ON THE GEOMETRY OF BIHARMONIC SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. We classify all proper-biharmonic Legendre curves in a Sasakian space form and point out some of their geometric properties. Then we provide a method for constructing anti-invariant proper-biharmonic submanifolds in Sasakian space forms. Finally, using the Boothby-Wang fibration, we determine all proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in complex projective spaces.

1. Introduction

As defined by Eells and Sampson in [14], harmonic maps \( f : (M, g) \rightarrow (N, h) \) are the critical points of the energy functional

\[
E(f) = \frac{1}{2} \int_M \|df\|^2 \, v_g
\]

and they are solutions of the associated Euler-Lagrange equation

\[
\tau(f) = \text{trace}_g \nabla df = 0
\]

where \( \tau(f) \) is called the tension field of \( f \). When \( f \) is an isometric immersion with mean curvature vector field \( H \), then \( \tau(f) = mH \) and \( f \) is harmonic if and only if it is minimal.

The bienergy functional (proposed also by Eells and Sampson in 1964, [14]) is defined by

\[
E_2(f) = \frac{1}{2} \int_M \|\tau(f)\|^2 \, v_g.
\]

The critical points of $E_2$ are called **biharmonic maps** and they are solutions of the Euler-Lagrange equation (derived by Jiang in 1986, [20]):

$$\tau_2(f) = -\Delta^f \tau(f) - \text{trace}_g R^N(df, \tau(f))df = 0$$

where $\Delta^f$ is the Laplacian on sections of $f^{-1}TN$ and $R^N(X,Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} - \nabla_{[X,Y]}$ is the curvature operator on $N$; $\tau_2(f)$ is called the **bitension field** of $f$. Since all harmonic maps are biharmonic, we are interested in studying those which are biharmonic but non-harmonic, called **proper-biharmonic** maps.

Now, if $f : M \to N_c$ is an isometric immersion into a space form of constant sectional curvature $c$, then

$$\tau(f) = mH \quad \text{and} \quad \tau_2(f) = -m\Delta^f H + cm^2H.$$  

Thus $f$ is biharmonic if and only if

$$\Delta^f H = mcH.$$  

In a different way, Chen defined the biharmonic submanifolds in an Euclidean space as those with harmonic mean curvature vector field ([10]). Replacing $c = 0$ in the above equation we just reobtain Chen’s definition. Moreover, let $f : M \to \mathbb{R}^n$ be an isometric immersion. Set $f = (f^1, \ldots, f^n)$ and $H = (H^1, \ldots, H^n)$. Then $\Delta^f H = (\Delta H^1, \ldots, \Delta H^n)$, where $\Delta$ is the Beltrami-Laplace operator on $M$, and $f$ is biharmonic if and only if

$$\Delta^f H = \Delta \left( \frac{-\Delta f}{m} \right) = -\frac{1}{m} \Delta^2 f = 0.$$  

There are several classification results for the proper-biharmonic submanifolds in Euclidean spheres and non-existence results for such submanifolds in the space forms manifolds $N_c$, $c \leq 0$ ([4, 5, 7–10, 13]), while in spaces of non-constant sectional curvature only a few results were obtained ([1, 12, 18, 19, 25, 29]).

We recall that the proper-biharmonic curves of the unit Euclidean two-dimensional sphere $S^2$ are the circles of radius $\frac{1}{\sqrt{2}}$, and the proper-biharmonic curves of $S^3$ are the geodesics of the minimal Clifford torus $S^1 \left( \frac{1}{\sqrt{2}} \right) \times S^1 \left( \frac{1}{\sqrt{2}} \right)$ with the slope different from $\pm 1$. The proper-biharmonic curves of $S^3$ are helices. Further, the proper-biharmonic curves of $S^n$, $n > 3$, are those of $S^3$ (up to a totally geodesic embedding). Concerning the hypersurfaces of $S^n$, it was conjectured in [4] that the only proper-biharmonic hypersurfaces are the open parts of $S^{n-1} \left( \frac{1}{\sqrt{2}} \right)$ or $S^{m_1} \left( \frac{1}{\sqrt{2}} \right) \times S^{m_2} \left( \frac{1}{\sqrt{2}} \right)$ with $m_1 + m_2 = n - 1$ and $m_1 \neq m_2$.

