ON SPECIAL TYPES OF MINIMAL AND TOTALLY GEODESIC UNIT VECTOR FIELDS

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Abstract. We present a new equation with respect to a unit vector field on Riemannian manifold $M^n$ such that its solution defines a totally geodesic submanifold in the unit tangent bundle with Sasakian metric and apply it to some classes of unit vector fields. We introduce a class of covariantly normal unit vector fields and prove that within this class the Hopf vector field is a unique global one with totally geodesic property. For the wider class of geodesic unit vector fields on a sphere we give a new necessary and sufficient condition to generate a totally geodesic submanifold in $T_1S^n$.

1. Introduction

This paper is organized as follows. In Section 2 we give definitions of harmonic and minimal unit vector fields, rough Hessian and harmonicity tensor for the unit vector field. In Section 3 we give definition of a totally geodesic unit vector field and prove a basic Lemma 2 which gives a necessary and sufficient condition for the unit vector field to be totally geodesic. Theorem 2 contains a necessary and sufficient condition on strongly normal unit vector field to be minimal. In Section 4 we apply Lemma 2 to the case of a unit sphere (Lemma 4) and describe the geodesic unit vector fields on the sphere with totally geodesic property (Theorem 5). We also introduce a notion of covariantly normal unit vector field and prove that within this class the Hopf vector field is a unique one with a totally geodesic property (Theorem 3). This theorem is a revised and simplified version of Theorem 2.1 in [27]. Section 5 contains an observation that the Hopf vector field on a unit sphere provides an example of global imbedding of Sasakian space form into Sasakian manifold as a Sasakian space form with a specific $\varphi$-curvature (Theorem 6).
2. Preliminaries

2.1. Sasakian Metric

Let \((M, g)\) be \(n\)-dimensional Riemannian manifold with metric \(g\). Denote by \(\langle \cdot, \cdot \rangle\) a scalar product with respect to \(g\). A natural Riemannian metric on the tangent bundle has been defined by S. Sasaki [20]. We describe it briefly in terms of the connection map.

At each point \(Q = (q, \xi) \in TM\) the tangent space \(T_QTM\) can be split into the so-called vertical and horizontal parts

\[ T_QTM = \mathcal{H}_QTM \oplus \mathcal{V}_QTM. \]

The vertical part \(\mathcal{V}_QTM\) is tangent to the fiber, while the horizontal part is transversal to it. If \((u^1, \ldots, u^n; \xi^1, \ldots, \xi^n)\) form the natural induced local coordinate system on \(TM\), then for \(X \in T_QTM^n\) we have

\[ \hat{X} = \hat{X}^i \partial/\partial u^i + \hat{X}^n+1 \partial/\partial \xi^i \]

with respect to the natural frame \(\{\partial/\partial u^i, \partial/\partial \xi^i\}\) on \(TM\).

Denote by \(\pi : TM \to M\) the tangent bundle projection map. Then its differential \(\pi_* : T_QTM \to T_qM\) acts on \(\hat{X}\) as \(\pi_* \hat{X} = \hat{X}^i \partial/\partial x^i\) and defines a linear isomorphism between \(\mathcal{V}_QTM\) and \(T_qM\).

The so-called connection map \(K : T_QTM \to T_qM\) acts on \(\hat{X}\) by the rule \(K \hat{X} = (\hat{X}^n+1 + \Gamma^i_{jk} \xi^i \hat{X}^k) \partial/\partial u^i\) and defines a linear isomorphism between \(\mathcal{H}_QTM\) and \(T_qM\). The images \(\pi_* \hat{X}\) and \(K \hat{X}\) are called horizontal and vertical projections of \(\hat{X}\), respectively. It is easy to see that \(\mathcal{V}_Q = \ker \pi_*|Q, \mathcal{H}_Q = \ker K|Q\).

Let \(\hat{X}, \hat{Y} \in T_QTM\). The Sasakian metric on \(TM\) is defined by the following scalar product

\[ \langle \langle \hat{X}, \hat{Y} \rangle \rangle|_Q = \langle \langle \pi_* \hat{X}, \pi_* \hat{Y} \rangle \rangle_q + \langle \langle K \hat{X}, K \hat{Y} \rangle \rangle_q \]

at each point \(Q = (q, \xi)\). Horizontal and vertical subspaces are mutually orthogonal with respect to Sasakian metric.

