X$ is a spin$^c$ action, then for each element $h \in G$, $h$ acts on $P$, $P_L$ and $\tilde{P}$ as a bundle automorphism which cover the action of $h$ on $X$.

Let $X$ be a closed, oriented, Riemannian four-manifold. Let $\tilde{P}$ be a principal spin$^c$ bundle associated to the line bundle $\tilde{L} \to X$. Let the action of a finite group $G$ on $\tilde{P} \to X$ be a spin$^c$ action. For each element $h \in G$ there are liftings $\tilde{h} : P \to \tilde{P}$ and $\tilde{h} : P \to \tilde{P}$. We define an action of $G$ on the twisted $\frac{1}{2}$-spinor bundle $W^+ = \tilde{P} \times \mathbb{C}^2$ by $h(\tilde{P}, a) = (h\tilde{P}, a)$ for each $h \in G$ and $(\tilde{P}, a) \in W^+$.
Since the action of $G$ on $\hat{P}$ commutes with the right action of $\text{Spin}^c(4)$ on $\hat{P}$, this action is well defined.

On the $W^+ = P \times \mathbb{C}^2$, the action of $\text{Spin}^c(4)$ on $\mathbb{C}^2$ is given by $(q_1, q_2, e^{i\theta}) a = q_1 a e^{i\theta}$ for each element

$$[q_1, q_2, e^{i\theta}] \in \text{Spin}^c(4) = (\text{SU}(2) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2, \quad a \in \mathbb{C}^2.$$

If $\hat{h} \tilde{\alpha} = \tilde{\alpha} a$ for some $\alpha = [q_1, q_2, e^{i\theta}] \in \text{Spin}^c(4)$, then

$$h(\tilde{\alpha}) = (\hat{h}) \tilde{\alpha} = (\tilde{\alpha} a) = (\tilde{\alpha} a a) = (\tilde{\alpha} q_1 a e^{i\theta}).$$

By the definition of the spin$^c$ action, the group $G$ acts on $X$ and $L$ as orientation-preserving isometries. The induced actions of $G$ on $p$ and $P_L$ commute with right actions of $\text{SO}(4)$ and $\text{U}(1)$ on $P$ and $P_L$. Thus the action of $G$ on $\hat{P}$ commutes with the lift of the connections to $\hat{P}$. If $\nabla$ is the connection of $W^+$ associated to the Levi-Civita connection on $P$ and a Riemannian connection $\hat{A}$ on $P_L$, then we have the formulas

$$h(\nabla, s) = \nabla_{h_e} h s, \quad h(D s) = h \left( \sum_i e_i \cdot \nabla_{e_i} s \right) = \sum_i h e_i \cdot \nabla_{h e_i} h s.$$

So $hD = Dh$, where $h \in G$, $v \in TX$, $s \in \Gamma(W^+)$ and $D : \Gamma(W^+) \rightarrow \Gamma(W^-)$ is the Dirac operator associated to the connection $\nabla$. Thus the Dirac operator $D$ is a $G$-equivariant elliptic operator. The $G$-index of $D$ is a virtual representation

$$L(G, X) = \ker D - \text{coker} D \in R(G).$$

For each element $h \in G$, the Lefschetz number is defined by

$$L(h, X) = \text{trace}(h|_{\ker D}) - \text{trace}(h|_{\text{coker} D}) \in \mathbb{C}.$$

**Theorem 2** (Atiyah–Singer). Let $X^h$ be the set of the fixed points of $h$ in $X$, and let $i : X^h \rightarrow X$ be the inclusion and $N^h$ the normal bundle of $X^h$ in $X$. Then the Lefschetz number $L(h, X)$ is

$$L(h, X) = (-1)^k \frac{\text{ch}_h(i^* (W^+ - W^-)) \text{td}(TX^h \otimes \mathbb{C})}{\text{e}(TX^h) \text{ch}_h(\Lambda^{-1} N^h \otimes \mathbb{C})} [X^h]$$

where $k = \frac{1}{2} \dim X^h$.