Since odd dimensional unit Euclidean spheres $S^{2n+1}$ are Sasakian space forms with constant $\varphi$-sectional curvature one, the next step is to study the biharmonic submanifolds of Sasakian space forms. In this paper we mainly gather the results obtained in [15–17].
We note that the proper-biharmonic submanifolds in pseudo-Riemannian mani-
folds are also intensively-studied (for example, see [2, 3, 11]).
For a general account of biharmonic maps see [22] and The Bibliography of Bio-
harmonic Maps [28].
Conventions. We work in the $C^\infty$ category, that means manifolds, metrics, con-
nections and maps are smooth. The Lie algebra of the vector fields on $N$ is denoted
by $C(TN)$.

2. Sasakian Space Forms

In this section we briefly recall some basic facts from the theory of Sasakian mani-
folds. For more details see [6].
A contact metric structure on a manifold $N^{2n+1}$ is given by $(\varphi, \xi, \eta, g)$, where
$\varphi$ is a tensor field of type $(1, 1)$ on $N$, $\xi$ is a vector field on $N$, $\eta$ is an one-form on
$N$ and $g$ is a Riemannian metric, such that
$$\varphi^2 = -I + \eta \otimes \xi,$$
$$\eta(\xi) = 1$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$g(X, \varphi Y) = d\eta(X, Y)$$
for any $X, Y \in C(TN)$.
A contact metric structure $(\varphi, \xi, \eta, g)$ is Sasakian if it is normal, i.e.,
$$N\varphi + 2d\eta \otimes \xi = 0$$
where for all $X, Y \in C(TN)$
$$N\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$$
is the Nijenhuis tensor field of $\varphi$.
The contact distribution of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in
TN; \eta(X) = 0\}$, and any integral curve of the contact distribution is called
Legendrian curve.
A submanifold $M$ of $N$ which is tangent to $\xi$ is said to be anti-invariant if $\varphi$ maps
any vector tangent to $M$ and normal to $\xi$ to a vector normal to $M$.
Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a two-plane
generated by $X$ and $\varphi X$, where $X$ is an unit vector orthogonal to $\xi$, is called
$\varphi$-sectional curvature determined by $X$. A Sasakian manifold with constant $\varphi$-
sectional curvature $c$ is called a Sasakian space form and it is denoted by $N(c)$.
A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is called regular if for any point $p \in N$
there exists a cubic neighborhood of $p$ such that any integral curve of $\xi$ passes
through the neighborhood at most once, and strictly regular if all integral curves are
homeomorphic to each other.
Let \((N, \varphi, \xi, \eta, g)\) be a regular contact metric manifold. Then the orbit space \(\tilde{N} = N/\xi\) has a natural manifold structure and, moreover, if \(N\) is compact then \(\tilde{N}\) is a principal circle bundle over \(\tilde{N}\) (the Boothby-Wang Theorem). In this case the fibration \(\pi : N \to \tilde{N}\) is called Boothby-Wang fibration. The Hopf fibration \(\pi : S^{2n+1} \to \mathbb{C}P^n\) is a well-known example of a Boothby-Wang fibration.

**Theorem 1** ([24]). Let \((N, \varphi, \xi, \eta, g)\) be a strictly regular Sasakian manifold. Then on \(\tilde{N}\) can be given the structure of a Kähler manifold. Moreover, if \((N, \varphi, \xi, \eta, g)\) is a Sasakian space form \(N(c)\), then \(\tilde{N}\) has constant sectional holomorphic curvature \(c + 3\).

