EFFECTIVE SOLUTIONS OF AN INTEGRABLE CASE OF THE HÉNON–HEILES SYSTEM

ANGEL ZHIVKOV and IOANNA MAKAVEEVA

Faculty of Mathematics and Informatics, Sofia University
5 James Bourchier Blvd, 1164 Sofia, Bulgaria

Abstract. We solve in two-dimensional theta functions the integrable case \( \ddot{r} = -ar + 2zr, \quad \ddot{z} = -bz + 6z^2 + r^2 \) (\( a \) and \( b \) are constant parameters) of the generalized Hénon–Heiles system. The general solution depends on six arbitrary constants, called algebraic–geometric coordinates. Three of them are coordinates on the degree two (and dimension three) Siegel upper half-plane and define two-dimensional tori \( T^2 \). Each trajectory of the Hénon–Heiles system lies on certain torus \( T^2 \). Next two arbitrary constants define the initial position on \( T^2 \). The speed of the flow depends multiplicatively on the last arbitrary constant.

Consider a galaxy which gravitational potential \( U_{gr} \) is time-independent and has an axis of symmetry. We are interested in the motion of a star in such a potential field.

Let us introduce a system of cylindrical coordinates \((r, \psi, z)\): \( Oz \) is the axis of symmetry, \( z \) is the height of the star, \( r := \sqrt{x^2 + y^2} \) is the distance between the star and the axis \( Oz \), \( \psi := \arctan \frac{y}{x} \) is the polar angle.

Two conservation laws (integrals) of the stellar motion are known:

\[
I_1 = U_{gr}(r, z) + \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\psi}^2 + \dot{z}^2 \right) = \text{total energy},
\]
\[
I_2 = mr^2 \dot{\psi} = \text{angular momentum of the star around } Oz \text{ axis},
\]

\( m \) is the mass of the star, \( \dot{\psi} = \frac{d}{dt} \) is the derivative with respect to the time \( t \).

With the help of the second integral \( I_2 \) we reduce the dynamics of the star on