The operations inverse to projections are called lifts. Namely, if \(X \in T_qM^n\), then \(X^h = X^i \partial/\partial u^i - \Gamma^i_{jk} \xi^j X^k \partial/\partial \xi^i\) is in \(\mathcal{H}_qTM\) and it is called a horizontal lift of \(X\), while \(X^v = X^i \partial/\partial \xi^i\), which is in \(\mathcal{V}_qTM\), is called a vertical lift of \(X\).

The Sasakian metric can be completely defined by scalar product of combinations of lifts of vector fields from \(M\) to \(TM\) as

\[ \langle \langle X^h, Y^h \rangle \rangle|_Q = \langle X, Y \rangle_q, \quad \langle \langle X^h, Y^v \rangle \rangle|_Q = 0, \quad \langle \langle X^v, Y^v \rangle \rangle|_Q = \langle X, Y \rangle_q. \]
2.2. Harmonic and Minimal Unit Vector Fields

Suppose, as above, that \( u := (u^1, \ldots, u^n) \) are the local coordinates on \( M^n \). Denote by \((u, \xi) := (u^1, \ldots, u^n; \xi^1, \ldots, \xi^n)\) the natural local coordinates in the tangent bundle \( TM^n \). If \( \xi(u) \) is a (unit) vector field on \( M^n \), then it defines a mapping

\[
\xi : M^n \to TM^n \quad \text{or} \quad \xi : M^n \to T_1M^n,
\]

when \( |\xi| = 1 \) given by \( \xi(u) = (u, \xi(u)) \).

For the mappings \( f : (M, g) \to (N, h) \) between Riemannian manifolds the energy of \( f \) is defined as

\[
E(f) := \frac{1}{2} \int_M |df|^2 \, \text{vol}_M
\]

where \( |df| \) is a norm of 1-form \( df \) in the co-tangent bundle \( T^*M \). Supposing on \( T_1M \) the Sasakian metric, the following definition becomes natural.

Definition 1. A unit vector field is called harmonic, if it is a critical point of energy functional of mapping \( \xi : M^n \to T_1M^n \).

Up to an additive constant, the energy functional of the mapping is a total bending of a unit vector field [24]

\[
B(\xi) := c_n \int_M |\nabla \xi|^2 \, \text{vol}_M
\]

where \( c_n \) is some normalizing constant and \( |\nabla \xi|^2 = \sum_{i=1}^n |\nabla_{e_i} \xi|^2 \) with respect to orthonormal frame \( e_1, \ldots, e_n \).

Introduce a point-wise linear operator \( A_\xi : T_1M^n \to \xi_\perp \), acting as

\[
A_\xi X = -\nabla_X \xi.
\]

In case of integrable distribution \( \xi_\perp \) the unit vector field \( \xi \) is called holonomic [1].

In this case the operator \( A_\xi \) is symmetric and is known as Weingarten or a shape operator for each hypersurface of the foliation. In general, \( A_\xi \) is not symmetric, but formally preserves the Codazzi equation. Namely, a covariant derivative of \( A_\xi \) is defined by

\[
-(\nabla_X A_\xi)Y = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi. \tag{1}
\]

Then for the curvature operator of \( M^n \) we can write down the Codazzi-type equation

\[
R(X, Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y.
\]

From this viewpoint, it is natural to call the operator \( A_\xi \) as non-holonomic shape operator. Remark, that the right hand side is, up to constant, a skew symmetric part of the covariant derivative of \( A_\xi \).

Introduce a symmetric tensor field

\[
\text{Hess}_\xi(X, Y) = \frac{1}{2} \left[ (\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y \right] \tag{2}
\]
which is the symmetric part of the covariant derivative of $A_\xi$. The trace

$$\sum_{i=1}^{n} \text{Hess}_\xi(e_i, e_i) := \Delta \xi$$

where $e_1, \ldots, e_n$ is an orthonormal frame, is known as rough Laplacian [2] of the field $\xi$. Therefore, one can treat the tensor field (2) as a rough Hessian of the field $\xi$.

With respect to the above given notations, the unit vector field is harmonic if and only if [24]

$$\Delta \xi = -|\nabla \xi|^2 \xi.$$

Introduce a tensor field

$$\text{Hm}_\xi(X, Y) = \frac{1}{2}[R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X]$$

which is a symmetric part of the tensor field $R(\xi, A_\xi X)Y$. The trace

$$\text{trace} \, \text{Hm}_\xi := \sum_{i=1}^{n} \text{Hm}_\xi(e_i, e_i)$$

is responsible for harmonicity of mapping $\xi : M^n \to T_1M^n$ in terms of general notion of harmonic maps [10]. Precisely, a harmonic unit vector field $\xi$ defines a harmonic mapping $\xi : M^n \to T_1M^n$ if and only if [11]

$$\text{trace} \, \text{Hm}_\xi = 0.$$

From this viewpoint, it is natural to refer to the tensor field (3) as harmonicity tensor of the field $\xi$.