Even if \(N\) is non-compact, we still call the fibration \(\pi : N \to \tilde{N}\) of a strictly regular Sasakian manifold, the Boothby-Wang fibration.

### 3. Biharmonic Legendre Curves in Sasakian Space Forms

Let \((N^n, g)\) be a Riemannian manifold and \(\gamma : I \to N\) a curve parametrized by arc length. Then \(\gamma\) is called a Frenet curve of osculating order \(r\), \(1 \leq r \leq n\), if there exists orthonormal vector fields \(E_1, E_2, \ldots, E_r\) along \(\gamma\) such that \(E_1 = \gamma' = T, \nabla_TE_1 = \kappa_1 E_2, \nabla_TE_2 = -\kappa_1 E_1 + \kappa_2 E_3, \ldots, \nabla_TE_r = -\kappa_{r-1} E_{r-1}\), where \(\kappa_1, \ldots, \kappa_{r-1}\) are positive functions on \(I\).

A geodesic is a Frenet curve of osculating order one, a *circle* is a Frenet curve of osculating order two with \(\kappa_1 = \text{const}\), a *helix of order} \(r\), \(r \geq 3\), is a Frenet curve of osculating order \(r\) with \(\kappa_1, \ldots, \kappa_{r-1}\) constants and a helix of order three is called, simply, helix.

In [16] we studied the biharmonicity of Legendre Frenet curves and we obtained the following results.

Let \((N^{2n+1}, \varphi, \xi, \eta, g)\) be a Sasakian space form with constant \(\varphi\)-sectional curvature \(c\) and \(\gamma : I \to N\) a Legendre Frenet curve of osculating order \(r\). Then \(\gamma\) is biharmonic if and only if

\[
\tau_2(\gamma) = \nabla_T^2 T - R(T, \nabla_T T) T
\]

\[
= (-3\kappa_1 \kappa'_1)E_1 + \left(\kappa''_1 - \kappa_1^3 - \kappa_1 \kappa_2^2 + \frac{(c + 3) \kappa_1}{4}\right) E_2
\]

\[
+ (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 + \frac{3(c - 1) \kappa_1}{4} g(E_2, \varphi T) \varphi T
\]

\[
= 0.
\]

The expression of the bitension field \(\tau_2(\gamma)\) imposed a case-by-case analysis as follows.
Case I \((c = 1)\)

**Theorem 2** ([16]). If \(c = 1\) then \(\gamma\) is proper-biharmonic if and only if \(n \geq 2\) and either \(\gamma\) is a circle with \(\kappa_1 = 1\) or \(\gamma\) is a helix with \(\kappa_1^2 + \kappa_2^2 = 1\).

Case II \((c \neq 1\) and \(E_2 \perp \varphi T)\)

**Theorem 3** ([16]). Assume that \(c \neq 1\) and \(E_2 \perp \varphi T\). We have

1) if \(c \leq -3\) then \(\gamma\) is biharmonic if and only if it is a geodesic;
2) if \(c > -3\) then \(\gamma\) is proper-biharmonic if and only if either
   a) \(n \geq 2\) and \(\gamma\) is a circle with \(\kappa_1^2 = \frac{c+3}{4}\), or
   b) \(n \geq 3\) and \(\gamma\) is a helix with \(\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}\).

Case III \((c \neq 1\) and \(E_2 \parallel \varphi T)\)

**Theorem 4** ([16]). If \(c \neq 1\) and \(E_2 \parallel \varphi T\), then \(\{T, \varphi T, \xi\}\) is the Frenet frame field of \(\gamma\) and we have

1) if \(c < 1\) then \(\gamma\) is biharmonic if and only if it is a geodesic
2) if \(c > 1\) then \(\gamma\) is proper-biharmonic if and only if it is a helix
   with \(\kappa_1^2 = c - 1\) and \(\kappa_2 = 1\).

