Special Geometric Structures in String Theory
Bonn, 8th–11th September, 2001

PARTIALLY-FLAT GAUGE FIELDS ON MANIFOLDS OF DIMENSION GREATER THAN FOUR

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Abstract. We describe two extensions of the notion of a self-dual connection in a vector bundle over a manifold M from $\dim M{=}4$ to higher dimensions. The first extension, Ω-self-duality, is based on the existence of an appropriate 4-form Ω on the Riemannian manifold M and yields solutions of the Yang-Mills equations. The second is the notion of half-flatness, which is defined for manifolds with certain Grassmann structure $T^{\mathbb{C}}M{\cong}E\otimes H$. In some cases, for example for hyper-Kähler manifolds M, half-flatness implies Ω-self-duality. A construction of half-flat connections inspired by the harmonic space approach is described. Locally, any such connection can be obtained from a free prepotential by solving a system of linear first order ODEs.

1. Self-duality and half-flatness

The Yang-Mills self-duality equations have played a very important role in field theory and in differential geometry. This talk is about ongoing work on two generalisations of the notion of *self-duality* from four to higher dimensional manifolds M. The first generalisation is the notion of

Ω-self-duality.

This is based on the existence of an appropriate 4-form Ω on a pseudo-Riemannian manifold (M,g). For $\Omega\in\Omega^4(M)$ we define a traceless symmetric endomorphism field $B_\Omega:\wedge^2T^*M\to\wedge^2T^*M$ by

$$B_\Omega\omega := *(*\Omega\wedge\omega)\ , \tag{1}$$

where $\omega\in\wedge^2T^*M$.

Talk given by C.D. on 11th September, 2001 at the Workshop on Special Geometric Structures in String Theory, Bonn. This work was supported by the 'Schwerpunktprogramm Stringtheorie' of the Deutsche Forschungsgemeinschaft.

Definition 1. *A 4-form* $\Omega \in \Omega^4(M)$ *on a pseudo-Riemannian manifold* M *is called* **appropriate** *if there exists a non-zero real constant eigenvalue* λ *of the endomorphism field* B_Ω.

We note that on a Riemannian manifold the eigenvalues of B_Ω are real for any 4-form Ω. We can now define a generalisation of the four dimensional notion of self-duality thus:

Definition 2. *Let* Ω *be an appropriate 4-form on a pseudo-Riemannian manifold* (M, g) *and* $\lambda \neq 0 \in \mathbb{R}$. *A connection* ∇ *in a vector bundle* $\nu : W \to M$ *is* **Ω-self-dual** *if its curvature* F^∇ *satisfies the linear algebraic system*

$$B_\Omega F^\nabla = \lambda F^\nabla \quad (2)$$

$$(d * \Omega) \wedge F^\nabla = 0 \, . \quad (3)$$

It is easy to see that an Ω-self-dual connection ∇ is a Yang-Mills connection, i.e. it satisfies the Yang-Mills equation:

$$d^\nabla * F^\nabla = \pm\tfrac{1}{\lambda} d^\nabla \left(*\Omega \wedge F^\nabla \right) = \pm\tfrac{1}{\lambda} \left((d * \Omega) \wedge F^\nabla + *\Omega \wedge d^\nabla F^\nabla \right) = 0$$

in virtue of eq. (3) and the Bianchi identity $d^\nabla F^\nabla = 0$. This extension of self-duality is based on early work of [CDFN] on gauge fields in flat spaces. Interesting examples of manifolds with appropriate (and parallel) 4-forms are provided, for instance by manifolds with special holonomy groups. So for gauge fields on these manifolds, Ω-self-duality may be defined. Various examples have been discussed in the literature, e.g. in [CS, BKS, DT, T].

The second generalisation of self-duality is the notion of

Half-flatness.

