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SYMMETRIES AND CONSERVATION LAWS OF PLATES AND SHELLS INTERACTING WITH FLUID FLOW

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Abstract. The present study is concerned with thin isotropic shallow shells interacting with inviscid fluid flow of constant velocity. It is assumed that the dynamic behaviour of the shells is governed by the Marguerre–von Kármán equations. The influence of the fluid flow is taken into account by introducing additional differential and external load terms in the shell equations. It is shown that the system of equations governing such a fluid-structure interaction is equivalent to the von Kármán equations. Thus, the symmetries and conservation laws of the considered fluid-structure system are established.

1. Introduction

A wide variety of mathematical models describing fluid-structure interactions have been suggested in the past 50 years. To the best of our knowledge, the first one is due to Niordson [12], where a single fourth-order linear partial differential equation governing the flow-induced vibration of pipes within Bernoulli–Euler beam theory is derived. Later on this equation has been obtained in another way by Benjamin [4] and used by many authors to ascertain substantial features of various beam-like structures contacting fluid flows. Fifteen years after the Niordson's paper, a study by Kornecki [9] on flow-induced vibrations of plates appeared, followed by the papers of Brazier–Smith and Scott [6] and Crighton and Oswell [7] on the same topic. In these papers the plate displacement is supposed to satisfy the fourth-order linear partial differential equation governing the dynamics of transversely loaded thin elastic plate. In this connection, we would like to mention that in earlier papers Benjamin [3] and Landahl [11] consider the motion of fluid bounded by an initially flat infinite flexible surface. However, in the latter papers, the surface motion is supposed to be prescribed and a fluid motion, coupled to this given surface motion, is sought. In contrast, Brazier–Smith and Scott [6] and Crighton and Oswell [7] consider surface motion that is a solution of the differential equation governing the dynamics of a plate contacting the moving fluid. In recent papers, Amabili *et al* [1, 2] have considered flow-induced vibration of cylindrical shells due to external or internal flows within the framework of the nonlinear Donnell–Mushtari–Vlasov (DMV) shell theory (for details see, e.g., Niordson [13]). The same phenomena, but within the Sanders nonlinear shell theory, consider Zhang *et al* [23, 24]. It should be underlined that the mathematical problems considered in all recent studies on flow-induced vibrations of shells turn out to be very complicated and therefore they are treated only numerically. The aim of the present study is to derive analytical results in this field of research – symmetries and conservation laws for the smooth solutions of the differential equations governing the large transverse vibrations of shallow shells contacting flowing fluids.

The symmetries of equations governing bending, stability or dynamics of structures have been studied for more than 20 years. All Lie symmetries of the von Kármán equations are obtained by Shwarz [18] and the respective Noether's conservation laws for the solutions of these equations are reported by Saccomandi and Salvatori [17] and Djondjorov and Vassilev [8]. The point Lie symmetries and the associated conservation laws for the equations governing the statics or dynamics of thin shells within the framework of DMV theory are presented in [19, 20]. Thus, the most important invariance properties (the point Lie symmetries, the conservation laws related to them through Noether's theorem and some invariant solutions) of the time-dependent and time-independent equations of thin shells within DMV theory in the absence of fluid are already clarified. To the best of our knowledge, invariance properties of equations describing fluid-structure interaction are studied only by Vassilev *et al* [21] and Vassilev and Djondjorov [22]. In these papers, the invariance properties of a class of fourth-order linear partial differential equations governing the flow-induced vibration of Bernoulli–Euler pipes on elastic foundations of Winkler type are ascertained. It seems natural to extend the foregoing studies exploring the invariance properties of differential equations governing the interaction of shallow shells with fluid flow, which is the subject of the present contribution.

2. Fluid-Structure Interaction Problem

Consider a thin isotropic elastic shell of uniform thickness h. Let (x^1, x^2, z) be a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space and $t \equiv x^3$ be the time. Let the shell middle-surface S be given by the equation

$$S : z = f(x^1, x^2), \qquad (x^1, x^2) \in \Omega \subset \mathbb{R}^2 \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded single-valued smooth function possessing as many derivatives as may be required on the domain Ω. Let us take (x^1, x^2) to serve as Gaussian coordinates on the surface S. Then, relative to this coordinate system, the components of the first $a_{\alpha\beta}$ and second $b_{\alpha\beta}$ fundamental tensors and the alternating tensor $\varepsilon^{\alpha\beta}$ of S are given by the expressions