**Remark 1.** In dimension 3 the result was obtained by Inoguchi in [19] and explicit examples are given in [15].

Case IV \((c \neq 1\) and \(g(E_2, \varphi T)\) is not constant 0, 1 or \(-1)\)

**Theorem 5** ([16]). Let \(c \neq 1\) and \(\gamma\) a Legendre Frenet curve of osculating order \(r\) such that \(g(E_2, \varphi T)\) is not constant 0, 1 or \(-1\). We have

1) if \(c \leq -3\) then \(\gamma\) is biharmonic if and only if it is a geodesic;
2) if \(c > -3\) then \(\gamma\) is proper-biharmonic if and only if \(r \geq 4\),
   \(\varphi T = \cos \alpha_0 E_2 + \sin \alpha_0 E_4\) and
   \[
   \kappa_1, \kappa_2, \kappa_3 = \text{const} > 0
   \]
   \[
   \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4} \cos^2 \alpha_0
   \]
   \[
   \kappa_2 \kappa_3 = -\frac{3(c-1)}{8} \sin(2\alpha_0)
   \]
   where \(\alpha_0 \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}\) is a constant such that
   \[
   c + 3 + 3(c-1) \cos^2 \alpha_0 > 0, \quad 3(c-1) \sin(2\alpha_0) < 0.
   \]
In order to obtain explicit examples of proper-biharmonic Legendre curves given by Theorem 2 we used the unit Euclidean sphere \(S^{2n+1}\) as a model of a Sasakian space form with \(c = 1\) and we proved the following
**Theorem 6** ([16]). Let \( \gamma : I \to S^{2n+1}(1) \), \( n \geq 2 \), be a proper-biharmonic Legendre curve parametrized by arc length. Then the parametric equation of \( \gamma \) in the Euclidean space \( \mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle \cdot, \cdot \rangle) \) is either

\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos \left( \sqrt{2}s \right) e_1 + \frac{1}{\sqrt{2}} \sin \left( \sqrt{2}s \right) e_2 + \frac{1}{\sqrt{2}} e_3
\]

where \( \{e_i, Ie_j\} \) are constant unit vectors orthogonal to each other, or

\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos (As)e_1 + \frac{1}{\sqrt{2}} \sin (As)e_2 + \frac{1}{\sqrt{2}} \cos (Bs)e_3 + \frac{1}{\sqrt{2}} \sin (Bs)e_4
\]

where \( A = \sqrt{1 + \kappa_1}, \ B = \sqrt{1 - \kappa_1}, \ \kappa_1 \in (0,1) \), \( \{e_i\} \) are constant unit vectors orthogonal to each other such that

\[
\langle e_1, Ie_3 \rangle = \langle e_1, Ie_4 \rangle = \langle e_2, Ie_3 \rangle = \langle e_2, Ie_4 \rangle = 0 \\
A\langle e_1, Ie_2 \rangle + B\langle e_3, Ie_4 \rangle = 0
\]

and \( I \) is the usual complex structure on \( \mathbb{R}^{2n+2} \).

**Remark 2.** For the Cases II and III we also obtained the explicit equations of proper-biharmonic Legendre curves in odd dimensional spheres endowed with the deformed Sasakian structure introduced in [27].

In [21] are introduced the complex torsions for a Frenet curve in a complex manifold. In the same way, for \( \gamma : I \to N \) a Legendre Frenet curve of osculating order \( r \) in a Sasakian manifold \( (N^{2n+1}, \varphi, \xi, \eta, g) \), we define the \( \varphi \)-torsions \( \tau_{ij} = g(E_i, \varphi E_j) = -g(\varphi E_i, E_j), \ i, j = 1, \ldots, r, \ i < j \).

It is easy to see that we can formulate

**Proposition 1.** Let \( \gamma : I \to N(c) \) be a proper-biharmonic Legendre Frenet curve in a Sasakian space form \( N(c), \ c \neq 1 \). Then \( c > -3 \) and \( \tau_{12} \) is a constant.