Consider now the image $\xi(M^n) \subset T_1M^n$ with a pull-back Sasakian metric.

**Definition 2.** A unit vector field $\xi$ on Riemannian manifold $M^n$ is called minimal if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is minimal submanifold in the unit tangent bundle $T_1M^n$ with Sasakian metric.

A number of results on minimal unit vector fields one can find in [4, 5, 6, 8, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23]. In [25], the author has found explicitly the second fundamental form of $\xi(M^n)$ and presented some examples of unit vector fields of constant mean curvature.

### 3. Totally Geodesic Unit Vector Fields

**Definition 3.** A unit vector field $\xi$ on Riemannian manifold $M^n$ is called totally geodesic if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is totally geodesic submanifold in the unit tangent bundle $T_1M^n$ with Sasakian metric.
Using the explicit expression for the second fundamental form [25], the author gave a full description of the totally geodesic (local) unit vector fields on two-dimensional Riemannian manifold.

**Theorem 1** ([28]). Let $(M^2, g)$ be a Riemannian manifold with a sign-preserving Gaussian curvature $K$. Then $M$ admits a totally geodesic unit vector field $\xi$ if and only if there is a local parametrization of $M$ with respect to which the metric $g$ is of the form

$$\dd s^2 = \dd u^2 + \sin^2 \alpha(u) \dd v^2$$

where $\alpha(u)$ solves the differential equation $\frac{\dd \alpha}{\dd u} = 1 - \frac{a + 1}{\cos \alpha}$. The corresponding local unit vector field $\xi$ is of the form

$$\xi = \cos(au + \omega_0) \partial_u + \frac{\sin(au + \omega_0)}{\sin \alpha(u)} \partial_v$$

where $a$ and $\omega_0$ are constants.

For the case of flat Riemannian two-manifold, the totally geodesic unit vector field is either parallel or moves helically along a pencil of parallel straight lines on a plane with a constant angle speed [26]. It is easy to see that the following corollary is true.

**Corollary 1.** Integral trajectories of a totally geodesic (local) unit vector field on the non-flat Riemannian manifold $M^2$ are locally conformally equivalent to the integral trajectories of totally geodesic unit vector field on a plane. Moreover, with respect to Cartesian coordinates $(x, y)$ on the plane, these integral trajectories are

$$x = c$$

$$y(x) = -\frac{1}{a} \ln|\sin(ax)| + c$$

for $a \neq 0$

where $c$ is a parameter.

In what follows, we present a new differential equation with respect to a unit vector field such that its solution generates a totally geodesic submanifold in $T_1M^n$.

In terms of horizontal and vertical lifts of vector fields from the base to its tangent bundle, the differential of mapping $\xi : M^n \to TM^n$ acting as

$$\xi_* X = X^h + (\nabla_X \xi)^v = X^h - (A_\xi X)^v$$

where $\nabla$ means Levi-Civita connection on $M^n$ and the lifts are considered to points of $\xi(M^n)$.

It is well known that if $\xi$ is a unit vector field on $M^n$, then the vertical lift $\xi^v$ is a unit normal vector field on a hypersurface $T_1M^n \subset TM^n$. Since $\xi$ is of unit length, $\xi_* X \perp \xi^v$ and hence in this case $\xi_* : TM^n \to T(T_1M^n)$.
Denote by $A^\xi_q : \xi_q \rightarrow T_q M^n$ a formal adjoint operator
\[
\langle A^\xi_q X, Y_q \rangle = \langle X, A^\xi_q Y \rangle_q.
\]
Denote by $\xi^\perp$ a distribution on $M^n$ with $\xi$ as its normal unit vector field. Then for each vector field $N \in \xi^\perp$, the vector field
\[
\hat{N} = (A^\xi_q N)^h + N^v
\]
is normal to $\xi(M^n)$. Thus, (5) presents the normal distribution on $\xi(M^n)$.

**Lemma 1.** Let $M^n$ be a Riemannian manifold and $T_1 M^n$ its unit tangent bundle with Sasakian metric. Let $\xi$ a smooth (local) unit vector field on $M^n$. The second fundamental form $\hat{\Omega}_N$ of $\xi(M^n) \subset T_1 M^n$ with respect to the normal vector field (5) is of the form
\[
\hat{\Omega}_N(\xi^\ast X, \xi^\ast Y) = -\langle \text{Hess}_\xi(X, Y) + A_\xi \text{Hm}_\xi(X, Y), N \rangle
\]
where $X$ and $Y$ are arbitrary vector fields on $M^n$.