$$a_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha f \partial_\beta f, \qquad b_{\alpha\beta} = a^{-1/2}\partial_\alpha\partial_\beta f, \qquad \varepsilon^{\alpha\beta} = a^{-1/2}e^{\alpha\beta} \tag{2}$$

where $a = \det(a_{\alpha\beta}) = 1 + (\partial_1 f)^2 + (\partial_2 f)^2$; $\delta_{\alpha\beta} = \delta^{\alpha\beta}$ is the Kronecker delta symbol; $e^{\alpha\beta}$ is the alternating symbol; ∂_α denote the partial derivatives with respect to the coordinates on S. Here and in what follows: Greek (Latin) indices range over 1, 2 (1, 2, 3), unless explicitly stated otherwise; the usual summation convention over a repeated index (one subscript and one superscript) is used.

Suppose that the mechanical behaviour of the shell is governed by the DMV shell theory. Within the framework of this theory, the large deflection of a thin isotropic elastic shell is described by the following system of two coupled nonlinear fourth-order partial differential equations

$$\begin{gathered} D\Delta^2 w - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}(\nabla_\alpha\nabla_\beta w + b_{\alpha\beta})\nabla_\mu\nabla_\nu F = p \\ \frac{1}{Eh}\Delta^2 F + \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}(\nabla_\alpha\nabla_\beta w + 2b_{\alpha\beta})\nabla_\mu\nabla_\nu w = 0 \end{gathered} \tag{3}$$

in two independent variables – the coordinates on the shell middle-surface S, and two dependent variables – the transversal displacement function w, and Airy's stress function F. Here p is the external transversal load; D and E are the bending rigidity and Young's modulus of the shell, respectively, which are supposed to be given constants, ∇_α denotes covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$ of the surface S and Δ is the **Laplace–Beltrami operator** on S.

In this paper, we shall restrict ourselves to shells with approximately Euclidean geometry. The latter means that the inequalities

$$|\partial_\alpha f|\,|\partial_\beta f| \le \varepsilon^2 \ll 1, \qquad \varepsilon = \text{const} \tag{4}$$

are supposed to hold for every point $(x^1, x^2) \in \Omega$. In this case the quadratic terms in the right-hand sides of expressions (2) are small compared to unity, they may be neglected, and thus allowing for a relative error of order $O(\varepsilon^2)$ one may regard the intrinsic geometry of the shell middle-surface S as Euclidean and (x^1, x^2) may be thought of as an Euclidean coordinate system on S, in which

$$a_{\alpha\beta} = \delta_{\alpha\beta}, \qquad b_{\alpha\beta} = \partial_\alpha\partial_\beta f, \qquad \varepsilon^{\alpha\beta} = e^{\alpha\beta} \tag{5}$$

and the **mean curvature** H of the surface S and its **Gaussian curvature** K read

$$\begin{aligned} H &= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}\delta^{\alpha\beta}\partial_\alpha\partial_\beta f \\ K &= \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}b_{\alpha\beta}b_{\mu\nu} = \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta f)(\partial_\mu\partial_\nu f) \end{aligned}$$

(note that the latter is not necessarily equal to zero within the allowed relative error). In this case the system (3) simplifies to

$$
\begin{aligned}
K_1[w,F] &\equiv D\Delta^2 w - e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta w + \partial_\alpha\partial_\beta f)\partial_\mu\partial_\nu F = p \\
K_2[w,F] &\equiv \frac{1}{Eh}\Delta^2 F + \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta w + 2\partial_\alpha\partial_\beta f)\partial_\mu\partial_\nu w = 0
\end{aligned} \tag{6}
$$

where $\Delta = \delta^{\alpha\beta}\partial_\alpha\partial_\beta$.

Introducing, according to d'Alembert principle, the inertia force $-m\partial_3\partial_3 w$ in the right-hand side of the first equation (6), $\partial_3\partial_3 w$ being the second derivative of the displacement field with respect to the time and m – the mass per unit area of the shell middle-surface, one can extend system (6) to

$$
K_1[w,F] + m\partial_3\partial_3 w = p, \qquad K_2[w,F] = 0 \tag{7}
$$

so as to describe the dynamic behaviour of the shells.