Moreover

**Proposition 2.** If \( \gamma \) is a proper-biharmonic Legendre Frenet curve in a Sasakian space form \( N(c), \ c > -3, \ c \neq 1 \), of osculating order \( r < 4 \), then it is a circle or a helix with constant \( \varphi \)-torsions.

**Proof:** From Theorems 3, 4 and 5 we see that if \( \gamma \) is a proper-biharmonic Legendre Frenet curve of osculating order \( r < 4 \), then \( \tau_{12} = 0 \) or \( \tau_{12} = \pm 1 \) and, obviously, we only have to prove that when \( \gamma \) is a helix then \( \tau_{13} \) and \( \tau_{23} \) are constants.

Indeed, by using the Frenet equations of \( \gamma \), we have

\[
\tau_{13} = g(E_1, \varphi E_3) = \frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2 + \kappa_1 E_1) = \frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2)
\]

\[
= \frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \varphi E_1) = \frac{1}{\kappa_2} g(E_2, \varphi \nabla_{E_1} E_1 + \xi) = 0
\]
since
\[
g(E_2, \xi) = \frac{1}{\kappa_1} g(\nabla_{E_1} E_1, \xi) = -\frac{1}{\kappa_1} g(E_1, \nabla_{E_1} \xi) = \frac{1}{\kappa_1} g(E_1, \varphi E_1) = 0.
\]

On the other hand, it is easy to see that for any Frenet curve of osculating order three we have \(\tau_{23} = \frac{1}{\kappa_1} (\tau'_{13} + \kappa_2 \tau_{12} + \eta(E_3))\) and
\[
\eta(E_3) = g(E_3, \xi) = \frac{1}{\kappa_2} (g(\nabla_{E_1} E_2, \xi) + \kappa_1 g(E_1, \xi)) = -\frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \xi)
\]
\[= -\frac{1}{\kappa_2} \tau_{12}.
\]
In conclusion \(\tau_{23} = \frac{1}{\kappa_1} (\tau'_{13} + \kappa_2 \tau_{12} - \frac{1}{\kappa_2} \tau_{12}) = \text{const.}\) □

**Proposition 3.** If \(\gamma\) is a proper-biharmonic Legendre Frenet curve in a Sasakian space form \(N(c)\) of osculating order \(r = 4\), then \(c \in \left(\frac{4}{3}, 5\right)\) and the curvatures of \(\gamma\) are
\[
\kappa_1 = \frac{\sqrt{c + 3}}{2}, \quad \kappa_2 = \frac{1}{2} \sqrt{\frac{6(c - 1)(5 - c)}{c + 3}}, \quad \kappa_3 = \frac{1}{2} \sqrt{\frac{3(c - 1)(3c - 7)}{c + 3}}.
\]
Moreover, the \(\varphi\)-torsions of \(\gamma\) are given by
\[
\tau_{12} = \mp \sqrt{\frac{2(5 - c)}{c + 3}}, \quad \tau_{13} = 0, \quad \tau_{14} = \pm \sqrt{\frac{3c - 7}{c + 3}}
\]
\[
\tau_{23} = \mp \sqrt{\frac{3c - 7}{\sqrt{3(c - 1)(c + 3)}}}, \quad \tau_{24} = 0, \quad \tau_{34} = \pm \sqrt{\frac{2(5 - c)(3c - 7)}{3(c - 1)(c + 3)}}.
\]