**Proof:** By definition, we have
\[
\hat{\Omega}_N(\xi^\ast X, \xi^\ast Y) = \langle \langle \nabla_{\xi^\ast X} \xi^\ast Y, \hat{N} \rangle \rangle_{\langle \xi, \xi(q) \rangle}
\]
where $\nabla$ is the Levi-Civita connection of Sasakian metric on $TM^n$. To calculate $\nabla_{\xi^\ast X} \xi^\ast Y$, we can use the formulas [18]
\[
\nabla_{\xi^\ast X} \xi^\ast Y = (\nabla_X Y)^h + \frac{1}{2} (\text{R}_X Y) V^v, \quad \nabla_{\xi^\ast X} Y^h = \frac{1}{2} (\text{R}(\xi, X) Y)^h
\]
\[
\nabla_{\xi^\ast X} Y^v = (\nabla_X Y)^v + \frac{1}{2} (\text{R}(\xi, Y) X)^h, \quad \nabla_{\xi^\ast X} Y^v = 0.
\]
A direct calculation yields
\[
\nabla_{\xi^\ast X} \xi^\ast Y = \left( \nabla_X Y + \frac{1}{2} \text{R}(\xi, \nabla_X \xi) Y + \frac{1}{2} \text{R}(\xi, \nabla_Y \xi) X \right)^h
\]
\[
+ \left( \nabla_X \nabla_Y \xi - \frac{1}{2} \text{R}(X, Y) \xi \right)^v.
\]
The derivative above is not tangent to $\xi(M^n)$. It contains a projection on “external” normal vector field, i.e. on $\xi^v$ which is a unit normal of $T_1 M^n$ inside $TM^n$. To correct the situation, we should subtract this projection, namely $-\langle \nabla_X \xi, \nabla_Y \xi \rangle_\xi$, from the vertical part of the derivative.

Therefore, we have
\[
\hat{\Omega}_N(\xi^\ast X, \xi^\ast Y) = \langle \nabla_X \nabla_Y \xi + \langle \nabla_X X, \nabla_Y \xi \rangle_\xi - \frac{1}{2} \text{R}(X, Y) \xi, N \rangle
\]
\[
+ \langle \nabla_X Y + \frac{1}{2} \text{R}(\xi, \nabla_X \xi) Y + \frac{1}{2} \text{R}(\xi, \nabla_Y \xi) X, A^\xi_q N \rangle
\]
or, equivalently,
\[
\hat{\Omega}(\xi, X, \xi Y) = \langle \nabla_X \nabla_Y \xi + \langle \nabla_X \xi, \nabla_Y \xi \rangle \xi - \frac{1}{2} R(X, Y) \xi \\
+ A_{\xi}(\nabla_X Y + \frac{1}{2} R(\xi, \nabla_X \xi) Y + \frac{1}{2} R(\xi, \nabla_Y \xi) X), N \rangle.
\]
Taking into account (1), (2), (3) and (5), and also
\[
R(X, Y) \xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]} \xi
\]
we can write
\[
\hat{\Omega}(\xi, X, \xi Y) = -\langle \text{Hess}_\xi(X, Y) + A_{\xi} \text{Hm}_\xi(X, Y), N \rangle
\]
which completes the proof. \(\square\)

**Lemma 2.** Let \(M^n\) be Riemannian manifold and \(T_1 M^n\) its unit tangent bundle with Sasakian metric. Let \(\xi\) be a smooth (local) unit vector field on \(M^n\). The vector field \(\xi\) generates a totally geodesic submanifold \(\xi(M^n) \subset T_1 M^n\) if and only if \(\xi\) satisfies
\[
\text{Hess}_\xi(X, Y) + A_{\xi} \text{Hm}_\xi(X, Y) - \langle A_{\xi} X, A_{\xi} Y \rangle \xi = 0 \tag{7}
\]
for all (local) vector fields \(X, Y\) on \(M^n\).

