HYPERCOMPLEX GEOMETRY

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1. Introduction

A manifold $M$ is said to be hypercomplex if there exist three integrable complex structures $I_1, I_2, I_3$ on $M$ satisfying the quaternion identities: $I_1I_2 = -I_2I_1 = I_3$.

Example 1. Let $\mathbb{H}$ denote the quaternion numbers and consider $(\mathbb{H}\setminus 0)^n = (S^3 \times \mathbb{R})^n$. Define a hypercomplex structure by

$$I_\lambda(q, \bar{x}) = (q\lambda, \bar{x})$$

for $(q, \bar{x}) \in (S^3)^n \times (\mathbb{R})^n$ and $\lambda \in \{i, j, k\}$. Note that this structure is left invariant. We get compact examples on $(S^3 \times S^1)^n$ via $(\mathbb{Z})^n$-quotients of $(S^3 \times \mathbb{R})^n$.

Thus, the Hopf surface $S^3 \times S^1$ is a simple example of a compact hypercomplex manifold. In the following we shall generalize this example in three directions. The Hopf surface together with the projection $S^3 \times S^1 \to S^3$ is an example of a special Kähler-Weyl 4-manifold $M^4$ with symmetry, fibering over an Einstein-Weyl 3-space $M^4 \to B^3$. This point of view leads to a construction of hypercomplex 4-manifolds via Abelian monopoles and geodesic congruences on Einstein-Weyl 3-manifolds [6].

We may also think of the Hopf surface as the Lie group $SU(2) \times S^1$ with a homogeneous hypercomplex structure. Spindel et al. [20] and independently Joyce [12] showed how such homogeneous structures may be constructed on $G \times T^k$ for $G$ a compact Lie group. Using the twistorial description of hypercomplex geometry [16], we may bring complex deformations to bear on these examples and obtain non-homogeneous structures on $G \times T^k$ [17].

The third theme we shall address is the following: to any quaternionic 4n-manifold $M$ we may associate a hypercomplex $(4n+4)$-space $\mathcal{V}(M)$ [18] generalizing the Swann bundle of a quaternionic Kähler manifold [21]. Joyce [12] showed how to twist this construction with an instanton $P \to M$ to obtain a hypercomplex manifold $\mathcal{V}_P(M)$ fibering over $M$ with fiber the Hopf surface $S^3 \times S^1$. Again such structures may be deformed using twistor theory [16].

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2. Kähler-Weyl 4-Manifolds

Consider a hypercomplex 4-manifold $M$. On $M$ we may define a conformal structure $[g]$; to each non-zero vector $X$ we declare $(X, I_1 X, I_2 X, I_3 X)$ to be orthonormal. Any hypercomplex manifold has a unique torsion-free connection preserving each of the complex structures, the Obata connection $D$ [14]. This connection clearly preserves the conformal structure, so we have a Weyl manifold $(M, [g], D)$ [6]. A Weyl manifold with vanishing trace-free-symmetric part of the Ricci curvature $S_{0r}D$ is called Einstein-Weyl [5]. In the following we shall see how Einstein-Weyl geometry in 3 and 4 dimensions interacts with hypercomplex geometry.

Let $V_\pm$ be the spin bundles and let $L$ be the bundle coming from the representation $A \mapsto |\text{det}(A)|^\frac{1}{2}$. Then the complexified tangent bundle $T_c M$ is equal to $V_+ \otimes V_- \otimes L$ and the curvature

$$R^D = W_+ + W_- + S_{0r}D + F^D_+ + F^D_- + s^D$$

of the Weyl connection $D$ is contained in

$$L^{-2} \otimes (S^4 V_+ \oplus S^4 V_- \oplus (S^2 V_+ \otimes S^2 V_-) \oplus S^2 V_+ \otimes S^2 V_- \oplus \mathbb{R}).$$

For a hypercomplex manifold, the structure is reduced to $\mathbb{R}_{>0} \times \text{SU}(2)_+$, so the curvature is contained in $L^{-2} \otimes (S^4 V \oplus S^2 V_+)$. Therefore, half of the Weyl curvature vanishes, $W_- = 0$, the trace-free-symmetric part of the Ricci curvature vanishes, $S_{0r}D = 0$, half of the Faraday curvature vanishes, $F^D_- = 0$, and the scalar curvature $s^D$ vanishes. In particular, a hypercomplex manifold is an example of a special selfdual 4-manifold (which is also Einstein-Weyl).

