GREEN’S FUNCTION FOR 5D $SU(2)$ MIC-KEPLER PROBLEM

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Abstract. The Green’s function for 5-dimensional counterpart of the MIC-Kepler problem (Kepler potential plus $SU(2)$ Yang–Mills instanton plus Zwanziger-like $1/R^2$ centrifugal term) is constructed on the basis of the Green’s function for the 8-dimensional harmonic oscillator.

1. Introduction

Coulomb Green’s functions in a $n$-dimensional Euclidean space have been constructed in [1]. The results for the cases $n = 2, 3, 5$ can be deduced from the oscillator Green’s functions in $N = 2, 4, 8$ dimensions due to Levi-Civita, Kustaanheimo–Stiefel [2] and Hurwitz transformations [3], respectively.

Moreover [4], the $N = 4$ oscillator representation allows to obtain Green’s function for 3-dimensional MIC-Kepler problem [5] (Kepler–Coulomb potential plus $U(1)$ Dirac monopole plus Zwanziger’s [6] $1/R^2$ centrifugal term).

In this paper we construct the Green’s function for 5-dimensional counterpart of the MIC-Kepler problem [7] (Kepler potential plus $SU(2)$ Yang–Mills instanton plus Zwanziger-like $1/R^2$ centrifugal term). We avoid a tedious procedure of path integration and deduce our result from the well-known expression for the 8-dimensional oscillator Green’s function by exploiting the Hurwitz correspondence between these 5- and 8-dimensional problems [7–9].
2. Correspondence Between 5- and 8-Dimensional Problems

Under the certain known conditions [7–9] there appears the correspondence between the 8-dimensional harmonic oscillator problem

$$H \psi^{(8)} = E \psi^{(8)}, \quad H = -\frac{1}{2} \Delta_8 + \frac{\omega^2}{2} (|u|^2 + |v|^2)$$

(1)

and 5-dimensional $SU(2)$ MIC-Kepler problem

$$\mathcal{H}^l \phi^l = \mathcal{E}^l \phi^l, \quad \mathcal{H}^l = \frac{\pi_\mu^2}{2} + \frac{l(l + 1)}{2R^2} - \frac{a}{R},$$

(2)

where the covariant derivative $\pi_\mu = -i \partial_\mu - A^a_\mu \Lambda^{2l+1}_a$ contains $SU(2)$ Yang–Mills instanton [10] as the gauge potential defined due to

$$A^a_\mu dr_\mu = \frac{1}{R(R + r_0)} \left(-r_4 dr_a + r_a dr_4 - \epsilon_{abc} r_b dr_c\right),$$

(3)

$$\mu = 0, \ldots, 4, \quad a, b, c = 1, 2, 3,$$

and $\Lambda^{2l+1}_a$ are the generators of the $(2l + 1)$-dimensional representation of $SU(2)$.

These conditions are the following.

1. The coordinates of 5D Euclidean space are expressed through that of 8D one by means of the Hurwitz transformation

$$r_0 = |u|^2 - |v|^2, \quad r = 2uv,$$

(4)

(5)

where $u = u_0 + u_a e_a$, $v = v_0 + v_a e_a$, $r = r_4 + r_a e_a$ ($a = 1, 2, 3$) are the real quaternions.

We recall that quaternion’s algebra

$$e_a e_b = -\delta_{ab} + \epsilon_{abc} e_c, \quad e_0 e_a = e_a e_0 = e_a$$

has the involution — quaternionic conjugation — which is an antiautomorphism of the algebra: $(uv) = v^* u$. One can define the norm $|u| = \sqrt{uu}$, scalar $(u)_s = 1/2 (u + \bar{u}) = u_0$ and vector $(u)_v = 1/2 (u - \bar{u}) = u_a e_a$ = u parts.

The Hurwitz transformation possesses the property

$$R \equiv \sqrt{r_0^2 + |r|^2} = |u|^2 + |v|^2.$$  

(6)

To make the change of coordinates (4)–(5) complete, we represent $u = |u|g$ (and, therefore, $v = |v|\bar{g}/|r|$) where $g$ is unimodular quaternion. It is relevant to note that there is the isomorphism between the unimodular
quaternions and the group $SU(2)$. We can introduce parameters (following [11] we shall call them vector parameters)

$$g = \pm \frac{1 + z}{\sqrt{1 + z^2}}, \quad z = \frac{u}{u_0},$$

(7)

and choose $z_a = u_a / u_0$ as an additional coordinates.