This is defined for manifolds with certain Grassmann structure. In what follows M denotes a complex manifold. A Grassmann structure on M is an isomorphism of the (holomorphic) tangent bundle, $TM \cong E \otimes H$, where E and H are two holomorphic vector bundles over M. This is just a generalised spinor decomposition, allowing representation of vector fields using two types of spinor indices, viz. $X_{a\alpha} := e_a \otimes h_\alpha$, where (e_a) and (h_α) are frames of E and H respectively. If M is four dimensional, both E and H have rank 2 and a, α are the familiar 2-spinor indices. Manifolds with Grassmann structure provide interesting generalisations of four dimensional manifolds. A Grassmann connection ∇ is a linear connection which preserves the

Grassmann structure:

$$\nabla(e\otimes h) = \nabla^E e\otimes h + e\otimes \nabla^H h\,,$$

where ∇^E, ∇^H are connections in the bundles E, H and e, h are local sections of E, H respectively. A Grassmann structure with Grassmann connection ∇ is called half-flat if the connection ∇^H in H is flat. A manifold with such a half-flat Grassmann structure is called a half-flat Grassmann manifold. Here we assume that rank $H = 2$ and that a ∇^H-parallel non-degenerate fibre-wise 2-form $\omega_H \in \Gamma(\wedge^2 H^*)$ in the bundle H is fixed, so these manifolds are generalisations of hyper-Kähler manifolds. The torsion of the Grassmann connection lives in the space

$$\begin{aligned} TM\otimes\wedge^2T^*M &= TM\otimes\big(S^2H^*\otimes\wedge^2E^* \ \oplus\ \wedge^2H^*\otimes S^2E^*\big)\\ &= EH\left(S^2H^*\wedge^2 E^* \ \oplus\ \omega_H S^2E^*\right)\\ &\cong \big(S^3H \ \oplus\ \omega_H H\big)\, E\wedge^2 E^* \ \oplus\ \omega_H HES^2E^*\,, \end{aligned}$$

where we omit the $\otimes$'s and we identify H^* with H using ω_H. Notice that since the bundle H has rank 2, the line bundle $\wedge^2 H$ is generated by the 2-form ω_H. (Choosing a frame (h_1, h_2) for H, we may write $\omega_H(h_\alpha, h_\beta) := \epsilon_{\alpha\beta}$, a skew matrix with $\epsilon_{12} = 1$.)

We now consider gauge fields on these manifolds. Let ∇ be a connection in a holomorphic vector bundle $\nu : W \to M$. (If ν has structure group G, we may choose a frame for it, in which the vector potential takes values in the gauge Lie algebra $\mathfrak{g}$). The curvature of the connection ∇ is a 2-form with values in the gauge algebra, $F^\nabla \in \wedge^2T^*M\otimes \mathrm{End}\, W$. Now, as above, in virtue of the Grassmann structure the space of bivectors decomposes as

$$\wedge^2TM = S^2E\otimes\wedge^2H\oplus\wedge^2E\otimes S^2H \ = S^2E\otimes\omega_H\oplus\wedge^2E\otimes S^2H\,.$$

If $e, e' \in \Gamma(E)$, we have the decomposition

$$F(e\otimes h_\alpha\,,\, e'\otimes h_\beta) = \omega_H(h_\alpha, h_\beta)F^{(e,e')} + F^{[e,e']}_{(\alpha\beta)}\,,$$

where $F^{(e,e')}$ is symmetric in e, e' and $F^{[e,e']}_{(\alpha\beta)}$ is skew in e, e' and symmetric in $\alpha\beta$. In 4 dimensions both E and H have rank 2, so this corresponds to the familiar decomposition into self-dual and anti-self-dual parts of the curvature.

Definition 3. *Let M be a manifold with Grassmann structure, not necessarily half-flat. A connection ∇ in a vector bundle $W\to M$ is called* **half-flat** *if its curvature*

$$F^\nabla \in \wedge^2H\otimes S^2E\otimes\mathrm{End}\, W = \omega_H\otimes S^2E\otimes \mathrm{End}\, W\ .$$

Quaternionic Kähler and hyper-Kähler manifolds are special cases of manifolds with (a locally defined) Grassmann structure, the Levi-Civita connection being the canonical Grassmann connection. In the hyper-Kähler case the Grassmann structure is half-flat. For these examples, half-flatness of a gauge connection implies Ω-self-duality with respect to the canonical parallel 4-form Ω. In other words, a half-flat curvature is one of the eigenstates of the endomorphism field B_Ω, as in 4 dimensions. Therefore, in these cases, the two generalisations of the idea of self-duality mentioned above coincide.

