TRANSFINITE SEQUENCES OF CONTINUOUS AND
BAIRE 1 FUNCTIONS ON SEPARABLE METRIC SPACES

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ABSTRACT. We investigate the existence of well-ordered sequences of
Baire 1 functions on separable metric spaces.

Any set $F$ of real valued functions defined on an arbitrary set $X$ is par-
tially ordered by the pointwise order, that is $f \leq g$ iff $f(x) \leq g(x)$ for
all $x \in X$. In other words put $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and
$f(x) \neq g(x)$ for at least one $x \in X$. Our aim will be to investigate the possi-
bile length of the increasing or decreasing well-ordered sequences of functions
in $F$ with respect to this order.

A classical theorem of Kuratowski asserts, that if $F$ is the set of contin-
uous or Baire 1 functions defined on a Polish space $X$, then there exists a
monotone sequence of length $\xi$ in $F$ iff $\xi < \omega_1$ (see [2, §24. III.2]). More-
over, P. Komjáth proved in [1] that the corresponding question concerning
Baire $\alpha$ functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

In the present paper we investigate what happens if we drop the condition
of completeness and replace the Polish space $X$ by a separable metric space.

Our main results are the following. Let $d(X)$ denote the density of a
space $X$.

**Theorem.** Let $(X, \varrho)$ be a metric space. Then there exists a well-ordered
sequence of length $\xi$ of continuous real-valued functions defined on $X$ iff
$\xi < d(X)^+$. 

**Corollary.** A metric space is separable iff every well-ordered sequence of
continuous functions defined on it is countable.

**Theorem.** There exists a separable metric space on which there exists a
well-ordered sequence of length $\omega_1$ of Baire 1 functions.

**Theorem.** The following statement: ‘There exists a separable metric space
on which there exists a well-ordered sequence of length $\omega_2$ of Baire 1 func-
tions’ is independent of ZFC + $-CH$.

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Remark. During and after the conference Kenneth Kunen answered one of my questions, and also improved some of the results and proofs. These results will appear in a forthcoming joint paper.

REFERENCES


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