Let the shell interacts with inviscid incompressible fluid of uniform density ρ, being a boundary of the fluid domain. Suppose that the motion of the fluid is potential, that is there exists a velocity potential function $\Phi(x^1, x^2, z, t)$, satisfying the Laplace equation

$$
\tilde{\Delta}\Phi = 0 \tag{8}
$$

in the fluid domain and such that the velocity field $\mathbf{V}(x^1, x^2, z, t)$ is given by the expression

$$
\mathbf{V} = \tilde{\nabla}\Phi \tag{9}
$$

and the pressure $P(x^1, x^2, z, t)$ – by the Bernoulli equation

$$
\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left(\tilde{\nabla}\Phi\right)^2 + \frac{P}{\rho} = q(t). \tag{10}
$$

Here $q(t)$ is an arbitrary function, $\tilde{\nabla} \equiv (\partial/\partial x^1, \partial/\partial x^2, \partial/\partial z)$ and $\tilde{\Delta}$ denotes the Laplace operator with respect to the variables x^1, x^2 and z. Finally, the so-called kinematic condition is supposed to hold at the undeformed shell middle-surface S, that is

$$
\partial_3(f+w) = \left.\frac{\partial\Phi}{\partial z}\right|_S - \delta^{\alpha\beta}\partial_\alpha(f+w)\,\partial_\beta\Phi|_S \tag{11}
$$

(see, e.g., Benjamin and Olver [5] and Lamb [10]).

Suppose that far enough from the shell (i.e., $z \to \infty$) the motion of the fluid represents a uniform flow with velocity $\mathbf{U} = (U, 0, 0)$, where U is a constant. Assume that in the whole fluid domain the difference between the fluid velocity $\mathbf{V}$ and the velocity $\mathbf{U}$ of the uniform flow is small compared to the latter. Then, the potential can be taken in the form

$$
\Phi(x^1, x^2, z, t) = Ux^1 + \varphi(x^1, x^2, z, t) \tag{12}
$$

where

$$\tilde{\Delta}\varphi = 0, \qquad \lim_{z\to\infty} \tilde{\nabla}\varphi = 0, \qquad \left|\tilde{\nabla}\varphi\right| \ll |\mathbf{U}|. \tag{13}$$

For such fluid motion, substituting the potential (12) in the Bernoulli equation (10) and taking into account the third equation in (13), one obtains the following expression for the pressure

$$P = -\rho L[\varphi] - \rho\frac{1}{2}U^2 + \rho q(t) \tag{14}$$

where the differential operator L is defined by

$$L = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x^1}. \tag{15}$$

Additionally, substituting the potential (12) in the kinematic condition (11) and linearizing it (neglecting the products of $\partial_\alpha\varphi$, $\partial_\alpha f$ and $\partial_\alpha w$ in comparison with the linear terms), one finds

$$\left.\frac{\partial\varphi}{\partial z}\right|_S = L[f + w]. \tag{16}$$

The pressure (14) is a part of the transverse loading on the shell and if there are no other prescribed transverse loadings we have

$$p = P|_S. \tag{17}$$

In this case, equations (7) read

$$K_1[w, F] + m\partial_3\partial_3 w = P|_S, \qquad K_2[w, F] = 0. \tag{18}$$

Thus, the displacement field w, Airy stress function F and the potential φ describe a coupled fluid-shell motion if they satisfy system (18) and condition (16) at the fluid-shell boundary S and equations (13) in the fluid domain. Of course, the functions w, F and φ should meet some appropriate boundary conditions depending on the particular problem. However, the specific form of these boundary conditions do not affect the results in this study and so they are omitted here.

Applying the Laplace operator $\tilde{\Delta}$ and the operator $\partial/\partial z$ to the equation (14), and taking into account the kinematic condition (16), one obtains

$$\tilde{\Delta}P = 0, \qquad \left.\frac{\partial P}{\partial z}\right|_S = -\rho L^2[f + w]. \tag{19}$$

Hence, the pressure P can be written in the form

$$P = -\tilde{M}\mathrm{e}^{\frac{z-f}{\tilde{M}}}\rho L^2[f + w] + \tilde{P} \tag{20}$$

where

$$\left.\frac{\partial\tilde{P}}{\partial z}\right|_S = 0 \tag{21}$$

and $\tilde{M}$ is a constant introduced to ensure the equality of the physical dimensions of both sides of equation (20). On account of (20), equations (18) governing the shell dynamics take the form

$$K_1[w,F] + ML^2[f+w] + m\partial_3\partial_3 w = \tilde{P}|_S, \qquad K_2[w,F] = 0 \tag{22}$$

where $M = \rho\tilde{M}$. Note that the physical dimension of the constant M is mass per unit surface.