Via the Penrose correspondence, a selfdual conformal 4-manifold $M$ with a conformal Killing vector $K$ corresponds to a 3-dimensional complex twistor space $Z$ with a complex holomorphic vector field $K_c$ [1]. The quotient $M/K$ is an Einstein-Weyl 3-space $B$ with a monopole $(w, A)$ consisting of a section $w$ of $L^{-1}$ and a 1-form $A$ such that $*D^B w = dA$ [10]. The quotient $Z/K_c$ is the minitwistor space $S$ of $B$ [8].

A conformal 4-manifold $(M, [g])$ with compatible complex structure $I$ has a natural weight-less anti-selfdual 2-form $\Omega$ ($\Omega \in L^{-2} \otimes \Lambda^2_+$) and a unique Weyl connection $D$ (i.e. a torsion-free connection preserving the conformal structure) such that $d^D \Omega = 0$ [6]. We called such a structure $(M, [g], I, D)$ a Kähler-Weyl manifold.

For a selfdual Kähler-Weyl manifold the twistor space $Z$ contains degree one divisors $D, \overline{D}$ corresponding to the complex structures $\pm I$. The line bundle $L_t = [D - \overline{D}]$ over $Z$ is clearly trivial on twistor lines. Via the Ward correspondence such a degree zero bundle gives an instanton [1], which in this case is the Ricci form $\rho^D$. Therefore, the 4-manifold is hypercomplex iff $L_t$ is trivial. When $L_t$ is trivial the meromorphic function defining the divisor $D - \overline{D}$ gives a map from $Z$ to $\mathbb{CP}^1$.

If a selfdual Kähler-Weyl manifold has a conformal Killing vector $K$, preserving the complex structure, then $D, \overline{D}$ project to divisors $C, \overline{C}$ contained in the minitwistor space $S$. The space $B$ parameterizes degree two rational curves in $S$ and points in $S$ correspond to oriented geodesics in $B$. The rational curve in $S$ corresponding to
a point $x$ in $B$ intersects $C, \overline{C}$ in a pair of points defining a geodesic in $B$ through $x$ with two orientations. In this way we obtain a shear-free geodesic congruence which may be formulated as a section $\chi$ of the bundle $L^{-1} \otimes TB$ satisfying

$$DB\chi = \tau(id - \chi \otimes \chi) + \kappa \ast \chi$$

where shear-free means that the conformal structure normal to $\chi$ is preserved. The sections $\tau, \kappa$ of $L^{-1}$ are monopoles representing the divergence and twist respectively of the congruence [6].

Conversely, from an Einstein Weyl space $(B^3, [h], DB)$ with a monopole $(w, A)$ we may construct a selfdual 4-metric

$$g = w^2h + (dt + A)^2.$$  

The twistor space $Z$ is the total space of the monopole line bundle over the minitwistor space $S$ of $B$. Choose a shear-free geodesic congruence $\chi$. This corresponds to a divisor in $S$ which lifts to a divisor in $Z$ defining a compatible complex structure on the 4-manifold. In fact this conformal 4-space is hypercomplex iff the divergence of $\chi$ is proportional to the monopole $w$ used to construct $g$. This can be seen as follows: the twistor space is the total space of $\mathcal{L}_\tau \xrightarrow{p} S$ and the pull back $p^*\mathcal{L}_\tau$ is trivial over $Z$, so the Ricci form vanishes. As an example we could take the Einstein-Weyl space given by the round 3-sphere and let $\chi$ be a left or right invariant congruence. Since these congruences have vanishing $\tau$ any sum $w$ of fundamental solutions to the Laplace equations would give a hypercomplex 4-space. The solution $w = 1$ (in the gauge given by the round sphere) gives the Hopf surface $S^3 \times S^1$.