2. The eigenvalues of one problem are expressed through the parameters of another one and vice versa:

$$E = 4a, \quad \omega^2 = -8E^l;$$

(8)

3. The equivariance condition

$$K^2 \psi^{(8)}(l + 1) = l(l + 1) \psi^{(8)}$$

(9)

is supposed to hold. It allows to establish the correspondence between the respective Hilbert spaces

$$\psi^{(8)}(u, v) = \text{trace}(\Psi^l(\vec{g}) \phi^l(\vec{r}_\mu)), \quad \Psi^l(\vec{g}) = \left[\Psi^l(g)\right]^\dagger.$$  

(10)

Here $\Psi^l(g)$ is the matrix of the $(2l + 1)$-dimensional representation of $SU(2)$ which components are the eigenfunctions of the mutually commuting operators $K^2, K_3, T_3$:

$$K^2 \psi^{l}_{mm'} = l(l + 1) \psi^{l}_{mm'}, \quad -K_3 \psi^{l}_{mm'} = m \psi^{l}_{mm'}$$

$$T_3 \psi^{l}_{mm'} = m' \psi^{l}_{mm'}, \quad -l \leq m, m' \leq l.$$  

(11)

When written in the vector parametrization, the operators $K_a$ and $T_a$ read [11]

$$K_a = -\frac{i}{2} \left( z_a z_b \frac{\partial}{\partial z_b} + \frac{\partial}{\partial z_a} + \varepsilon_{abc} z_b \frac{\partial}{\partial z_c} \right),$$  

$$T_a = \frac{i}{2} \left( z_a z_b \frac{\partial}{\partial z_b} - \frac{\partial}{\partial z_a} - \varepsilon_{abc} z_b \frac{\partial}{\partial z_c} \right).$$  

(12)

(13)

The well-known formula for the $SU(2)$ matrix elements [12]

$$\psi^{l}_{mm'}(g) = \sqrt{\frac{(l - m)!(l - m')!}{(l + m)!(l + m')!} \delta^{m+m'}}$$

$$\times \sum_{j=\max(m,m')}^l \frac{(l + j)!(\beta \gamma)^j}{(l - j)!(j - m)!(j - m')!},$$

(14)
where \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with \( \{\alpha, \beta, \gamma, \delta \in \mathbb{C}; \alpha \delta - \beta \gamma = 1\} \) can be expressed in terms of vector parameters if we choose
\[
g = \pm \frac{1}{\sqrt{1 + z^2}} \begin{pmatrix} 1 - i z_3 & -i (z_1 - i z_2) \\ -i (z_1 + i z_2) & 1 + i z_3 \end{pmatrix} = \pm \frac{1 - i \sigma_a z_a}{\sqrt{1 + z^2}} \tag{15}\]
(compare with (7). Note that there is the representation for quaternion’s basis \( e_a = -i \sigma_a \)).

In the spherical coordinates
\[
\begin{align*}
z_1 &= n_1 \tan \chi = \tan \chi \sin \theta \cos \varphi, \\
z_2 &= n_2 \tan \chi = \tan \chi \sin \theta \sin \varphi, \\
z_3 &= n_3 \tan \chi = \tan \chi \cos \theta, \\
0 &\leq \chi < \pi, \quad 0 \leq \theta < \pi, \quad 0 \leq \varphi < 2\pi,
\end{align*}
\tag{16}
\]
the group element \( g \) and its representation \( \Psi^l (g) \) are parametrized
\[
g = \exp (n \chi) = \cos \chi - i \sigma_a n_a \sin \chi
\]
\[
= \begin{pmatrix} \cos \chi - i \sin \chi \cos \theta & -i \sin \chi \sin \theta \exp (-i \varphi) \\ -i \sin \chi \sin \theta \exp (i \varphi) & \cos \chi + i \sin \chi \cos \theta \end{pmatrix}
\tag{17}
\]
and
\[
\psi_{m,m'}^l (g) = \sqrt{\frac{(l-m)!(l-m')!}{(l+m)!(l+m')!}} \left( \frac{\cos \chi + i \sin \chi \cos \theta}{-i \sin \chi \sin \theta} \right)^{m+m'} e^{i(m-m')\varphi}
\times \sum_{j=\max(m,m')}^{l} \frac{(l+j)!(\sin \chi \sin \theta)^{2j}}{(l-j)!(j-m)!(j-m')!},
\tag{18}
\]
respectively.