**Proof:** Taking into account (6), the condition on \(\xi\) to be totally geodesic takes the form
\[
- \text{Hess}_\xi(X, Y) - A_{\xi} \text{Hm}_\xi(X, Y) = \lambda \xi.
\]
Multiplying the equation above by \(\xi\), we can find easily \(\lambda = -\langle A_{\xi} X, A_{\xi} Y \rangle\). \(\square\)

Following [16], we call a unit vector field \(\xi\) **strongly normal** if
\[
\langle (\nabla_X A_{\xi}) Y, Z \rangle = 0
\]
for all \(X, Y, Z \in \xi^\perp\). In other words, \((\nabla_X A_{\xi}) Y = \lambda \xi\) for all \(X, Y \in \xi^\perp\). It is easy to find the function \(\lambda\). Indeed, we have
\[
\lambda = \langle (\nabla_X A_{\xi}) Y, \xi \rangle = \langle \nabla_X Y \xi - \nabla_Y \nabla_X \xi, \xi \rangle \\
= -\langle \nabla_X \nabla_Y \xi, \xi \rangle = \langle \nabla_X \xi, \nabla_Y \xi \rangle.
\]
Thus, the strongly normal unit vector field can be characterized by the equation
\[
(\nabla_X A_{\xi}) Y = \langle A_{\xi} X, A_{\xi} Y \rangle \xi \tag{8}
\]
for all \(X, Y \in \xi^\perp\).

The strong normality condition highly simplifies the second fundamental form of \(\xi(M^n) \subset T_1 M^n\). An orthonormal frame \(e_1, e_2, \ldots, e_n\) is called **adapted** to the field \(\xi\) if \(e_1 = \xi\) and \(e_2, \ldots, e_n \in \xi^\perp\).
Lemma 3. Let $\xi$ be a unit strongly normal vector field on Riemannian manifold $M^n$. With respect to the adapted frame, the matrical components of the second fundamental form of $\xi(M^n) \subset T_1(M^n)$ simultaneously take the form

$$
\hat{\Omega}_N = \begin{pmatrix}
* & * & \ldots & * \\
* & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & 0 & \ldots & 0
\end{pmatrix}.
$$

Proof: Set $N_\sigma = e_\sigma$, $\sigma = 2, \ldots, n$. The condition (8) implies

$$
R(X, Y)\xi = 0, \quad \text{Hess}_\xi(X, Y) = (A_\xi X, A_\xi Y)\xi, \quad \text{Hm}_\xi(X, Y) \sim \xi
$$

for all $X, Y \in \xi \perp$. Therefore, with respect to the adapted frame

$$
\hat{\Omega}_\sigma(\xi, e_\alpha, \xi, e_\beta) = 0, \quad \alpha, \beta = 2, \ldots, n
$$

for all $\sigma = 2, \ldots, n$. □

The following assertion is a natural corollary of the Lemma 3.

Theorem 2. Let $\xi$ be a unit strongly normal vector field. Denote by $k$ the geodesic curvature of its integral trajectories and by $\nu$ the principal normal unit vector field of the trajectories. The field $\xi$ is minimal if and only if

$$
k[\xi, \nu] + \xi(k)\nu - kA_\xi R(\nu, \xi)\xi + k^2\xi = 0
$$

where $[\xi, \nu] = \nabla_\xi \nu - \nabla_\nu \xi$.

Proof: Indeed,

$$
\hat{\Omega}_\sigma(\xi, e_1, \xi, e_1) = -\langle \text{Hess}_\xi(\xi, \xi) + A_\xi \text{Hm}_\xi(\xi, \xi), e_\sigma \rangle.
$$

Denote by $\nu$ a vector field of the principal normals of $\xi$-integral trajectories and by $k$ their geodesic curvature function. Then

$$
\text{Hess}_\xi(\xi, \xi) = \nabla_{\nabla_\xi} - \nabla_\xi \nabla_\xi = k\nabla_\nu \xi - k\nabla_{\xi(k)}(k)\nu = k[\nu, \xi] + \xi(k)\nu
$$

and we get

$$
\hat{\Omega}_\sigma(\xi, e_1, \xi, e_1) = \langle k[\xi, \nu] + \xi(k)\nu - kA_\xi R(\nu, \xi)\xi, e_\sigma \rangle.
$$

Finally, to be minimal, the field $\xi$ should satisfy

$$
k[\xi, \nu] + \xi(k)\nu - kA_\xi R(\nu, \xi)\xi = \lambda \xi.
$$

Multiplying by $\xi$, we get

$$
\lambda = k[\xi, \nu], \xi = k(\nabla_\xi \nu, \xi) = -k^2
$$

which completes the proof. □
Thus, we get the following

**Corollary 2** ([16]). *Every unit strongly normal geodesic vector field is minimal.*

Most of examples of minimal unit vector fields in [16] are based on this Corollary.

### 4. The Case of a Unit Sphere

If the manifold is a unit sphere $S^{n+1}$, the equation (7) can be simplified essentially.

