Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

$Sp(1)^n$-INVARIANT QUATERNIONIC KÄHLER METRIC

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We study $Sp(1)^n$-invariant hyperKähler or quaternionic Kähler manifolds of real
dimension $4n$. In the case of $n = 1$, Hitchin classified these kinds of metrics associated
with special functions. They are written as

$$g = dt^2 + \sum_{i=1}^{3} f_i(t)\sigma_i^2 \quad \text{on} \quad \mathbb{R} \times Sp(1),$$

where $\sigma_1, \sigma_2, \sigma_3$ are canonical 1-forms associated with $i, j, k \in \mathfrak{sp}(1)$. We obtain a
generalization of the Hitchin’s result ([2]).

Theorem 0.1. Let $\mathbb{H}$ be the Hamilton’s quaternion number field $\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.
Then $\mathbb{H}^n$ has a natural quaternionic structure $I, J, K$ induced by the action of $i, j, k$.
Since $\mathbb{H} \setminus \{0\}$ is diffeomorphic to $\mathbb{R} \times Sp(1)$ canonically, $(\mathbb{H} \setminus \{0\})^n$ is diffeomorphic to
$\mathbb{R}^n \times (Sp(1))^n$. We denote the coordinate of $\mathbb{R}^n$ by $(t_1, t_2, \ldots, t_n)$. Let a Riemannian
metric $g$ be written as

$$g = \sum_{i=1}^{n} (dt_i^2 + \sum_{j=1}^{3} f_{ij}(t_1, t_2, \ldots, t_n)\sigma_{ij}^2),$$

where $\sigma_{i1}, \sigma_{i2}, \sigma_{i3}$ are canonical 1-forms associated with $i, j, k \in \mathfrak{sp}(1)$. Then we obtain the following:
(i) If $g$ is hyperKählerian with respect to the quaternionic structure $I, J, K$, then
each $f_{ij}(t_1, t_2, \ldots, t_n)$ depend only on $t_i$. Hence the Riemannian metric is an $n$-times
product of hyperKähler metric obtained by Hitchin.
(ii) If $g$ is quaternionic Kählerian with respect to the quaternionic structure $\mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$, then $g$ is hyperKählerian.

By Hitchin, the coefficient functions $f_{ij}$ satisfy

$$\begin{cases}
\frac{df_{i1}}{dt} = 2f_{i2}f_{i3}, \\
\frac{df_{i2}}{dt} = 2f_{i3}f_{i1}, \\
\frac{df_{i3}}{dt} = 2f_{i1}f_{i2}.
\end{cases}$$
These equations imply the first integral
\[
\begin{cases}
  f_{i1} - f_{i2} = a_i, \\
  f_{i1} - f_{i3} = b_i,
\end{cases}
\]
where \( a_i, b_i \) are constant. Associated to \( (a_i \neq 0, b_i \neq 0), \) \( (a_i = 0, b_i \neq 0) \) and \( (a_i = 0, b_i = 0), \) the metric is the type of Belinski-Gibbons-Page-Pope metric, Eguchi-Hanson metric and conformally flat metric.

One of our backgrounds is a natural metric on a moduli space of self-dual connections on \( \mathbb{H} \). It coincides to a framed moduli space of self-dual connections on \( S^4 \). The quaternionic Kähler manifold \( \mathbb{H} \) has an isometry \( Sp(1) \cdot Sp(1) \), that acts on the framed moduli space \( \mathcal{M}_k \) on a Hermitian vector bundle \( V \) of rank 2 with the second Chern class \( k \).