Representation \( \Psi^l (g) \) coincides with that used in [7] up to the complex conjugation.

3. Green’s Function

The equation defining the Green’s function of the 8-dimensional harmonic oscillator is
\[
(H - E) G(u, v, u', v'; E) = -i \delta^{(4)} (u - u') \delta^{(4)} (v - v').
\tag{19}
\]
Its solution is well-known [3]

\[
G = \int_0^\infty dt \exp\left(\frac{i4at}{2\pi \sin \omega t}\right)^4 \times \exp\left[\frac{i\omega}{2\sin \omega t} \left( (|u|^2 + |v|^2 + |u'|^2 + |v'|^2) \cos \omega t - 2(u\bar{u'} + v\bar{v'})_S \right) \right].
\]

Let us express it in \((r_\mu, z)\)-coordinates. In this section we now assume \(u = |u|h\) and \(u' = |u|h'\). The notation \(g\) we shall reserve for \(g = h\bar{h}'\).

First of all, note that

\[
2(u\bar{u'} + v\bar{v'})_S = 2 \left( |u| |u'| \bar{h}\bar{h}' + |v| |v'| \frac{r}{|r|} \bar{h}\bar{h}' \frac{r'}{|r'|} \right)_S = 2 \left( \left( |u| |u'| + |v| |v'| \frac{r'}{|r'|} \frac{r}{|r|} \right) \bar{h}\bar{h}' \right)_S = (\bar{F}g)_S
\]

where

\[
F = 2 \left( |u| |u'| + |v| |v'| \frac{r}{|r'|} \frac{r'}{|r|} \right) = 2|u| |u'| \left( 1 + \frac{r \bar{r}'}{4|u|^2 |u'|^2} \right) = \frac{RR' + Rr'_0 + r_0 R' + r_\mu r'_\mu + (r \bar{r}')_V}{\sqrt{(R + r_0)(R' + r'_0)}}.
\]

The norm of the quaternion \(F\) is

\[
|F| = \sqrt{2 \left( RR' + r_\mu r'_\mu \right)} = 2\sqrt{RR'} \cos \frac{\Theta}{2},
\]

and then we can introduce the unimodular quaternion \(f\) which is

\[
f \equiv \frac{F}{|F|} = \frac{RR' + Rr'_0 + r_0 R' + r_\mu r'_\mu + (r \bar{r}')_V}{\sqrt{2 \left( RR' + r_\mu r'_\mu \right)} (R + r_0)(R' + r'_0)}.
\]

Then

\[
G(r_\mu, r'_\mu; g) = \int_0^\infty dt \left( \frac{\omega}{2\pi \sin \omega t} \right)^4 \exp \left[ i4at + \frac{i\omega}{2} (R + R') \cot \omega t \right] \times \exp \left( -\frac{i\omega |F|}{2\sin \omega t} (\bar{F}g)_S \right).
\]
To obtain the expression for the 5-dimensional Green’s function we make the following simple manipulations on Eq. (19):

\[ 4R \Psi^l (\tilde{h}) \left( \mathcal{H}^l - \mathcal{E}^l \right) \Psi^l (h) G = -i \delta^{(4)} (u - u') \delta^{(4)} (v - v') , \]  

(26)

then

\[ \left( \mathcal{H}^l - \mathcal{E}^l \right) \Psi^l (h \tilde{h}') G = -\frac{1}{4R} i \delta^{(4)} (u - u') \delta^{(4)} (v - v') \Psi^l (h \tilde{h}') . \]  

(27)