4. **Finite Group Actions and Seiberg–Witten Equations**

Let $X$ be a closed, oriented, Riemannian four-manifold. Let a finite group $G$ acts on $X$ as orientation preserving isometries. Let $\pi : L \rightarrow X$ be a complex line bundle satisfying $c_1(L) \equiv w_2(TX) \mod 2$. Let the group $G$ acts on $L$ such that the projection $\pi$ is a $G$-map. Choose a metric on $L$ on which $G$ acts by isometries. Let $\mathcal{A}(L)$ be the set of all Riemannian connections on $L$. The space $\mathcal{A}(L)$ is an affine space modelled on $\Omega^2(i\mathbb{R})$. The set of all bundle automorphisms of $L$ forms
a group $\mathcal{G}(L)$ which is identified with the set of smooth maps from $X$ into $S^1$. The group $\mathcal{G}(L)$ acts on $\mathcal{A}(L) \times \Gamma(W^+)$ by $g(A, \phi) = (A - g^{-1}dg, g^{\frac{1}{2}}\phi)$ for $g \in \mathcal{G}(L), (A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$. The $\left(\pm \frac{1}{2}\right)$-spinor bundles $S^{\pm}$ of $X$ and the square root $L^{\frac{1}{2}}$ of the bundle $L$ may not exist globally but exist locally. Since $c_1(L) \equiv w_2(TX)$ mod 2, the twisted $\left(\pm \frac{1}{2}\right)$-spinor bundles $W^{\pm} = S^{\pm} \otimes L^{\frac{1}{2}}$ do exist globally.

For each $h \in G$, $g \in \mathcal{G}(L)$, $\nabla \in \mathcal{A}(L)$, $v_i \in \Gamma(TX)$, $\sigma \in \Gamma(L)$, $\varphi \in \Omega^k$ and $\phi \in \Gamma(W^+)$, we define the actions of $G$ on these as follows:

1) $h(\nabla)_v \sigma = h(\nabla_{h^{-1}v}(h^{-1}\sigma)), h(\sigma) = h\sigma h^{-1}$
2) $h(\varphi)_{v_1, \ldots, v_k} = h(\varphi_{h^{-1}v_1, \ldots, h^{-1}v_k})$
3) $h(g) = h \circ g \circ h^{-1}$
4) $h(\phi) = \phi h^{-\frac{1}{2}},$ where $h^{\frac{1}{2}} : X \to U(2)$ is given by the lift $h^{\frac{1}{2}}(x) = \tilde{h}(x)$ of $h(x) : P_{h^{-1}(x)} \to P_x$.

**Proposition 2.** The spaces $\mathcal{A}(L), \mathcal{G}(L), \Omega^k, \Gamma(L)$ and $\Gamma(W^+)$ are closed under the action of $G$.

For each connection $A \in \mathcal{A}(L)$ on $L$ we have a Dirac operator

$$D_A : \Gamma(W^+) \to \Gamma(W^-)$$

whose symbol is given by the Clifford multiplication. The Clifford multiplication produces an isomorphism

$$\varrho : \Lambda^+ \otimes \mathbb{C} \to \text{End}(W^+)_0$$

between the complexified self-dual two-forms and the traceless endomorphisms of $W^+$. There is a pairing $\tau : W^+ \times W^+ \to \text{End}(W^+)_0$ defined by $\tau(\phi_1, \phi_2) = \phi_1 \circ \phi_2 - \frac{1}{2} \text{tr}(\phi_1 \circ \phi_2) \text{Id}$. The Seiberg–Witten equations are defined by

$$D_A \phi = 0, \quad \varrho(F^+_A) = \tau(\phi, \bar{\phi})$$

where $(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$ are invariant under the action of $G$.

**Proposition 3.** If the action of $G$ on the spin$^c$ structure $W^+$ is a spin$^c$ action, then $G$ acts on the solutions of the Seiberg–Witten equations.