There is an interesting special case of Grassmann structure, considered by [AG]:

Definition 4. *A* **spin $\frac{m}{2}$ Grassmann structure** *on a (complex) manifold M is a holomorphic Grassmann structure of the form $TM \cong E \otimes F = E \otimes S^m H$, with a holomorphic Grassmann connection $\nabla^{TM} = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F$, where H is a rank 2 holomorphic vector bundle over M with symplectic form ω_H and holomorphic symplectic connection ∇^H, and ∇^F is the connection in $F = S^m H$ induced by ∇^H. M is called* **half-flat spin $\frac{m}{2}$ Grassmann manifold** *if the connection ∇^F is flat.*

The bundle $S^m H$ is associated with the spin $\frac{m}{2}$ representation of the group $\mathrm{Sp}(1, \mathbb{C})$. Any frame (h_1, h_2) for H defines a frame for $S^m H$, $(h_A := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_m})$, where the multi-index $A := \alpha_1 \alpha_2 \ldots \alpha_m$, $\alpha_i = 1, 2$. The symplectic form ω_H induces a bilinear form ω_H^m on $F = S^m H$ given by

$$\omega_H^m(h_A, h_B) := \mathfrak{S}_A \mathfrak{S}_B \omega_H(h_{\alpha_1}, h_{\beta_1}) \omega_H(h_{\alpha_2}, h_{\beta_2}) \cdots \omega_H(h_{\alpha_m}, h_{\beta_m}) ,$$

where $\mathfrak{S}_A$ denotes the sum over all permutations of the α's. This form is skew-symmetric if m is odd and symmetric if m is even. To any section $e \in \Gamma(E)$ and multi-index A we associate the vector field $X_A^e := e \otimes h_A$ on M and we put $X_{aA} := X_A^{e_a}$.

Let $(M, \nabla^{\mathrm{TM}})$ be a half-flat spin $\frac{m}{2}$ Grassmann manifold and ∇ a connection in a holomorphic vector bundle $W \to M$. The space of bivectors $\wedge^2 TM$ has the following decomposition into $GL(E) \otimes \mathrm{Sp}(1, \mathbb{C})$-submodules:

$$\wedge^2 TM = \wedge^2 \Big(E \otimes S^m H \Big) = \wedge^2 E \otimes S^2 S^m H \ \oplus \ S^2 E \otimes \wedge^2 S^m H \, ,$$

where

$$\begin{aligned} S^2 S^m H \ &= \ S^{2m} H \ \oplus \ \omega_H^2 S^{2m-4} H \ \oplus \ \cdots \oplus \omega_H^{2[\frac{m}{2}]} S^{2m-4[\frac{m}{2}]} H \\ \wedge^2 S^m H \ &= \ \omega_H S^{2m-2} H \ \oplus \ \omega_H^3 S^{2m-6} H \oplus \cdots \oplus \omega_H^{2[\frac{m}{2}]+1} S^{2m-4[\frac{m}{2}]-2} H \, . \end{aligned}$$

Here we use the convention that $S^l H = 0$ if $l < 0$. The corresponding decomposition of the curvature of ∇ is:

$$\begin{aligned} &F\big(X^e_A, X^{e'}_B\big) \\ &= \mathfrak{S}_A \mathfrak{S}_B \sum_{k=0}^{[m/2]} \big(\omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k}}, h_{\beta_{2k}}) \overset{(2k)}{F}{}^{[ee']}_{\alpha_{2k+1}\cdots\alpha_m \beta_{2k+1}\cdots\beta_m} \\ &\qquad + \omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k+1}}, h_{\beta_{2k+1}}) \overset{(2k+1)}{F}{}^{(ee')}_{\alpha_{2k+2}\cdots\alpha_m \beta_{2k+2}\cdots\beta_m}\big) , \end{aligned}$$

where the tensors $\overset{(2k)}{F} \in \Gamma(\wedge^2 E^* \otimes S^{2m-4k} H^*)$ and $\overset{(2k+1)}{F} \in \Gamma(S^2 E^* \otimes S^{2m-4k-2} H^*)$ are the curvature components in irreducible $GL(E)\cdot\mathrm{Sp}(1,\mathbb{C})$-submodules.

