GENERALIZED ACTIONS

MOHAMMAD R. MOLAEI

Department of Mathematics, Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
P.O. Box 76135-133, Kerman, Iran

Abstract. In this paper a generalization of the concept of action is considered. This notion is based on a new algebraic structure called generalized groups. An action is deduced by imposing an Abelian condition on a generalized group. Generalized actions on normal generalized groups are also considered.

1. Basic Notions

The theory of generalized groups was first introduced in [1]. A generalized group means a non-empty set \( G \) admitting an operation

\[
G \times G \rightarrow G \\
(a, b) \mapsto ab
\]

called multiplication which satisfies the following conditions:

i) \((ab)c = a(bc)\) for all \(a, b, c\) in \(G\);

ii) For each \(a \in G\) there exists a unique \(e(a) \in G\) such that \(ae(a) = e(a)a = a\);

iii) For each \(a \in G\) there exists \(a^{-1} \in G\) such that \(aa^{-1} = a^{-1}a = e(a)\).

Theorem 1.1. [1] For each \(a \in G\) there exists a unique \(a^{-1} \in G\).

Theorem 1.2. [2] Let \(G\) be a generalized group and \(ab = ba\) for all \(a, b\) in \(G\). Then \(G\) is a group.

Example 1.1. Let \(G = \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}\), where \(\mathbb{R}\) is the set of real numbers. Then \(G\) with the multiplication \((a_1, b_1, c_1)(a_2, b_2, c_2) = (b_1a_1, b_1b_2, b_1c_2)\) is a generalized group.
In this paper we consider a generalized action of a generalized group on a set.

Definition 1.1. We say that a generalized group $G$ acts on a set $S$ if there exists a function

$$G \times S \rightarrow S$$

$$(g, x) \mapsto gx$$

which is called a generalized action such that:

- $(g_1 g_2) x = g_1 (g_2 x)$ for all $g_1, g_2 \in G$, and $x \in S$;
- For all $x \in S$ there exists $e(g) \in G$ such that $e(g)x = x$.

Example 1.2. Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \ a, b, c \text{ and } d \text{ are real numbers} \right\}.$$  

Then $G$ with the product

$$\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \mapsto \begin{bmatrix} a & f \\ g & d \end{bmatrix}$$

is a generalized group, and the function

$$G \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

$$\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} , (e, f, g, h) \right) \mapsto (a, f, g, d)$$

is a generalized action of $G$ on $\mathbb{R}^4$.

Theorem 1.3. Let $\tau : G \times S \rightarrow S$ be a genaralzed action, and $G$ be an Abelian generalized group. Then $G$ is a group and $\tau$ is an action.

Proof: By theorem 1.2, $G$ is a group. So $\tau$ is an action.

2. Elementary Results on Generalized Actions

If $G$ acts on a set $S$, then the relation $\sim$ defined by:

$$x_1 \sim x_2 \Leftrightarrow (g_1 x_1 = x_2 \text{ and } g_2 x_2 = x_1 \text{ for some } g_1, g_2 \in G)$$

is an equivalence relation.

Definition 2.1. If $x \in S$, then $O(x) = \{y \in S; \ x \sim y \}$ is called the generalized orbit of $x$.

Now we deduce a generalized subgroup by a generalized action.

Theorem 2.1. Let a generalized group $G$ act on a set $S$. Then for every $x \in S$, the set $I_x = \{g \in G; \ gx = x \}$ is a generalized subgroup of $G$. 
\textbf{Proof:} For \( g \in I_x \) we have:
\[
\begin{align*}
gx &= x \Rightarrow (e(g)g)x = x \Rightarrow e(g)(gx) = x \\
&\Rightarrow e(g)x = x \Rightarrow e(g) \in I_x,
\end{align*}
\]
and
\[
\begin{align*}
g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = e(g)x = x.
\end{align*}
\]
So
\[
g^{-1} \in I_x.
\]
If
\[
g_1, g_2 \in I_x,
\]
then
\[
(g_1g_2)x = g_1(g_2x) = g_1x = x.
\]
Hence
\[
g_1g_2 \in I_x.
\]
Thus \( I_x \) is a generalized subgroup of \( G \). \( \square \)

\textbf{Theorem 2.2.} Let \( f : G \to E \) be a generalized group homomorphism. Then
\[
\begin{align*}
\tau : G \times E &\to E \\
(g, h) &\mapsto f(g)h
\end{align*}
\]
is a generalized action.