Let $\mathcal{G}(L)^G = \{g \in \mathcal{G}(L) ; h \circ g \circ h^{-1} = g, h \in G\}$ be the $G$-invariant subgroup of $\mathcal{G}(L)$. Since $G$ acts on $L$ as orientation preserving isometries and the structure group of $L$ is the abelian group $U(1)$, $h \circ g \circ h^{-1} = g$ on the fixed point set $X^G$, for each element $h \in G$ and $g \in \mathcal{G}(L)$. Let $SW(L)$ be the set of solutions of the Seiberg–Witten equations.
If $b_2^+ > 0$, then the space of solutions to the Seiberg–Witten equations does not contain $(A, \phi)$ which $\phi \equiv 0$ for a generic metric of perturbation of the second equation by an imaginary-valued $G$-invariant self-dual two-form on $X$.

5. Some Results of Finite Group Action on Seiberg–Witten Invariants

Let $X$ be a closed symplectic four-manifold. The tangent bundle $TX$ of $X$ admits an almost complex structure which is an endomorphism $J : TX \to TX$ with $J^2 = -I$. The almost complex structure $J$ defines a splitting

$$T^*X \otimes \mathbb{C} = T^{1,0} \otimes T^{0,1}$$

where $J$ acts on $T^{1,0}$ and $T^{0,1}$ as multiplication by $-i$ and $i$, respectively. The canonical bundle $K_X$ on $X$ associated to the almost complex structure $J$ is defined by $K_X = \Lambda^2 T^{1,0}$.

A symplectic structure $\omega$ on $X$ is defined as a closed two-form with $\omega \wedge \omega \neq 0$ everywhere. An almost complex structure $J$ on $X$ is said to be compatible with the symplectic structure $\omega$ if $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ and $\omega(v, Jv) > 0$ for a non-zero tangent vector $v$.

The space of compatible almost complex structure of a given symplectic structure on $X$ is non-empty and constructible. If an almost complex structure $J$ is compatible with $\omega$. Then for any $v, w \in TX$, $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric on $X$. For such a metric on $X$, the symplectic structure $\omega$ is self-dual and $\omega \wedge \omega$ gives the orientation on $X$. On the other hand, any metric on $X$ for which $\omega$ is self-dual can define an almost complex structure $J$ which is compatible with the symplectic structure $\omega$.

Let $(X, \omega)$ be a closed, symplectic, four-manifold. A diffeomorphism $\sigma$ on $X$ is symplectic, anti-symplectic if $\sigma$ satisfies $\sigma^* \omega = \omega$ or $\sigma^* \omega = -\omega$, respectively.

An involution $\sigma$ on $X$ is symplectic, anti-symplectic if and only if it satisfies $\sigma_* J = J \sigma_*$ or $\sigma_* J = -J \sigma_*$, respectively, for some compatible almost complex structure $J$ on $X$ with the symplectic structure $\omega$. If $(X, \omega)$ is a Kähler surface with Kähler form $\omega$, then an involution $\sigma$ on $X$ is symplectic, anti-symplectic if and only if it is holomorphic, anti-holomorphic, respectively.

Now we assume that $X$ is a closed, smooth and oriented four-manifold with a finite fundamental group. Let $G$ be a finite group.

**Theorem 3** ([6]). If $G$ acts smoothly and freely on the four-manifold $X$ which has a non-vanishing SW-invariant, then the quotient $X/G$ cannot be decomposed as a smooth connected sum $X_1 \# X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$.

**Theorem 4** ([6]). Let $X$ be a closed and symplectic four-manifold with a finite fundamental group, $c_1(X)^2 > 0$ and $b_2^+(X) > 3$. 
If \( \sigma : X \to X \) is a free anti-symplectic involution on \( X \), then the SW-invariants vanish on the quotient \( X/\sigma \). In particular, the \( X/\sigma \) does not have any symplectic structure.

**Theorem 5** ([7]). Let \( X \) be a manifold with a nontrivial SW-invariant, \( b_2^+ (X) > 1 \), and let \( Y \) be a manifold with negative definite intersection form. If \( n_1, \ldots, n_k \) are even integers such that \( 4b_3(Y) = 2n_1 + \cdots + 2n_k + n_1^2 + \cdots + n_k^2 \) and \( \pi_3(Y) \) has a nontrivial finite quotient, then the connected sum \( X \# Y \) has a nontrivial SW-invariant but does not admit any symplectic structure.