**Lemma 4.** A unit (local) vector field $\xi$ on a unit sphere $S^{n+1}$ generates a totally geodesic submanifold $\xi(S^{n+1}) \subset T S^{n+1}$ if and only if $\xi$ satisfies

$$
(\nabla_X A_\xi)Y = \frac{1}{2} \left[ (\mathcal{L}_\xi g)(X,Y) A_\xi \xi + \langle \xi, X \rangle (A_\xi^2 Y + Y) \\
+ \langle \xi, Y \rangle (A_\xi^2 X - X) \right] + \langle A_\xi X, A_\xi Y \rangle \xi
$$

(9)

where $(\mathcal{L}_\xi g)(X,Y) = \langle \nabla_X \xi, Y \rangle + \langle X, \nabla_Y \xi \rangle$ is a Lie derivative of metric tensor in a direction of $\xi$.

**Proof:** Indeed, on the unit sphere

$$
(\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y = R(X,Y)\xi = \langle \xi, Y \rangle X - \langle \xi, X \rangle Y.
$$

Hence,

$$
\text{Hess}_\xi(X, Y) = (\nabla_X A_\xi)Y + \frac{1}{2} \langle \xi, Y \rangle X - \langle \xi, X \rangle Y
$$

For $\text{Hm}_\xi(X, Y)$ we have

$$
\text{Hm}_\xi(X, Y) = \frac{1}{2} \left[ \langle \nabla_X \xi, Y \rangle \xi - \langle \xi, Y \rangle \nabla_X \xi + \langle \nabla_Y \xi, X \rangle \xi - \langle \xi, X \rangle \nabla_Y \xi \right]
$$

$$
= \frac{1}{2} (\mathcal{L}_\xi g)(X,Y) \xi + \frac{1}{2} \left[ \langle \xi, Y \rangle A_\xi X + \langle \xi, X \rangle A_\xi Y \right].
$$

Finally, we find

$$
(\nabla_X A_\xi)Y = \frac{1}{2} \left[ (\mathcal{L}_\xi g)(X,Y) A_\xi \xi + \langle \xi, X \rangle (A_\xi^2 Y + Y) + \langle \xi, Y \rangle (A_\xi^2 X - X) \right]
$$

$$
+ \langle A_\xi X, A_\xi Y \rangle \xi.
$$

Remind that the operator $A_\xi$ is symmetric if and only if the field $\xi$ is holonomic, and is skew-symmetric if and only if the field $\xi$ is a Killing vector field. Both types of these fields can be included into a class of **covariantly normal unit vector fields**.
**Definition 4.** A regular unit vector field on a Riemannian manifold is said to be covariantly normal if the operator \( A_\xi : TM \rightarrow \xi^\perp \) defined by \( A_\xi X = -\nabla_X \xi \) satisfies the normality condition

\[
A_\xi^t A_\xi = A_\xi A_\xi^t
\]

with respect to some orthonormal frame.

The integral trajectories of holonomic and Killing unit vector fields are always geodesic. Every covariantly normal unit vector field possesses this property.

**Lemma 5.** Integral trajectories of a covariantly normal unit vector field are geodesic lines.

**Proof:** Suppose \( \xi \) is a unit covariantly normal vector field on a Riemannian manifold \( M^{n+1} \). Find a unit vector field \( \nu_1 \) such that

\[
\nabla_\xi \xi = -k \nu_1.
\]

Geometrically, the function \( k \) is a geodesic curvature of the integral trajectory of the field \( \xi \).

Complete up the pair \((\xi, \nu_1)\) to the orthonormal frame \((\xi, \nu_1, \ldots, \nu_n)\). Then we can set

\[
\nabla_\xi \xi = -k \nu_1, \quad \nabla_{\nu_\alpha} \xi = -a_\alpha^\beta \nu_\beta
\]

where \( \alpha, \beta = 1, \ldots, n \). With respect to the frame \((\xi, \nu_1, \ldots, \nu_n)\) the matrix \( A_\xi \) takes the form

\[
-A_\xi = \begin{pmatrix}
0 & k & 0 & \cdots & 0 \\
0 & a_1^1 & a_2^1 & \cdots & a_n^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_1^n & a_2^n & \cdots & a_n^n
\end{pmatrix}
\]

and, therefore,

\[
A_\xi A_\xi^t = \begin{pmatrix}
k^2 & ka_1^1 & \cdots & ka_n^1 \\
ka_1^1 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
ka_n^1 & * & \cdots & *
\end{pmatrix}, \quad A_\xi^t A_\xi = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}
\]

which allows to conclude that \( k = 0 \). \( \square \)

