TWISTOR INTEGRAL REPRESENTATIONS OF SOLUTIONS OF THE SUB-LAPLACIAN

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Abstract. The twistor integral representations of solutions of the Laplacian on the complex space are well-known. The purpose of this article is to generalize the results above to that of the sub-Laplacian on the odd-dimensional complex space with the standard contact structure.

Introduction

The twistor integral representations of solutions of the complex Laplacian on the complex space $\mathbb{C}^{2n}$ of even dimension $2n$ are well-known. We also showed them on $\mathbb{C}^{2n-1}$ of odd dimension $2n - 1$ before. The purpose of this article is to generalize the results above to that of the complex sub-Laplacian on $\mathbb{C}^{2n-1}$ with the standard contact structure. The details and further discussion will appear elsewhere.

Let $(x_i, y_i, z)$ $i = 1, \ldots, n-1$ be the standard coordinate system of $\mathbb{M} = \mathbb{C}^{2n-1}$. We give $\mathbb{M}$ a contact structure defined by

$$\theta = dz - \sum_{i=1}^{n-1} (y_i \, dx_i - x_i \, dy_i)$$

called a contact form. The contact distribution $D$ on $\mathbb{M}$ is defined by $\theta = 0$. The vector fields

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z}, \quad i = 1, \ldots, n - 1$$

furnish a basis of $D$. Let us join $Z = \frac{\partial}{\partial z}$ to them. By $[Y_i, X_i] = 2Z$; $i = 1, \ldots, n - 1$ they form a basis of the Heisenberg algebra.
Let $g$ be a complex sub-Riemannian metric on $D$ such that

$$g(X_i, Y_j) = \delta_{ij},$$
$$g(X_i, X_j) = 0, \quad g(Y_i, Y_j) = 0.$$  

Let $\mathbb{P}$ be the set of all totally null affine $(n - 1)$-planes in $\mathbb{M}$ in the sense of the Heisenberg group. The space $\mathbb{P}$ is called the twistor space of $\mathbb{M}$. Either of the following equations represents a generic element belonging to $\mathbb{P}$:

$$\mathbb{P}_1 : \left\{ \begin{array}{l}
  y_i = \sum_{j=1}^{n-1} a_{ij} x_j + b_i, \quad a_{ij} = -a_{ji} \quad i = 1, \ldots, n - 1 \\
  z = \sum_{j=1}^{n-1} b_j x_j + c \\
  = \sum_{j=1}^{n-1} x_j y_j + c
\end{array} \right. \quad$$

$$\mathbb{P}_2 : \left\{ \begin{array}{l}
  y_i = \sum_{j=1}^{n-1} a_{ij} x_j + b_i, \quad a_{ij} = -a_{ji} \quad i = 1, \ldots, n - 1 \\
  z = -\sum_{j=1}^{n-1} b_j x_j + c \\
  = -\sum_{j=1}^{n-1} x_j y_j + c
\end{array} \right. \quad$$

Remark that each totally null affine $(n - 1)$-plane is not tangent to $D$, but the projection to the $(x_i, y_i)$-space is totally null affine $(n - 1)$-plane in the usual sense. We can take $(a_{ij}, b_i, c)$ as generic parameters of $\mathbb{P}$. Therefore the dimension of $\mathbb{P}$ is $\frac{n^2 - n + 2}{2}$. By the natural projection $(a_{ij}, b_i, c) \mapsto (a_{ij})$, the $(a_{ij})$-space is of dimension $\frac{(n - 1)(n - 2)}{2}$.