3. Lie Groups and Hypercomplex Geometry

The hypercomplex structure of the Hopf surface defined in the example in the introduction may be considered as a left invariant structure on the Lie group $S^1 \times SU(2)$. Consider the Lie group $SU(3)$. The Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ decomposes as $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1$ where

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{d}_1 & \mathfrak{f}_1 \\ \mathfrak{f}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{su}(2) & \mathbb{C}^2 \\ \mathbb{C}^2 & \mathfrak{u}(1) \end{pmatrix} = \mathfrak{su}(3).$$

Think of $\mathfrak{b} \oplus \mathfrak{d}_1$ as $\mathbb{H}$ and think of $\mathfrak{d}_1$ as the imaginary quaternions acting on $\mathfrak{f}_1$ via the adjoint representation. Applying left translations we obtain in this way a hypercomplex structure on $SU(3)$. Now, let $G$ be a compact semi-simple Lie group. The Lie algebra $\mathfrak{g}$ decomposes as follows

$$\mathfrak{g} = \mathfrak{b} \oplus_{j=1}^n \mathfrak{d}_j \oplus_{j=1}^n \mathfrak{f}_j,$$

where $\mathfrak{b}$ is Abelian, $\mathfrak{d}_j$ is isomorphic to $\mathfrak{su}(2)$ and $[\mathfrak{d}_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$. The rank $r$ of $G$ is equal to $n + \dim \mathfrak{b}$ and if we add $2n - r$ Abelian factors we can think of $(2n - r)\mathfrak{u}(1) \oplus \mathfrak{b} \oplus_{j=1}^n \mathfrak{d}_j$ as $\mathbb{H}^n$. Since $[\mathfrak{d}_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$ we can proceed as with $SU(3)$ above to get a left
invariant hypercomplex structure on $T^{2n-r} \times G$ [12]. In this way we get homogeneous hypercomplex structures on for example

$$
SU(2\ell + 1), T^1 \times SU(2\ell), T^\ell \times SO(2\ell + 1), T^\ell \times Sp(\ell), T^{2\ell} \times SO(4\ell),
$$

$$
T^{2\ell-1} \times SO(4\ell + 2), T^2 \times E_6, T^7 \times E_7, T^8 \times E_8, T^4 \times F_4 \text{ and } T^2 \times G_2.
$$

The issue is now how to get more than these homogeneous examples. For a general hypercomplex manifold $(M^{4n}, I_1, I_2, I_3)$ we note that we have a 2-sphere of complex structures $I_v = v_1 I_1 + v_2 I_2 + v_3 I_3$ for $v = (v_1, v_2, v_3) \in S^2$. The twistor space of $M$ is the space $W = M \times S^2$ of these compatible complex structures [15, 16]. This space is a complex manifold of dimension $2n + 1$: the complex structure $\mathcal{I}$ at $(x, v) \in M \times S^2$ is standard along the 2-sphere and it is equal to $I_v(x)$ along $T_x M$. The integrability of $\mathcal{I}$ is a consequence of $M$ being hypercomplex. The holomorphic projection $W \xrightarrow{p} S^2 = \mathbb{CP}^1$ has fiber $p^{-1}(z)$ which is $M$ together with the complex structure determined by the point $z \in \mathbb{CP}^1$. The non-holomorphic projection $W \xrightarrow{\pi} M$ has as fibers, rational curves of normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

The idea is to deform the hypercomplex structure on $M$ by deforming the map $W \xrightarrow{p} \mathbb{CP}^1$ [17]. Consider the sheaf $\mathcal{D}$ defined by the exact sequence

$$0 \rightarrow \mathcal{D} \rightarrow \Theta_W \xrightarrow{dp} p^* \Theta_{\mathbb{CP}^1} \rightarrow 0.
$$

where $\Theta$ is the tangent sheaf. The deformations of the map $p$ (and therefore the deformations of the hypercomplex geometry on $M$) is measured by the cohomology groups of the sheaf $\mathcal{D}$ [9]: $H^0(W, \mathcal{D})$ is the space of hypercomplex symmetries, $H^1(W, \mathcal{D})$ is the parameter space of deformations and $H^2(W, \mathcal{D})$ is the obstruction space.

For $M = T^k \times G$ the twistor space $W$ is a homogeneous complex manifold and one may expect that $H^q(W, \mathcal{D})$ is computable via Bott-Borel-Weil-Hirzebruch theory for representations and cohomology. Consider the natural map $\Phi$ from $W$ to $G/U$ where $U$ is a maximal torus in $G$. The spaces $Z = G/U$ is a complex manifold and is called the Borel flag [2, 7]. The cohomology of the Borel flag has indeed been studied using representation theory and this will help us getting information about the cohomology on $W$: let $X$ be $M$ with a complex structure $X = p^{-1}(z)$. The restriction of $\Phi$ to $X$ has fiber $E$ which is a product of elliptic curves. We may compute $H^q(X, \mathcal{O}_X)$, say, using a Leray spectral sequence

$$E_2^{pq} = H^p(Z, R^q \Phi_* \mathcal{O}_X), E_\infty^{pq} = H^{p+q}(X, \mathcal{O}_X).
$$