The case $\tilde{P}|_S = 0$ in equations (22) implies that the transverse load on the structure is of form $p = -ML^2[f+w]$ and this is the most widely used approach in the dynamics of pipes and rods subjected to parallel fluid flow (see [4, 16], etc). In these studies, the influence of the flow on the structure is only due to the mean flow. The disturbances, which the structure induces in the fluid are considered negligible for the structure response. Therefore, extending this understanding over the problem of fluid-shell interaction, one can interpret the terms in equations (22) as follows: $ML^2[f+w]$ is the influence of the mean flow and $\tilde{P}|_S$ is the influence of the mean flow disturbances on the shell response.

3. Symmetries and Conservation Laws

It is a simple matter to verify (though it was not so easy to arrive at this conclusion) that equations (22) are equivalent to the von Kármán equations. Indeed, the following transformation of the independent and dependent variables

$$\begin{gathered} x^1 \to x^1 - \frac{M}{m+M}Ux^3, \qquad x^2 \to x^2, \qquad x^3 \to x^3 \\ w \to w + f, \qquad F \to F - \frac{1}{2}\frac{mMU^2}{m+M}(x^2)^2 \end{gathered} \tag{23}$$

maps this system to the nonhomogeneous von Kármán equations

$$\begin{gathered} D\Delta^2 w - e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta w)(\partial_\mu\partial_\nu F) + (m+M)\partial_3\partial_3 w = \tilde{P}|_S + 2D\delta^{\mu\nu}\partial_\mu\partial_\nu H \\ \frac{1}{Eh}\Delta^2 F + \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta w)(\partial_\mu\partial_\nu w) = K. \end{gathered} \tag{24}$$

Thus, the problem of invariance of system (22) converts into the problem of invariance of system (24) as a change of the variables does not affect the group properties of a system of differential equations (see [15]). As mentioned before, the invariance properties of system (24) are already established. Its symmetry groups are determined by Shwarz [18] for the homogeneous case and by Vassilev [20] for the nonhomogeneous one. The corresponding conservation laws for the solutions of the homogeneous equations (24) are reported in [17] and [8]. The densities and fluxes of the basic conservation laws are listed in [8, Table 1]. In the nonhomogeneous case, each of the aforementioned conservation laws transforms to a conservation law with an appropriate source term (see [20]).

Hence, in order to write down explicitly the generators of the Lie symmetries and the currents of the conservation laws that hold on the smooth solutions of equations (22) one has to apply the transformation (23) and its inverse to the corresponding results, reported in [8] and [20]. For the sake of brevity, the explicit expressions are not presented here. As an example, the density Ψ and the flux P^α of the energy conservation law are only given. Up to a trivial conservation law (see [14]), they read

$$\begin{aligned}\Psi &= \frac{1}{2}(m+M)(\partial_3 w)^2 + \frac{1}{2}MU[\partial_3(w\partial_1 w) + Uw\partial_1\partial_1 w]\\ &+ \frac{D}{2}\Big[(\Delta w)^2 - (1-\nu)e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta w)(\partial_\mu\partial_\nu w)\Big]\\ &+ \frac{1}{2Eh}\Big[(\Delta F)^2 - (1+\nu)e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta F)(\partial_\mu\partial_\nu F)\Big]\\ &+ \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}(\partial_\alpha\partial_\beta F)(\partial_\mu w)(\partial_\nu w)\\ P^\alpha &= -Q^\alpha\partial_3 w - (\partial_3 F)\partial_\nu G^{\alpha\nu} + M^{\alpha\beta}\partial_3\partial_\beta w + G^{\alpha\beta}\partial_3\partial_\beta F\\ &+ \frac{1}{2}MU\Big(-U(\partial_1 w)\partial_3 w - (\partial_3 w)^2 + Uw\partial_1\partial_3 w + w\partial_3\partial_3 w\Big)\end{aligned}$$

where

$$\begin{aligned}G^{\alpha\beta} &= \frac{1}{Eh}\Big[(1+\nu)\delta^{\alpha\mu}\delta^{\beta\nu} - \nu\delta^{\alpha\beta}\delta^{\mu\nu}\Big]\partial_\mu\partial_\nu F - \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}(\partial_\mu w)\partial_\nu w\\ M^{\alpha\beta} &= -D\Big[(1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\alpha\beta}\delta^{\mu\nu}\Big]\partial_\mu\partial_\nu w\\ Q^\alpha &= \partial_\mu M^{\alpha\mu} + e^{\alpha\sigma}e^{\mu\nu}(\partial_\mu w)\partial_\sigma\partial_\nu F.\end{aligned}$$

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