Let $\Box_R$, $\Box_L$ and $\Box$ be complex sub-Laplacians associated with $g$ as follows:

$$\Box_R \phi = \left( \sum_{i=1}^{n-1} Y_i X_i \right) \phi$$

$$\Box_L \phi = \left( \sum_{i=1}^{n-1} X_i Y_i \right) \phi$$

$$\Box \phi = (\Box_L + \Box_R) \phi = \sum_{i=1}^{n-1} (X_i Y_i + Y_i X_i) \phi$$
Let \( f = f(a_{ij}, b_i, c) \) be a suitable analytic function on \( \mathbb{P} \). Then we can define a function

\[
\phi(x, y, z) = \int_{\Delta} f(a_{ij}, y_i - \sum_{j=1}^{n-1} a_{ij} x_j, z + \sum_{j=1}^{n-1} x_j y_j) \wedge da_{ij}
\]

where \( b_i = y_i - \sum_{j=1}^{n-1} a_{ij} x_j, c = z + \sum_{j=1}^{n-1} x_j y_j \), and \( \wedge da_{ij} \) is an exterior \( k \)-form by any of \( da_{ij} \) while \( \Delta \) is a \( k \)-chain. The function \( \phi \) on \( \mathbb{M} \) is not necessarily a solution of \( \Box_R, \Box_L, \Box \) for any \( f \).

First, we have the following.

**Proposition 1.** Take a form \( f = f(a_{ij}, b_i) = f(a_{ij}, b_i, \gamma) \), where \( \gamma \) is a constant. We have \( \phi(x, y, z) = \varphi(x, y) \). Then we have

\[
\Box_R \phi = 0, \quad \Box_L \phi = 0.
\]

These are nothing but the twistor integral representations of solutions of the complex Laplacian on \( \mathbb{C}^{2n-2} \). We call them type 1 and write them as \( f_1 \) and \( \phi_1 \).

Next, we have the following.

**Proposition 2.** Take a form \( f = f(c) = f(\alpha_{ij}, \beta_i, c) \), where \( \alpha_{ij} \) and \( \beta_i \) are constants. We have \( \phi(x, y, z) = \varphi(z + \sum_{j=1}^{n-1} x_j y_j) \). Then we have

i) for \( \phi = \varphi \left( z - \sum_{j=1}^{n-1} x_j y_j \right) \)

\[
X_i \phi = 0 \ (i = 1, \ldots, n-1), \quad i.e. \quad \Box_R \phi = 0,
\]

ii) for \( \phi = \varphi \left( z + \sum_{j=1}^{n-1} x_j y_j \right) \)

\[
Y_i \phi = 0 \ (i = 1, \ldots, n-1), \quad i.e. \quad \Box_L \phi = 0.
\]

We call them type 2 and write them as \( f_2 \) and \( \phi_2 \).

Combining the above two propositions, we have the following.

**Theorem 1.** Take a form

\[
f = f(a_{ij}, b_i, c) = f_1(a_{ij}, b_i) + f_2(c) = f_1 + f_2
\]
on \( \mathbb{P}_1 \). We have
\[
\phi(x_i, y_i, z) = \phi_1(x_i, y_i) + \phi_2 \left( z - \sum_{j=1}^{n-1} x_j y_j \right) = \phi_1 + \phi_2
\]

on \( \mathbb{M} \). Then we have
\[
\Box_R \phi = 0.
\]
Conversely, a solution \( \phi \) of \( \Box_R \phi = 0 \) is represented by \( \phi = \phi_1 + \phi_2 \) by some \( f = f_1 + f_2 \). Similarly, from \( f = f_1 + f_2 \) on \( \mathbb{P}_2 \), \( \phi = \phi_1 + \phi_2 \) satisfies \( \Box_L \phi = 0 \).

We embed \((a_{ij}, b_i, c, c')\) into \( \mathbb{P}_0 \times \mathbb{P}_2 \) as \((a_{ij}, b_i, c) \times (a_{ij}, b_i, c')\). Taking a function
\[
F = F(a_{ij}, b_i, c, c') = F(c, c') = (cc')^{-\frac{n-1}{2}}
\]
on \( \mathbb{P}_1 \times \mathbb{P}_2 \), we have
\[
\Phi(x_i, y_i, z) = \text{const} \left( \sum_{i=1}^{n-1} x_i y_i \right)^2 - z^2 \right)^{-\frac{n-1}{2}}.
\]
This is the (complex) fundamental solution of \( \Box \).

References