**Proof:** Let \(\gamma\) be a proper-biharmonic Legendre Frenet curve in \(N(c)\) of osculating order \(r = 4\). Then \(c \neq 1\) and \(\tau_{12}\) is different from 0, 1 or -1. From Theorem 5 we have \(\varphi E_1 = \cos \alpha_0 E_2 + \sin \alpha_0 E_4\). It results that
\[
\tau_{12} = -\cos \alpha_0, \quad \tau_{13} = 0, \quad \tau_{14} = -\sin \alpha_0 \quad \text{and} \quad \tau_{24} = 0.
\]
In order to prove that \(\tau_{23}\) is constant we differentiate the expression of \(\varphi E_1\) along \(\gamma\) and using the Frenet equations we obtain
\[
\nabla_{E_1} \varphi E_1 = \cos \alpha_0 \nabla_{E_1} E_2 + \sin \alpha_0 \nabla_{E_1} E_4
\]
\[= -\kappa_1 \cos \alpha_0 E_1 + (\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) E_3.
\]
On the other hand, \(\nabla_{E_1} \varphi E_1 = \kappa_1 \varphi E_2 + \xi\) and therefore we have
\[
\kappa_1 \varphi E_2 + \xi = -\kappa_1 \cos \alpha_0 E_1 + (\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) E_3.
\]
(1)

We take the scalar product in (1) with \(\xi\) and obtain
\[
(\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) \eta(E_3) = 1.
\]
(2)
In the same way as in the proof of Proposition 2 we get
\[
\eta(E_3) = g(E_3, \xi) = \frac{1}{\kappa_2} \left( g(\nabla E_1 E_2, \xi) + \kappa_1 g(E_1, \xi) \right)
\]
\[
= -\frac{1}{\kappa_2} g(E_2, \nabla E_1 \xi)
\]
\[
= -\frac{1}{\kappa_2} \tau_2 = \frac{\cos \alpha_0}{\kappa_2}
\]
and then, from (2), \( \kappa_2 \sin \alpha_0 = -\kappa_3 \cos \alpha_0 \). Therefore \( \alpha_0 \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi) \).

Next, from Theorem 5, we have
\[
\kappa_1 = \frac{c + 3}{4}, \quad \kappa_2 = \frac{3(c - 1)}{4} \cos^2 \alpha_0, \quad \kappa_3 = \frac{3(c - 1)}{4} \sin^2 \alpha_0
\]
and so \( c \) must be greater than one.

Now, we take the scalar product in (1) with \( E_3, \varphi E_2 \) and \( \varphi E_4 \), respectively, and we get
\[
\kappa_1 \tau_{23} = - (\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) + \eta(E_3) = -\frac{\kappa_2}{\cos \alpha_0} + \frac{\cos \alpha_0}{\kappa_2}
\]
(3)
\[
\kappa_1 \sin^2 \alpha_0 = - (\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) \tau_{23} = - \frac{\kappa_2}{\cos \alpha_0} \tau_{23}
\]
(4)
\[
0 = \kappa_1 \cos \alpha_0 \sin \alpha_0 + (\kappa_2 \cos \alpha_0 - \kappa_3 \sin \alpha_0) \tau_{34}
\]
\[
= \kappa_1 \cos \alpha_0 \sin \alpha_0 + \frac{\kappa_2}{\cos \alpha_0} \tau_{34}
\]
(5)
and then, equations (3) and (4) lead to \( \kappa_1^2 \sin^2 \alpha_0 = \frac{\kappa_2^2}{\cos^2 \alpha_0} - 1 \). We come to the conclusion that \( \sin^2 \alpha_0 = \frac{3c - 7}{c + 3} \), so \( \alpha \in \left( \frac{7}{3}, 5 \right) \), and then we obtain the expressions of the curvatures and the \( \varphi \)-torsions.

Remark 3. The proper-biharmonic Legendre curves given by Theorem 6 (for the case \( c = 1 \)) have also constant \( \varphi \)-torsions.

4. A Method to Obtain Biharmonic Submanifolds in a Sasakian Space Form

In [16] we gave a method to obtain proper-biharmonic anti-invariant submanifolds in a Sasakian space form from proper-biharmonic integral submanifolds.