For manifolds with spin $\frac{m}{2}$ Grassmann structure it is natural to define half-flat connections as those satisfying

$$\overset{(2i)}{F} = 0 \ , \quad \text{for all } i \in \mathbb{N}.$$

However, for these manifolds it is useful to consider a more refined notion:

Definition 5. *A connection ∇ in the vector bundle $\nu : W \to M$ is called* **k-partially flat** *if $\overset{(i)}{F} = 0$ for all $i \le 2k$. Here $0 \le k \le [\frac{m+2}{2}]$.*

Clearly, $[\frac{m+2}{2}]$-partially flat connections are simply flat connections. We note that for $m{=}1$, 0-partially flat connections are precisely half-flat connections. For general odd $m{=}2p{+}1$, 0-partially flat connections in a vector bundle ν over flat spaces with spin $\frac{m}{2}$ Grassmann structure were considered in [W]. Some other k-partially flat connections on flat spaces were considered in [DN].

The penultimate case $k = [\frac{m}{2}]$ is particularly interesting for odd m.

Theorem 1. *Let M be a half-flat spin $\frac{m}{2}$ Grassmann manifold M. If m is odd and the vector bundle $E \to M$ admits a ∇^E-parallel symplectic form ω_E, then M has canonical* Sp(E)·Sp(H)*-invariant metric $g = \omega_E \otimes \omega_H^m$ and* 4*-form $\Omega \neq 0$. If Ω is co-closed with respect to the metric g, then any $\frac{m-1}{2}$-partially flat connection ∇ in a vector bundle W over M is Ω-selfdual and hence it is a Yang-Mills connection, i.e. it satisfies the Yang-Mills equation.*

Sketch of proof: To describe Ω we use the following notation: e_a is a basis of E, h_α is a basis of H, h_A is the corresponding basis of $S^m H$ and $X_{aA} := e_a \otimes h_A$ the corresponding basis of $TM = E \otimes S^m H$. With respect to these bases, the skew symmetric forms ω_E, ω_H and ω_H^m are represented by the matrices ω_{ab}, $\omega_{\alpha\beta}$ and ω_{AB}

respectively. We define Ω by

$$\Omega := \sum \omega_{ab}\omega_{cd}\omega_{AC}\omega_{BD} X^{aA} \wedge X^{bB} \wedge X^{cC} \wedge X^{dD} ,$$

where X^{aA} is the dual basis of the basis X_{aA}. The connection ∇ is $\frac{m-1}{2}$-partially flat if and only if its curvature F belongs to the space $S^2E^* \otimes \omega_H^m \otimes \mathrm{End}\,(W)$. Contracting a tensor $S = S_{ab}\omega_{AB} e^a e^b \otimes h^A h^B$ in $S^2E^* \otimes \omega_H^m$ with Ω, after some algebra, yields λS with $\lambda = -4(m+1) \neq 0$. Hence any $\frac{m-1}{2}$-partially flat connection is Ω-selfdual and is a Yang-Mills connection. □

2. Construction

We now briefly sketch a construction of half-flat connections on half-flat Grassmann manifolds using a variant of the *harmonic space* approach (c.f. [GIOS1]). The method affords application to a construction of partially flat connections on a half-flat spin $\frac{m}{2}$ Grassmann manifold as well. Details and proofs will appear in [ACD].

We denote by S_H the $\mathrm{Sp}(1,\mathbb{C})$-principal holomorphic bundle over M consisting of symplectic bases of $H_m \cong \mathbb{C}^2$, $m \in M$,

$$S_H = \{ s = (h_+, h_-) \mid \omega_H(h_+, h_-) = 1 \} .$$

The bundle $S_H \to M$ is called **harmonic space**. A fixed parallel symplectic frame (h_1, h_2) of H, such that $m \mapsto s_m = (h_1(m), h_2(m)) \in S_H$ yields a trivialisations $M \times \mathrm{Sp}(1,\mathbb{C}) \cong S_H$ defined by

$$(m, \mathfrak{U}) \mapsto s_m \mathfrak{U} = \Big(h_+ = \sum h_\alpha u_+^\alpha \ , \ h_- = \sum h_\alpha u_-^\alpha \Big) \, , \, \mathfrak{U} = \begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix} ; \, \det \mathfrak{U} = 1 \, .$$

This is precisely the harmonic space of [GIKOS, GIOS2]. The matrix coefficients $u_\pm^\alpha$ of $\mathrm{Sp}(1,\mathbb{C})$ are considered as holomorphic functions on S_H and together with local coordinates of M, we obtain a system of local coordinates on S_H constrained by the relation $u_+^1 u_-^2 - u_-^1 u_+^2 = 1$. We denote by $\partial_{++}, \partial_{--}, \partial_0$ the fundamental vector fields on S_H generated by the standard generators of $\mathrm{Sp}(1,\mathbb{C})$, satisfying the relations

$$[\partial_{++} \, , \, \partial_{--}] = \partial_0 \ , \quad [\partial_0 \, , \, \partial_{++}] = 2\partial_{++} \ , \quad [\partial_0 \, , \, \partial_{--}] = -2\partial_{--} \ .$$

