CHAINABLE SUBCONTINUA

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By R.H. Bing’s theorem eleven [2] if a metric continuum $X$ contains a non-degenerate subcontinuum $H$ which is hereditarily decomposable, hereditarily unicoherent, and atriodic, then $H$ is chainable.

The following papers give examples of continua with the property that each non-degenerate subcontinuum is not chainable. G.T. Whyburn [16], R.D. Anderson and G. Choquet [1], A. Lelek [6] gives an example of a planar weakly chainable continuum each non-degenerate subcontinuum of which separates the plane and thus contains no non-degenerate chainable subcontinuum. W.T. Ingram [5] gives an example of an hereditarily indecomposable tree-like continuum such that each non-degenerate subcontinuum has positive span and hence is not chainable.

C.E. Burgess in [3] shows if a continuum $M$ is almost chainable and $K$ is a proper subcontinuum of $M$ which contains an endpoint $p$ of $M$, then $K$ is linearly chainable with $p$ as an end point. A continuum $M$ is almost chainable if, for every positive number $\varepsilon$, there exists an $\varepsilon$-covering $G$ of $M$ and a linear chain $C(L_1, L_2, \ldots, L_n)$ of elements of $G$ such that no $L_i$ $(1 \leq i < n)$ intersects an element of $G - C$ and every point of $M$ is within a distance $\varepsilon$ of some element of $C$. He also shows if $M$ is almost chainable, then $M$ is not a triod and $M$ is unicoherent and irreducible between some two points. Examples show $M$ can contain a triod or a non-unicoherent sub continuum.

If $X$ and $Y$ are metric continua and if $X$ can be $\varepsilon$-mapped onto $Y$ for all positive $\varepsilon$ and $Y$ has a non-degenerate chainable continuum then so does $X$. This result suggests considering inverse limit spaces. At this stage we refer to a result from the paper of S. Mardešić and J. Segal [9] Theorem

2000 Mathematics Subject Classification. 54F20.
Key words and phrases. chainable continuum.
Every \( \pi \)-like continuum \( X \) is the inverse limit of an inverse sequence \( \{P_i; \pi_{ij}\} \) with bonding maps \( \pi_{ij} \) onto and with polyhedra \( P_i \in \pi \). A continuum is \( \pi \)-like if it can be \( \varepsilon \)-mapped onto some polyhedron in \( \pi \) for each positive \( \varepsilon \). E. Duda and P. Krupski \[4\] showed that a \( k \)-junctioned metric continuum, \( k \) a non-negative integer, has at most \( k \) points such that any continuum which contains none of the \( k \) points is chainable. A metric continuum is said to be \( k \)-junctioned if it is the inverse limit of graphs each of which has at most \( k \) branch points, with surjective bonding maps. A continuum is called finitely junctioned if it is \( k \)-junctioned for some non-negative integer \( k \).

Suppose now \( X \) is a tree-like continuum. Then for each \( \varepsilon > 0 \) \( X \) can be mapped onto a tree. By a result quoted above \( X \) is the inverse limit of a sequence of trees with surjective bonding maps. \( X = \lim\leftarrow \{T_n, f_{nm}\} \). Let \( f_n : X \to T_n \) be the standard projection map and let

\[ P_n = U \{f_n^{-1}(q) | q \text{ is a branch point}\}. \]

Since \( T_n \) has at most a finite number of branch points (points of order \( \geq 2 \)) \( P_n \) is closed in \( X \). If the union of the \( P_n \) is not dense in \( X \) then \( X \) contains a non-degenerate chainable continuum. Actually it is sufficient that \( \{P_n\} \) have a subsequence whose union is not dense in \( X \).

Let now consider a non-degenerate metric continuum in \( X \) with span equal to zero. The notion of span was defined by A. Lelek \[7\]. In the paper \[8\] he showed continua with span zero are atriodic and tree-like.

There is a series of papers by L.G. Oversteegen and E.D. Tymchatyn which develop properties of spaces with spans equal to zero or sufficient conditions that a space have a span equal to zero \[12, 15, 13, 14\]. Also by L.G. Oversteegen \[11\].

It is interesting to note that a chainable continuum \( X \) can be \( \varepsilon \)-mapped onto any fixed dendrite. Thus for any tree \( T \), by the result of Mardešić and Segal quoted above, \( X \) is the inverse of a sequence of \( T \)'s.

In the paper \[10\] P. Minc shows an inverse limit of trees with simplicial bonding maps having surjective span zero is chainable.

\textbf{References}


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