We find $R^q \Phi_* \mathcal{O}_X = \mathcal{O}_Z \otimes H^q(E, \mathcal{O}_E)$ and since $H^p(Z, \mathcal{O}_Z)$ vanishes for $p \geq 1$ [3], the spectral sequence is easy to handle and we get

$$H^q(X, \mathcal{O}_X) = E_\infty^{0q} = E_2^{0q} = H^q(E, \mathcal{O}_E) \cong \Lambda^q \mathbb{C}^n.$$
In much the same way we can compute the cohomology $H^j(W, \mathcal{O}_W)$, $H^j(W, \Phi^*\Theta_Z)$ etc. via vanishing results of Bott [4]. Then using the sequences

$$0 \to \mathcal{O}_W \to p^*\Theta_{\mathbb{CP}^1} \to \mathcal{O}_{X_1 \cup X_2} \to 0$$

$$0 \to D \to \Theta_W \xrightarrow{dp} p^*\Theta_{\mathbb{CP}^1} \to 0,$$

we are able to find $H^j(W, D)$.

It turns out that the obstruction space $H^2(W, D)$ is non-trivial. Therefore we study the possible obstructions using Kuranishi theory [13]. However, we can prove that for the $U$-invariant part of $H^1(W, D)$ the obstruction vanishes and we obtain (see [17] for a more precise formulation of the theorem):

**Theorem 1.** Suppose $G$ is a compact semi-simple Lie group of rank $r$ and containing $n$ factors of $\mathfrak{sp}(1)$. Then the local moduli at a generic deformation of left-invariant hypercomplex structures on $T^{2n-r} \times G$ is a smooth manifold of dimension $n(n+r)$. The identity component of the group of hypercomplex symmetries of a generic deformation is the Abelian group $T^{2n}$.

In the introduction we defined one hypercomplex structure on $(S^3 \times S^1)^n$. Inspired by the theory of Abelian varieties, we shall now construct a family of hypercomplex structures on $(S^3 \times S^1)^n$ and use the theorem above to secure completeness. Let $(q_1, \ldots, q_n; x_1, \ldots, x_n) = (\mathbf{q}, \mathbf{x})$ be coordinates for $(S^3)^n \times \mathbb{R}^n$. Here the $q_j$ are unit quaternions. Choose a hypercomplex structure on $\mathbb{H}^n$ by right multiplication of unit quaternions. Then we define a hypercomplex structure on $(S^3 \times \mathbb{R})^n$ through the embedding into $\mathbb{H}^n$.

For $1 \leq j \leq n$, define an action generated by

$$\gamma_j(\mathbf{q}, \mathbf{x}) = (e^{2\pi i \theta_{1j}}q_1, \ldots, e^{2\pi i \theta_{nj}}q_n; \mathbf{x} + \mathbf{v}_j).$$

The action of $\gamma_j$ is represented by the column vectors $\mathbf{v}_j$ and $\Theta_j = (\theta_{1j}, \ldots, \theta_{nj})^T$, where $\theta_{ij}$ are in $\mathbb{R}/\mathbb{Z}$.

Assume that the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent. Let $\Gamma \cong \mathbb{Z}^n$ be the group generated by $\{\gamma_1, \ldots, \gamma_n\}$. We call

$$(\Theta|V) = (\Theta_1, \ldots, \Theta_n|\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

the period matrix of the manifold $(S^3 \times \mathbb{R})^n/\Gamma$. Thus the groups $\Gamma$ are parameterized by the space $\mathbb{R}/\mathbb{Z})^n \times GL(n, \mathbb{R})$. However, different period matrices may generate the same group. In fact, the period matrices $(\Theta|V)$ and $(\hat{\Theta}|\hat{V})$ generate the same group if and only if there is a matrix $M = (m_{ij})$ in $GL(n, \mathbb{Z})$ such that

