EXPLORING AN UNDERSTANDING OF EQUALS AS QUANTITATIVE SAMENESS WITH 5 YEAR OLD STUDENTS

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Many students persistently experience difficulties in their understanding of the equal sign with the common misconception of equal signifying a place to put the answer being prevalent at all levels of schooling. It is conjectured that one reason for this is the types of activities that occur in the early years. This paper reports on the results of a teaching experiment conducted with forty 5 year olds. A purposeful sample of 4 students participated in two clinical interviews. The results of these interviews indicated that not only are young students capable of understanding equal as quantitative sameness but they can represent this using real world contexts and in symbolic form.

INTRODUCTION

The question that persists in the algebraic domain is: why do so many adolescents face difficulties with algebra? Is it an issue of readiness and/or that the teaching or curriculum to which students have been exposed has been preventing them from developing foundational mathematical ideas and representations? There is a wealth of evidence emerging? that is beginning to support the later (e.g., Carraher, Schlieman, Brizuela & Earnest, 2006). Research has shown that students are not only capable of engaging in functional thinking at a young age but also that carefully chosen tasks, materials and conversations support them in these discussions. One area that has been given little attention is the area of equivalence, and in particular the meaning of equals. While much has been written about students' misconceptions with regard to the equal sign, there has been little research on young students' understanding of equivalence and in particular identifying teaching and tasks that begin to support this understanding.

In mathematics, the use of the equal sign appears to fall into four main categories. These are (a) the result of a sum (e.g., 3+4=7), (b) quantitative sameness (e.g., 1+3=2+2), (c) a statement that something is true for all values of the variable (e.g., $x+y=y+x$), and (d) a statement that assigns a value to a new variable (e.g., $x+y=z$) (Freudenthal, 1983). With regard to quantitative sameness, "equals" means that both sides of an equation are the same and that information can be from either direction in a symmetrical fashion (Kieran & Chalouh, 1992). Most adolescent students do not have this understanding; rather they have a persistent idea that the equals sign is either a *syntactic indicator* (i.e., a symbol indicating where the answer should be written) or an *operator sign* (i.e., a stimulus to action or "to do something") (Behr, Erlwanger & Nichols, 1980; Filloy & Rojano, 1989). Carpenter and Levi (2000) claim that many students complete elementary school with a very narrow view of 'equal' and with an emphasis on finding the answer. Given that in the elementary classroom,

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misunderstandings regarding the equal sign develop at an early age and remain entrenched as students' progress through school (Warren, 2006). The question is are young children capable of equivalence thinking and if so what types of activities begin to support the development of this thinking?

In the early years the equal sign is closely linked to the development of the concepts of addition and subtraction. There has been a vast amount of research relating to this area of mathematics (e.g., Nesher, Greeno, & Riley, 1978; Verngaud, 1982; Verschaffel & De Corte, 1996). Each has classified addition and subtraction problems into various categories. For example, Nesher et al. (1978) identified 14 types of addition and subtraction word problems, each falling under the major categories of change, combine or compare. Change problems refer to dynamic situations in which some event changes the value of the quantity. Combine problems relate to static situations where there are two amounts that are considered either separately or in combination. Compare problems involve two amounts that are compared and the difference between them ascertained. Further distinctions were made depending on the identity of the unknown quantity. Other groups identified different classification systems (e.g., composition of two measures, transformations linking two measures, and static relationships between two measures, Verngaud, 1982). In all of these instances the focus was on first interpreting the problem, second ascertaining the appropriate representation for the problem, and third finding the answer. Textbooks and classrooms tend to be permeated with these types of problems, particularly rudimentary change and compare problems. While these problem types support the development of computational understanding and give insights into the interpretation of word problems, their primary focus is on computation. Little research has occurred on what types of problems support the development of equivalence and particularly the notion of quantitative sameness. The focus of the research reported in this paper was developing an understanding of equal as quantitative sameness, hence the choice of using compare type problems involving two quantities of the same value as the basis for classroom activity.

This research project investigates young children's development of algebraic thinking utilising unmeasured quantities (i.e., length, volume, and area) in conjunction with number and the operations. The advantage of unmeasured quantities is that numbers are not required to investigate ideas such as equivalence and non-equivalence or generalisations such as $a=c+d$ then $c+d=a$ (Davydov, 1982).These can be explored by using concrete models such as streamers of differing lengths. Thus young children can investigate and conjecture about the 'big' ideas of mathematics, focus on processes rather than products, and develop relevant language and representations before they even formally begin number.