**Theorem 6** ([10]). Let \( (X, w) \) be a closed, symplectic four-manifold and let \( \sigma \) be an anti-symplectic involution on \( (X, w) \) with fixed loci \( X^\sigma = \Pi_i \Sigma_i \) as a disjoint union of Lagrangian surfaces. If one of the components of \( X^\sigma \) is a surface of genus \( g > 1 \) and \( b_2^+ (X/\sigma) > 1 \), then the quotient \( X/\sigma \) has vanishing SW-invariants.

**Example.** Let \( X = (\Sigma_g \times \Sigma_g, w \oplus w) \), and an anti-symplectic involution

\[
f : \Sigma_g \to \Sigma_g \quad \text{with} \quad f^\ast w = w.
\]

Let \( \sigma_f : \Sigma_g \times \Sigma_g \to \Sigma_g \times \Sigma_g \) be given by

\[
\sigma_f(x, y) = (f^{-1}(y), f(x)).
\]

Then \( \sigma_f \) is an anti-symplectic involution with fixed points \( (\Sigma_g \times \Sigma_g)^{\sigma_f} \cong \Sigma_g \). By the Hirzebruch signature theorem we have

\[
b_2^+ (X/\sigma_f) = \frac{1}{2} (b_2^+ (X) - 1) = g^2 > 1, \quad \text{if} \quad g > 1.
\]

In [6] Cho shows that if the cyclic group \( \mathbb{Z}_2 \) acts smoothly and freely on a closed, oriented, smooth four-manifold \( X \) with a finite fundamental group and with a nonvanishing Seiberg–Witten invariant, then the quotient \( X/\mathbb{Z}_2 \) cannot be decomposed as a smooth connected sum \( X \# X_i \) with \( b_2^+ (X_i) > 0 \), \( i = 1, 2 \). In [28] Wang shows that if \( X \) is a Kähler surface with \( b_2^+ (X) > 3 \) and \( K_X^2 > 0 \) and if \( \sigma : X \to X \) is an anti-holomorphic involution (i.e., \( \sigma_\ast \circ J = -J \circ \sigma_\ast \)) without fixed point set then the quotient \( X/\sigma \) has vanishing Seiberg–Witten invariant.

**Theorem 7** ([9]). Let \( X \) be a Kähler surface with \( b_2^+ (X) > 3 \) and \( H_2 (X, \mathbb{Z}) \) have no two-torsion. Suppose that \( \sigma : X \to X \) is an anti-holomorphic involution with fixed point set \( \Sigma \) which is a Lagrangian surface with genus greater than 0 and \( [\Sigma] \in 2H_2 (X, \mathbb{Z}) \). If \( K_X^2 > 0 \) or \( K_X^2 = 0 \) and \( g(\Sigma) > 1 \), then the quotient \( X/\sigma \) has a vanishing Seiberg–Witten invariant.

Let \( X \) be a closed smooth four-manifold with \( \mathbb{Z}_p \) action, where \( p \) is a prime. Suppose \( H_1 (X, \mathbb{R}) = 0 \) and \( b_2^+ (X) > 1 \).
Theorem 8 ([13]). Suppose $\mathbb{Z}_p$ acts trivially on $H^2(X^+, \mathbb{R})$ of self-dual harmonic two-forms and for any $\mathbb{Z}_p$-equivariant spin$^c$-structure $L$ on $X$, the index of the equivariant Dirac operator

$$D_A : \Gamma(W^+) \longrightarrow \Gamma(W^-)$$

has the form

$$\text{ind}_{\mathbb{Z}_p}(D_A) = \sum_{j=0}^{p-1} k_j t^j \in R(\mathbb{Z}_p) = \mathbb{Z}[t]/(t^p = 1).$$

Then the Seiberg–Witten invariant satisfies

$$\text{SW}(L) = 0 \pmod{p} \quad \text{if} \quad k_j \leq \frac{1}{2} (b_2^+(X) - 1), \quad j = 0, 1, \ldots, p - 1.$$

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References

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