\textbf{Proof:} Let \( g, g' \in G \) and \( h \in E \). Then:
\[
\tau(g, \tau(g', h)) = f(g)\tau(g', h) = f(g)(f(g')h) = f(gg')h = \tau(gg', h).
\]
Moreover if \( g \in f^{-1}(\{e(h)\}) \), then:
\[
\tau(e(g), h) = f(e(g))h = e(f(g))h = e(e(h))h = e(h)h = h.
\]
Thus \( \tau \) is a generalized action. \( \square \)

\textbf{Example 2.1.} Let \( G = \mathbb{R} \times \mathbb{R}\setminus\{0\} \) with multiplication \((a, b)(c, d) = (bc, bd)\). Since
\[
\begin{align*}
f : G &\to \mathbb{R} \\
(a, b) &\mapsto \frac{a}{b}
\end{align*}
\]
is a homomorphism, when the multiplication of \( \mathbb{R} \) is \( ab = b \), the function
\[
\begin{align*}
G \times \mathbb{R} &\to \mathbb{R} \\
((a, b), c) &\mapsto \frac{ac}{b}
\end{align*}
\]
is a generalized action.
3. Generalized Action of Normal Generalized Groups on a Set

A generalized group $G$ is called a normal generalized group if $c(ab) = c(a)c(b)$ for all $a, b \in G$. In this section we assume that $G$ is a normal generalized group.

**Definition 3.1.** [3] A generalized subgroup $N$ of a generalized group $G$ is called a generalized normal subgroup if there exist generalized group $E$ and a homomorphism $f : G \to E$ such that for all $a \in G$,

$$N_a = \phi \quad \text{or} \quad N_a = \text{kernel } f_a,$$

where $N_a = N \cap G_a, G_a = \{g \in G; e(g) = e(a)\}$, and $f_a = f|_{G_a}$.

**Example 3.1.** Let $G$ be the generalized group of Example 2.1. Then $N = \{(a, b); a = b \quad \text{or} \quad a = 3b\}$ is a generalized normal subgroup of $G$.

**Theorem 3.1.** Let $G$ be a normal generalized group, and $f : G \to G$ be a generalized groups homomorphism. Moreover let $N = \ker f$. Then

$$\tau : \frac{G}{N} \times f(G) \to f(G)$$

$$(gN_g, x) \mapsto f(g)x$$

is a generalized action.

**Proof:**

i) If $(g_1N_{g_1}, x_1) = (g_2N_{g_2}, x_2)$, then $g_1N_{g_1} = g_2N_{g_2}$ and $x_1 = x_2$.

So $N_{g_1} = N_{g_2}, x_1 = x_2$ and $g_1 = ng_2$ for some $n \in N_{g_1}$. Hence

$$f(g_1) = f(ng_2) = f(n)f(g_2) = f(e(g_2))f(g_2) = f(g_2).$$

Therefore $f(g_1)x_1 = f(g_2)x_2$. Thus $\tau$ is well defined;

ii) Let $g_1N_{g_1}, g_2N_{g_2} \in \frac{G}{N}$ and $x \in S$ are given. Then

$$\tau(g_1N_{g_1}, \tau(g_2N_{g_2}, x)) = f(g_1)\tau(g_2N_{g_2}, x) = f(g_1)(f(g_2)x)$$

$$= f(g_1g_2)x = \tau((g_1N_{g_1})(g_2N_{g_2}), x);$$

iii) If $x \in f(G)$, then $x = f(g)$ for some $g \in G$, and we have

$$\tau(e(g)N_g, x) = f(e(g))x = f(e(g))f(g) = f(g) = x.$$

$\square$

**Notation.** We denote the set

$$\left\{ \varphi_g : S \to S , \ g \in G \right\}$$

$$x \mapsto gx$$
by $H(S)$.  

The following example shows that $H(S)$ with multiplication $\varphi_{g_1} \varphi_{g_2} := \varphi_{g_1} \circ \varphi_{g_2}$ is not a generalized group.

**Example 3.2.** Let $S = \mathbb{R} \times \mathbb{R} \setminus \{0\}$, and $G = \mathbb{R} \times \{1\}$ with multiplication: $(a, 1)(b, 1) = (b, 1)$. Then the function

$$G \times S \rightarrow S$$

$$((a, 1), (c, d)) \mapsto (c, d)$$

is a generalized action, but $H(S)$ is not a generalized group. Because the inverse of an element is not unique.

**Theorem 3.2.** Let a generalized group $G$ act on a set $S$, and the function

$$G \rightarrow H(S)$$

$$g \mapsto \varphi_g$$

be a one-to-one mapping. Then $H(S)$ with the multiplication $\varphi_{g_1} \varphi_{g_2} := \varphi_{g_1} \circ \varphi_{g_2}$ is a generalized group. Moreover, if $G$ be normal, then $H(S)$ is a normal generalized group.

**Proof:** Suppose that $\varphi_g \in H(S)$ is given. If $\varphi_g \varphi_h = \varphi_h \varphi_g = \varphi_g$, then $\varphi_{gh} = \varphi_{hg} = \varphi_g$. So $gh = hg = g$. Hence the identity of $\varphi_g$ is $\varphi_{e(g)}$. Thus $h = e(g)$.

One can easily deduce other properties of generalized group. Now let $G$ be a normal generalized group, and $\varphi_{g_1}, \varphi_{g_2} \in H(S)$. Then

$$e(\varphi_{g_1} \varphi_{g_2}) = e(\varphi_{g_1} \varphi_{g_2}) = \varphi_{e(g_1) e(g_2)} = e(\varphi(g_1)) e(\varphi(g_2)).$$

□

**References**