\[
\mathcal{M}_k = \{ \nabla : \text{self - dual connection on } V, c_2(V) = k \}/\text{gauge group}.
\]
The tangent space of \( \mathcal{M}_k \) is represented as the first cohomology of the following elliptic complex:
\[
0 \longrightarrow \text{End}(V) \xrightarrow{\nabla} \text{End}(V) \otimes T^* \mathbb{R}^4 \xrightarrow{pr_+ \circ dV} \text{End}(V) \otimes \wedge_- \longrightarrow 0
\]
where \( \wedge^2 T^* \mathbb{R}^4 \) is decomposed into the self-dual part \( \wedge_+ \) and the anti-self-dual part \( \wedge_- \), \( pr_- : \wedge^2 T^* \mathbb{R}^4 \longrightarrow \wedge_- \) is the natural projection. The tangent space of the moduli space is represented as a subset of \( \text{End}(V) \)-valued 1-forms. The \( L_2 \)-metric of \( \text{End}(V) \)-valued 1-forms induces a Riemannian metric on the moduli space \( \mathcal{M}_k \)
\[
\langle \alpha, \beta \rangle = \int_{\mathbb{R}^4} \text{tr}(\alpha \wedge \beta).
\]

Furthermore the quaternionic structure \( I, J, K \) induces a hyperKählerian structure with respect to the Riemannian metric. It is known that the dimension of \( \mathcal{M}_k \) is \( 8k \). These are represented as elements of
\[
\mathcal{M}_{k,k+1}(\mathbb{H}) = \{ (A, B)|A \in M_{k,1}(\mathbb{H}), \ B \in M_{k,k}(\mathbb{H}) \}
\]
by the A.D.H.M. construction. We denote
\[
\mathcal{M}^0_{k,k+1}(\mathbb{H}) = \{ (A, B)|(A, B) \in \mathcal{M}_{k,k+1}(\mathbb{H}), \ \text{tr}(B) = 0 \}.
\]
It corresponds to a hyperKähler submanifold in \( \mathcal{M}_k \), whose dimension is equal to \( 8k - 4 \). We denote it by \( \mathcal{M}^0_k \). The conformal group \( (Sp(1) \times Sp(1))/\mathbb{Z}_2 \times \mathbb{R}^+ \times \mathbb{H} \) on \( \mathbb{H} \) and the gauge group \( Sp(1)/\mathbb{Z}_2 \) at the infinity act on \( \mathcal{M}^0_k \)
\[
\begin{align*}
  \text{i.} & \quad (q, p) \in (Sp(1) \times Sp(1))/\mathbb{Z}_2, \quad x \mapsto qx^{-1} \quad (A, B) \mapsto (Aq, pB), \\
  \text{ii.} & \quad \lambda \in \mathbb{R}^+, \quad x \mapsto \frac{1}{\lambda} x \quad (A, B) \mapsto (\lambda A, \lambda B), \\
  \text{iii.} & \quad a \in \mathbb{H}, \quad x \mapsto x - a \quad (A, B) \mapsto (A, B + a), \\
  \text{iv.} & \quad r \in Sp(1)/\mathbb{Z}_2, \quad (A, B) \mapsto (rA, B).
\end{align*}
\]
We denote vector fields generated from the action i, ii by \( V_1(\lambda), \ V_2(a) \). Then the norms of \( V_1(\lambda), \ V_2(a) \) are constant on each orbit.
Proposition.  

\[ \|V_1(\lambda)\|^2 = \lambda^2 C_1 \]
\[ \|V_2(a)\|^2 = \sum_{i,j=0}^3 C_{2ij}a_ia_j, \]

where \( C_1, C_2 \) are constant, \( a = a_0 + ia_1 + ja_2 + ka_3 \).

The \( Sp(1) \times \mathbb{R}^+ \) acts on \( \mathcal{M}_k^0 \). The reduced space \( \mathbb{P}(\mathcal{M}_k^0) \) is known to be quaternionic Kählerian ([1]). These are not smooth manifolds, they have singularities. Now in the case \( k = 2 \), \( \mathcal{M}_2^0 \) and \( \mathbb{P}(\mathcal{M}_k^0) \) are examples that are hyperKähler or quaternionic Kähler space of dimension \( 4n \) with \( Sp(1)^n \)-symmetry. In fact \( \mathcal{M}_2^0 \) is a hyperKähler space of dimension \( 3 \times 4 \) with \( (Sp(1) \times Sp(1))/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2 \)-symmetry and \( \mathbb{P}(\mathcal{M}_2^0) \) is a quaternionic Kähler space of dimension \( 2 \times 4 \) with \( Sp(1)/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2 \)-symmetry.

References