$$(\hat{\Theta}|\hat{V}) = (\Theta M|VM).$$

The quotient space $(S^3 \times \mathbb{R})^n / \Gamma$ is a hypercomplex manifold because the actions of $\Gamma$ commute with the right multiplications of the quaternions on $(q_1, \ldots, q_n)$. The quotient space is clearly diffeomorphic to $(S^3 \times S^1)^n$. Using the fact that symmetries
lifts to holomorphic maps of the twistor space (which is built out of a complex projective space), it is seen that hypercomplex manifolds \((S^3 \times \mathbb{R})/\Gamma\) and \((S^3 \times \mathbb{R})/\Gamma'\) are equivalent if and only if there exist period matrices \((\Theta|V)\) and \((\Theta'|V')\) for \(\Gamma\) and \(\Gamma'\) respectively such that \(V = V'\), and \(\Theta_j = \pm \Theta'_j\). Thus we obtain

**Theorem 2.** The quotient space \(((\mathbb{R}/\mathbb{Z})^{n^2} \times \text{GL}(n, \mathbb{R}))/((\mathbb{Z}^2 \times \text{GL}(n, \mathbb{Z}))\) is a complete moduli space for hypercomplex structures on the product manifold \((S^3 \times S^1)^n\).

The constructions above are currently being modified to work for the case of nilpotent automorphisms and for combinations of the semi-simple and the nilpotent situation in joint work with Grantcharov and Poon.

### 4. The Swann Bundle

Now we turn to the third theme where \(S^3 \times S^1\) appears as the fiber of a bundle. The definition of a hypercomplex manifold is equivalent to requiring that the holonomy group lies in \(\text{GL}(n, \mathbb{H})\). More generally for a quaternionic manifold \(M\) the frame bundle has a torsion free connection with holonomy in

\[
\text{GL}(n, \mathbb{H}) \text{ GL}(1, \mathbb{H}) = (\mathbb{R}_{>0} \times \text{SL}(n, \mathbb{H}) \times \text{Sp}(1))/\{\pm 1\}.
\]

This group acts on \(\mathbb{H}/\mathbb{Z}_2\) by

\[
\rho(\lambda, A, q)(\eta) = \lambda \eta^2 \eta q^{-1}.
\]

The associated bundle is denoted by \(\mathcal{U}(M)\) and was studied by Swann for \(M\) a quaternionic Kähler manifold [21]. For \(M\) quaternionic \(\mathcal{U}(M)\) is hypercomplex [18]. The group \(\mathbb{H}^+\) acts from the left on \(\mathcal{U}(M)\) and the center \(\mathbb{Z}\) preserves the hypercomplex structure. The quotient \(\mathcal{U}(M)/\mathbb{Z}\) is denoted \(\mathcal{V}(M)\) and is a compact hypercomplex manifold which we call the **Swann bundle** [18], [19]. Now, let \(P\) be an \(S^1\)-instanton on \(M\). Then Joyce [12] introduces the twisted bundle \(\mathcal{V}_P(M) = P \times_{S^1} \mathcal{V}(M)\) which again provides us with an example of a compact hypercomplex manifold. The fiber from \(\mathcal{V}_P(M)\) to \(M\) is \(S^3 \times S^1\).

**Example 2.** Let \(M\) be the complex projective plane and let \(P \rightarrow M\) be the instanton given by the Hopf fibration \(S^5 \rightarrow \mathbb{C}P^2\). Then in this case the hypercomplex manifold \(\mathcal{V}_P(M)\) is equal to \(\text{SU}(3)/\mathbb{Z}_2\).

We may now apply complex deformation theory to these twisted Swann bundles. The twistor space \(W\) of \(\mathcal{V}_P(M)\) fibers over the twistor space \(Z\) of \(M\) and via the Leray spectral sequence we are able to compute the cohomology \(H^2(W, \mathcal{D})\) in terms of the cohomology on \(Z\) [16].

**Example 3.** Let \(M\) be the connected sum \(2\mathbb{C}P^2\) equipped with a Poon conformal structure \(c_\lambda, \lambda \in (0, 1)\). Then the deformation theory gives a 4-parameter space of \(T^3\)-symmetric hypercomplex structures on the 8-manifold \(\mathcal{V}(2\mathbb{C}P^2)\). Furthermore, we can integrate and find these hypercomplex manifolds locally as a (Joyce-) hypercomplex
quotient [11] of $\mathbb{H}^4$ with a $T^2$ action. The space is realized as a subspace of $\mathbb{C}^6 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ given by simple equations [16].

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