Theorem 7 ([16]). Let \( (\mathbb{N}^{2n+1}, \varphi, \xi, \eta, g) \) be a strictly regular Sasakian space form with constant \( \varphi \)-sectional curvature \( c \) and let \( i : M \rightarrow N \) be an r-dimensional integral submanifold of \( N \), \( 1 \leq r \leq n \). Consider
\[
F : \widetilde{M} = I \times M \rightarrow N, \quad F(t, p) = \phi_t(p) = \phi_p(t)
\]
where $I = S^1$ or $I = \mathbb{R}$ and $\{\phi_t\}_{t \in I}$ is the flow of the vector field $\xi$. Then $F: (\widetilde{M}, \widetilde{g}) = dt^2 + \gamma^*g) \to N$ is a Riemannian immersion and it is proper-biharmonic if and only if $M$ is a proper-biharmonic submanifold of $N$.

The previous Theorem provides a classification result for proper-biharmonic surfaces in a Sasakian space form, which are invariant under the flow-action of $\xi$.

**Theorem 8** ([16]). Let $M^2$ be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field $\xi$. Then $M$ is proper-biharmonic if and only if, locally, it is given by $x(t, s) = \phi_t(\gamma(s))$, where $\gamma$ is a proper-biharmonic Legendre curve.

Also, using the standard Sasakian 3-structure on $S^7$, by iteration, Theorem 7 leads to examples of three-dimensional proper-biharmonic submanifolds of $S^7$.

### 5. Biharmonic Hopf Cylinders in a Sasakian Space Form

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold and $\bar{\gamma}: \bar{M} \to \bar{N}$ a submanifold of $\bar{N}$. Then $M = \pi^{-1}(\bar{M})$ is the Hopf cylinder over $\bar{M}$, where $\pi: N \to \bar{N} = N/\xi$ is the Boothby-Wang fibration.

In [19] the biharmonic Hopf cylinders in a three-dimensional Sasakian space form are classified.

**Theorem 9** ([19]). Let $S_{\bar{\gamma}}$ be a Hopf cylinder, where $\bar{\gamma}$ is a curve in the orbit space of $N^3(c)$, parametrized by arc length. We have

1) if $c \leq 1$, then $S_{\bar{\gamma}}$ is biharmonic if and only if it is minimal;

2) if $c > 1$, then $S_{\bar{\gamma}}$ is proper-biharmonic if and only if the curvature $\bar{k}$ of $\bar{\gamma}$ is constant $\bar{k}^2 = c - 1$.

In [17] we obtained a geometric characterization of biharmonic Hopf cylinders of any codimension in an arbitrary Sasakian space form. A special case of our result is the case when $\bar{M}$ is a hypersurface.

**Proposition 4** ([17]). If $\bar{M}$ is a hypersurface of $\bar{N}$, then $M = \pi^{-1}(\bar{M})$ is biharmonic if and only if

$$\Delta^\perp H = \left( -\|B\|^2 + \frac{c(n + 1) + 3n - 1}{2} \right) H$$

$$2 \text{ trace } A_{\nabla^\perp H}(\cdot) + n \text{ grad}(\|H\|^2) = 0$$

where $B$, $A$ and $H$ are the second fundamental form of $M$ in $N$, the shape operator and the mean curvature vector field, respectively, and $\nabla^\perp$ and $\Delta^\perp$ are the normal connection and Laplacian on the normal bundle of $M$ in $N$. 

Proposition 5 ([17]). If \( \tilde{M} \) is a hypersurface and \( \| \tilde{H} \| = \text{const} \neq 0 \), then \( M = \pi^{-1}(\tilde{M}) \) is proper-biharmonic if and only if
\[
\| \tilde{H} \|^2 = \frac{c(n + 1) + 3n - 5}{2}.
\]

Remark 4. From the last result we see that there exist no proper-biharmonic hypersurfaces of constant mean curvature \( M = \pi^{-1}(\tilde{M}) \) in \( N(c) \) if \( c \leq \frac{5 - 3n}{n+1} \), which implies that such hypersurfaces do not exist if \( c \leq -3 \), whatever the dimension of \( N \) is.