On the analogy to the symbolic identity \( \delta(x)f(x) = \delta(x)f(0) \) we can write

\[ \delta^{(4)} (u - u') \Psi^l \left( \frac{uu'}{|u||u'|} \right) = \delta^{(4)} (u - u') \Psi^l (1) = \delta^{(4)} (u - u') . \]  

(28)

Integrating (27) over the group we obtain

\[ \left( \mathcal{H}^l - \mathcal{E}^l \right) \int d\tau (g) \Psi^l (g) G = -\frac{1}{4R} i \int d\tau (g) \delta^{(4)} (u - u') \delta^{(4)} (v - v') . \]  

(29)

Because the identity proven in [3]

\[ \int d\tau (g) \delta^{(4)} (u - u') \delta^{(4)} (v - v') = \frac{16R}{\pi^2} \delta^{(5)} (r_\mu - r'_\mu) \]  

(30)

we are led to the equation defining the Green’s function for the 5-dimensional problem

\[ \left( \mathcal{H}^l - \mathcal{E}^l \right) G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = -i \delta^{(5)} (r_\mu - r'_\mu) . \]  

(31)

It can be solved easily by evaluation of the integral

\[ G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = \frac{\pi^2}{4} \int d\tau (g) \Psi^l (g) G \left( r_\mu, r'_\mu, g; E \right) . \]  

(32)

Due to the properties of the invariant measure \( d\tau (g) \) the next expression is valid

\[ G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = \frac{\pi^2}{4} \Psi^l (f) \int d\tau (g) \Psi^l (g) G \left( r_\mu, r'_\mu, fg; E \right) . \]  

(33)

To achieve the final result we have to perform the integration over the group volume in the expression

\[ G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = \frac{\pi^2}{4} \Psi^l (f) \int_0^\infty dt \int d\tau (g) \Psi^l (g) \exp \left( i x (g) S \right) \]  

\[ \times \left( \frac{\omega}{2\pi \sin \omega t} \right)^4 \exp \left[ i4at + \frac{i\omega}{2} (R + R') \cot \omega t \right] . \]  

(34)
where it is introduced

\[ x = -\frac{\omega |F|}{2 \sin \omega t}. \]  

(35)

Due to the identity

\[ \int d\tau (g) \Psi^l (g) \exp (ix(x)_s) = i^{2l} \frac{2}{x} J_{2l+1} (x), \]

where \( J_{2l+1} (x) \) is the Bessel function, we obtain

\[ G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = \Psi^l (f) \left( \frac{(-i)^{2l} \omega^3}{16 \pi^2 |F|} \right) \int_0^\infty dt J_{2l+1} \left( \frac{\omega |F|}{2 \sin \omega t} \right) \]

\[ \times \exp \left[ i4at + \frac{i\omega}{2} (R + R') \cot \omega t \right] \frac{\sin^3 \omega t}{\sin \omega t}. \]  

(36)

To bring our result to the notations of [1] we introduce \( q = -i\omega t, \omega = 2ik, p' = -ia/k \) and finally have

\[ G^l \left( r_\mu, r'_\mu; \mathcal{E}^l \right) = \Psi^l (f) \left( \frac{(-i)^{2l} k^2}{8 \pi^2 \sqrt{RR'} \cos \frac{\Theta}{2}} \right) \int_0^\infty dq J_{2l+1} \left( \frac{2k \sqrt{RR'} \cos \frac{\Theta}{2}}{\sinh q} \right) \]

\[ \times \exp \left[ -2p'q + ik (R + R') \coth q \right] \frac{1}{\sinh^3 q}. \]  

(37)

For the case of the trivial constraints \( l = 0 \) the expression

\[ G^0 \left( r_\mu, r'_\mu; \mathcal{E}^0 \right) = \frac{k^2}{8 \pi^2 \sqrt{RR'} \cos \frac{\Theta}{2}} \int_0^\infty dq J_1 \left( \frac{2k \sqrt{RR'} \cos \frac{\Theta}{2}}{\sinh q} \right) \]

\[ \times \exp \left[ -2p'q + ik (R + R') \coth q \right] \frac{1}{\sinh^3 q} \]  

(38)

appears to be the same as the respective result in [1] for \( n = 5 \).

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References