Now we can easily prove the following

**Theorem 3.** Let \( \xi \) be a global covariantly normal unit vector field on a unit sphere \( S^{n+1} \). Then \( \xi \) is a totally geodesic if and only if \( n = 2m \) and \( \xi \) is a Hopf vector field.
**Proof:** Suppose $\xi$ is covariantly normal and totally geodesic. Then

$$A_\xi \xi = -\nabla_\xi \xi = 0$$

by Lemma 5 and the equation (9) takes the form

$$(\nabla_X A_\xi) Y = \frac{1}{2} \left[ \langle \xi, X \rangle (A_\xi^2 Y + Y) + \langle \xi, Y \rangle (A_\xi^2 X - X) \right] + \langle A_\xi X, A_\xi Y \rangle \xi.$$  (10)

Setting $X = Y = \xi$ we get an identity. Set $Y = \xi$ and take arbitrary unit $X \perp \xi$. Then we get

$$2(\nabla_X A_\xi) \xi + X = A_\xi^2 X.$$  

On the other hand, directly

$$(\nabla_X A_\xi) \xi = - (\nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi) = A_\xi^2 X.$$  

Hence,

$$A_{\xi}^2 \big|_{\perp} = -E.$$  

Therefore, $n = 2m$. Since $A_{\xi}$ is real normal linear operator, there exists an orthonormal frame such that

$$A_{\xi} = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & \ddots \\
& & 0 & 1 \\
& & -1 & 0
\end{pmatrix}$$

with zero all other entries. Therefore, $A_{\xi} + A_{\xi}^\dagger = 0$ and $\xi$ is a Killing vector field. Since $\xi$ is supposed global, $\xi$ is a Hopf vector field.

Finally, if we take $X, Y \perp \xi$, we get the equation

$$(\nabla_X A_\xi) Y = \langle A_\xi X, A_\xi Y \rangle \xi.$$  

But for a Killing vector field $\xi$ we have [16]

$$(\nabla_X A_\xi) Y = R(\xi, X) Y = \langle X, Y \rangle \xi.$$  

Since $\xi$ is a Hopf vector field, $\langle A_\xi X, A_\xi Y \rangle = \langle X, Y \rangle$. So, in this case we have an identity.

If we suppose now that $\xi$ is a Hopf vector field on a unit sphere, then $\xi$ is covariantly normal as a Killing vector field and totally geodesic [27] as a characteristic vector field of a standard contact metric structure on $S^{2m+1}$.

Theorem 3 is a correct and simplified version of Theorem 2.1 [27], where the normality of the operator $A_{\xi}$ was implicitly used in a proof.

In the case of a weaker condition on the field $\xi$ to be only a geodesic one, the result is not so definite. We begin with some preparations.
The almost complex structure on $TM^n$ is defined by

$$JX^h = X^v, \quad JX^v = -X^h$$

for all vector field $X$ on $M^n$. Thus, $TM^n$ with Sasakian metric is an almost Kählerian manifold. It is Kählerian if and only if $M^n$ is flat [9].

The unit tangent bundle $T_1M^n$ is a hypersurface in $TM^n$ with a unit normal vector $\xi^v$ at each point $(q, \xi) \in T_1M^n$. Define a unit vector field $\tilde{\xi}$, a 1-form $\tilde{\eta}$ and a $(1, 1)$ tensor field $\tilde{\varphi}$ on $T_1M^n$ by

$$\tilde{\xi} = -J\xi^v = \xi^h, \quad JX = \tilde{\varphi}X + \tilde{\eta}(X)\xi^v.$$ 

The triple $(\tilde{\xi}, \tilde{\eta}, \tilde{\varphi})$ form a standard almost contact structure on $T_1M^n$ with Sasakian metric $g_\tilde{S}$. This structure is not almost contact metric one. By taking

$$\tilde{\xi} = 2\tilde{\xi} = 2\xi^h, \quad \tilde{\eta} = \frac{1}{2}\tilde{\eta}, \quad \tilde{\varphi} = \varphi, \quad g_{em} = \frac{1}{4}g_\tilde{S}$$

at each point $(q, \xi) \in T_1M^n$, we get the almost contact metric structure $(\tilde{\xi}, \tilde{\eta}, \tilde{\varphi})$ on $(T_1M^n, g_{em})$.

In a case of a general almost contact metric manifold $(\tilde{M}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi}, \tilde{g})$ the following definition is known [7].