In [26] Takagi classified all homogeneous real hypersurfaces in the complex projective space \( \mathbb{C}P^n \), \( n > 1 \), and found five types of such hypersurfaces (see also [23]). The first type (with subtypes A1 and A2) are described in the following.

We shall consider \( u \in (0, \frac{\pi}{2}) \) and \( r \) a positive constant given by \( \frac{1}{r^2} = \frac{c + 3}{4} \).

Theorem 10 ([26]). The geodesic spheres (Type A1) in complex projective space \( \mathbb{C}P^n(c + 3) \) have two distinct principal curvatures: \( \lambda_2 = \frac{1}{r} \cot u \) of multiplicity \( 2n - 2 \) and \( a = \frac{2}{r} \cot 2u \) of multiplicity one.

Theorem 11 ([26]). The hypersurfaces of Type A2 in complex projective space \( \mathbb{C}P^n(c + 3) \) have three distinct principal curvatures: \( \lambda_1 = -\frac{1}{r} \tan u \) of multiplicity \( 2p \), \( \lambda_2 = \frac{1}{r} \cot u \) of multiplicity \( 2q \), and \( a = \frac{2}{r} \cot 2u \) of multiplicity one, where \( p > 0, q > 0 \), and \( p + q = n - 1 \).

We note that if \( c = 1 \) and \( \tilde{M} \) is of type A1 or A2 then \( \pi^{-1}(\tilde{M}) = S^1(\cos u) \times S^{2n-1}(\sin u) \subset S^{2n+1} \) or \( \pi^{-1}(\tilde{M}) = S^{2p+1}(\cos u) \times S^{2q+1}(\sin u) \), respectively.

By using Takagi’s result we classified in [17] the biharmonic Hopf cylinders \( M = \pi^{-1}(\tilde{M}) \) in a Sasakian space form \( N^{2n+1} \) over homogeneous real hypersurfaces in \( \mathbb{C}P^n \), \( n > 1 \).

Theorem 12 ([17]). Let \( M = \pi^{-1}(\tilde{M}) \) be the Hopf cylinder over \( \tilde{M} \).

1) If \( \tilde{M} \) is of Type A1, then \( M \) is proper-biharmonic if and only if either
   a) \( c = 1 \) and \( \tan^2 u = 1 \), or
   
   \[ b) \quad c \in \left( \frac{-3n^2 + 2n + 1 + 8\sqrt{2n - 1}}{n^2 + 2n + 5}, +\infty \right) \setminus \{1\} \quad \text{and} \]
   
   \[ \tan^2 u = n + \frac{2c - 2}{c + 3} + \frac{\sqrt{c^2(n^2 + 2n + 5)^2 + 2c(3n^2 - 2n - 1) + 9n^2 - 30n + 13}}{c + 3} \]

2) If \( \tilde{M} \) is of Type A2, then \( M \) is proper-biharmonic if and only if either
   a) \( c = 1 \), \( \tan^2 u = 1 \) and \( p \neq q \), or
\[ b) \ c \in \left[ \frac{-3(p-q)^2 - 4n+4 + 8 \sqrt{(p+1)(2q+1)}}{(p-q)^2 + 4n+4}, +\infty \right) \setminus \{1\} \text{ and} \]
\[ \tan^2 u = \frac{n}{2p+1} + \frac{2c-2}{(c+3)(2p+1)} \pm \frac{\sqrt{c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4}}{(c+3)(2p+1)}. \]

**Theorem 13** ([17]), There are no proper-biharmonic hypersurfaces \( M = \pi^{-1}(\bar{M}) \) when \( \bar{M} \) is a hypersurface of Type B, C, D or E in the complex projective space \( \mathbb{C}\mathbb{P}^n(c+3) \).

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**References**