For any section $e \in \Gamma(E)$ we define vector fields $X_\pm^e \in \mathfrak{X}(S_H)$ by the formula

$$X_\pm^e|_{(h_+,h_-)} = \widetilde{e \otimes h_\pm} \ ,$$

where $\widetilde{Y}$ is the horizontal lift of a vector field Y on M. They satisfy the relations

$$[\partial_0, X^e_\pm] = \pm X^e_\pm \ , \quad [\partial_{\pm\pm}, X^e_\pm] = 0 \ , \quad [\partial_{\pm\pm}, X^e_\mp] = X^e_\pm \ ,$$

$$[X^e_\pm, X^{e'}_\pm] \ = \ X_\pm^{\nabla_{\pi_* X^e_\pm} e'} - X_\pm^{\nabla_{\pi_* X^{e'}_\pm} e} - \widetilde{T}(\pi_* X^e_\pm, \pi_* X^{e'}_\pm) \, ,$$

$$[X^e_+, X^{e'}_-] \ = \ X_-^{\nabla_{\pi_* X^e_+} e'} - X_+^{\nabla_{\pi_* X^{e'}_-} e} - \widetilde{T}(\pi_* X^e_+, \pi_* X^{e'}_-) \, ,$$

where T is the torsion of the Grassmann connection, $\widetilde{T}(X,Y) := \widetilde{T(X,Y)}$ denotes the horizontal lift of the vector $T(X,Y)$ and we have used the abbreviation $\nabla_X e := \nabla^E_X e$. We denote by $\mathcal{D}_\pm$ the distributions generated by vector fields of the form $X^e_\pm$, $e \in \Gamma(E)$.

The holomorphic Grassmann structure is called admissible if the torsion of the Grassmann connection ∇ has no component in $S^3H \otimes E \otimes \wedge^2 E^*$. This means, in particular, that the torsion can be written as a sum of tensors linear in ω_H. Now, it is not difficult to prove that if a Grassmann structure is admissible, the distribution $\mathcal{D}_+$ (associated to any parallel frame (h_1, h_2)) on S_H is integrable.

The basic idea of the harmonic space construction is to lift the geometric data from M to S_H via the projection $\pi : S_H \to M$. The pull back $\pi^*\nabla$ of a half-flat connection ∇ in the trivial vector bundle $\nu : W = \mathbb{C}^r \times M \to M$ is a connection in the vector bundle $\pi^*\nu : \pi^*W \to S_H$ which satisfies equations defining the notion of a half-flat gauge connection on S_H. One can also define the weaker notion of an almost half-flat connection in $\pi^*\nu : \pi^*W \to S_H$ by considering only the equations on the curvature involving ∂_0, ∂_{++} and X^e_+ in one of the arguments.

Now, the main result, for which the integrability of $\mathcal{D}_+$ is crucial, can be stated:

Theorem 2. *Let M be a manifold with a half-flat admissible Grassmann structure. Consider a matrix-valued function A_{++} on S_H (the "analytic prepotential"), constant along the leaves of $\mathcal{D}_+$ with the homogeneity property $\partial_0 A_{++} = 2A_{++}$, and an invertible matrix-valued solution of the system of first-order linear ODE's, $\partial_{++}\Phi = -A_{++}\Phi$, $\partial_0\Phi = 0$. The pair (A_{++}, Φ) defines an almost half-flat connection $\nabla^{(A_{++},\Phi)}$ in the bundle $\pi^*\nu : \pi^*W \to S_H$, which depends only on A_{++}, Φ and their first and second partial derivatives. The potential $A(X^e_+)$ of this connection with respect to the frame $e\Phi = e_i\Phi^i_j$, where $e = (e_i)$ is the standard frame of $W = \mathbb{C}^r \times M$, determines a half-flat connection in the bundle $\nu : W \to M$. Conversely, any half-flat connection over M can be obtained using this construction.*

The proof and further details may be found in [ACD].

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