**Definition 5.** A submanifold $N$ of a contact metric manifold $(\tilde{M}, \tilde{\xi}, \tilde{\eta}, \tilde{\varphi}, \tilde{g})$ is called invariant if $\tilde{\varphi}(T_pN) \subset T_pN$ and anti-invariant if $\tilde{\varphi}(T_pN) \subset (T_pN)^\perp$ for every $p \in N$.

If $N$ is the invariant submanifold, then the characteristic vector field $\tilde{\xi}$ is tangent to $N$ at each of its points.

After all mentioned above, the following definition is natural [3].

**Definition 6.** A unit vector field $\xi$ on a Riemannian manifold $(M^n, g)$ is called invariant (anti-invariant) is the submanifold $\xi(M^n) \subset (T_1M^n, g_{em})$ is invariant (anti-invariant).

It is easy to see from (4) that the invariant unit vector field is always a geodesic one, i.e. its integral trajectories are geodesic lines.

Binh, Boeckx and Vanhecke [3] have considered this kind of unit vector fields and proved the following

**Theorem 4.** A unit vector field $\xi$ on $(M^n, g)$ is invariant if and only if $(\tilde{\xi} = \xi, \tilde{\eta} = (\tilde{\xi}, \tilde{\varphi} = A_\xi)$ is an almost contact structure on $M^n$. In particular, $\xi$ is a geodesic vector field on $M^n$ and $n = 2m + 1$.

Now we can formulate the result.

**Theorem 5.** A unit geodesic vector field $\xi$ on $S^{n+1}$ is totally geodesic if and only if $n = 2m$ and $\xi$ is a strongly normal invariant unit vector field.
Proof: Suppose \( \xi \) is a geodesic and totally geodesic unit vector field. Then \( A_\xi \xi = 0 \) and the equation (9) takes the form (10). Follow the proof of Theorem 3, we come to the following conditions on the field \( \xi \)

\[
A_\xi^2 X = -X, \quad (\nabla_X A_\xi)Y = \langle A_\xi X, A_\xi Y \rangle \xi
\]

(11)

for all \( X, Y \in \xi^\perp \). From the left equation in (11) we conclude that \( n = 2m \). Comparing the right one with (8), we see that \( \xi \) is a strongly normal vector field. Consider now a \((1, 1)\) tensor field \( \varphi = A_\xi = -\nabla \xi \) and a 1-form \( \eta = \langle \cdot, \xi \rangle \). Taking into account the left equation in (11) and \( A_\xi \xi = 0 \), we see that

\[
\varphi^2 X = -X + \eta(X)\xi, \quad \varphi \xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(X) = 1
\]

for any vector field \( X \) on the sphere. Therefore, the triple

\[
\varphi = A_\xi, \quad \xi = \xi, \quad \tilde{\eta} = \langle \cdot, \xi \rangle
\]

form an almost contact structure with the field \( \xi \) as a characteristic vector field of this structure. By Theorem 4, the field \( \xi \) is invariant.

Conversely, suppose \( \xi \) is strongly normal and invariant vector field on \( S^{n+1} \). Then, by Theorem 4, \( \xi \) is geodesic and \( n = 2m \). The rest of the proof is a direct checking of formula (10).

\[\square\]

5. A Remarkable Property of the Hopf Vector Field

It is well-known that for a unit sphere \( S^n \) the standard contact metric structure on \( T_1 S^n \) is a Sasakian one. If \( \xi \) is a Hopf unit vector field on \( S^{2m+1} \), then \( \xi \) is a characteristic vector field of a standard contact metric structure on the unit sphere \( S^{2m+1} \). By Theorem 4, the submanifold \( \xi(S^{2m+1}) \) is invariant submanifold in \( T_1 S^{2m+1} \). Therefore, \( \xi(S^{2m+1}) \) is also Sasakian with respect to the induced structure [29]. Since the Hopf vector field is strongly normal, by Theorem 5, the submanifold \( \xi(S^{2m+1}) \) is totally geodesic. The sectional curvature of the submanifold \( \xi(S^{2m+1}) \) was found in [27] and implies a remarkable corollary.

Theorem 6. Let \( \xi \) be a Hopf vector field on the unit sphere \( S^{2m+1} \). With respect to the induced structure, the manifold \( \xi(S^{2m+1}) \) is a Sasakian space form of \( \varphi \)-curvature \( 5/4 \).

In other words, the Hopf vector field provides an example of embedding of a Sasakian space form of \( \varphi \)-curvature 1 into Sasakian manifold such that the image is contact, totally geodesic Sasakian space form of \( \varphi \)-curvature \( 5/4 \) with respect to the induced structure.
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