The particular aims were to (a) use unmeasured quantities to develop a language base to describe equal situations, and (b) transfer this language to create compare stories with a focus on quantitative sameness. The conjecture was that a focus on these two aspects in the early years assists students to broaden their understanding of equals beyond 'the result of a sum'.

METHODS

Three schools volunteered to participate in the Professional learning. Each school was requested to select two Year 1 teachers on the understanding that they would be working together to collaboratively develop learning experiences for implementation in their classrooms. Thus a total of six teachers participated in the project, two from each site. The teachers were also aware that their learning would be fully supported by the researcher. The focus of this paper is on one aspect of the larger project, the students who participated in the classrooms where the focus was on equivalence. The average age of the participating students was 5 years.

The teaching cycle consisted of four dimensions, collaborating planning 4 lessons, with the researcher critically reflecting on the lessons, implementing the adjusted lessons in their classroom, and, sharing the outcomes with the whole group. This cycle was repeated twice. During the teaching phase electronic contact was maintained between the pair of teachers and the researcher. All lessons were video taped. In the equivalence classrooms, the teaching experiment consisted of two phases. Phase 1 focused on developing the language of equals (e.g., same as, different from, equal) using unmeasured quantities in comparative situations such as comparing the amount of liquid in two containers, the height of two children, the weight of two objects. Each situation involved comparing two quantities with one attribute difference, for example colour or height, introducing the language of different from and comparing two quantities and in situations where one attribute was the same, introducing the language of same as and equal. Physical balance scales were used to compare the masses of various objects. Phase 2 aimed to transfer these contexts and understandings to number situations with a focus on the attribute of number (e.g., 2 parrots add 3 galahs is the same as 3 parrots add 2 galahs 2+3=3+2).

To ascertain students' learning four students were purposefully selected by the teachers as representing the spread of ability of the participating students. At the end of each cycle these students participated in a clinical interview. Because of the age of the students the data gathering was based on Piaget's Clinical Interview, where inferences are drawn from young student's behaviour in activities as much as from what they say. All interviews were audio-taped and detailed notes were kept. Interview 1 focused on ascertaining the students' understanding of equality, and their ability to apply this understanding to a situation involving numbers. Interview 2 moved to seeing if students could transfer this understanding to addition situations. There was approximately a two month period between each interview.

RESULTS

In the first interview students were asked what they thought the words equal, same, different, and balance meant and to give examples of each of these. They were then asked to look at the two diagrams (see Table 2) and state whether they were true or false giving a reason for their answer. Table 1 summarises their explanations for the meaning of the words. The order in which the students appear in the result is according to their perceived ability. Brianna represented the weaker students in the class and Olivia represented the more able students.

Table 1.

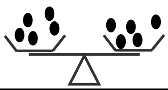
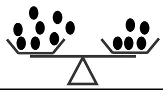
Student's explanations of the words equal, same and different.

Student	Equal	Same	Different
Abby	Same as each other.	Two blue teddies and everything the same are equal.	Two people with different coloured eyes would not be equal.
Brianna	If something is the same.	If something is wood and something else is wood they are the same as.	If something is blue and something is yellow they are not the same as.
Ethan	Equal means the same height, the same number.	Same height as each other.	Different height and different number.
Olivia	Equal means the same as like that.	The same as sought of means the same as equals.	Like one is red and one is blue.

The students were given a selection of coloured bears and were asked to use the bears to show what each of these words meant. Nearly all of them focused on the attribute of colour to demonstrate their understanding of equal, same and different. The students were also asked to explain the word balanced. In each instance the students used gestures to represent balance. Abby put her arms out horizontally and said *If you were walking along a bridge and in that hand had three cans and the other hand you had three cans you would be balanced.* The other three children used their hands to show balance. In each instance they put their hands at the same height. Olivia added *you could have two bananas in each. They would balance.* Ethan said *If you weren't balanced one would be up and one would be down. They are not equal.* For the two diagrams in Table 2, they were asked if they were balanced or not balanced and to explain how they knew.

Table 2

Student's explanation of the two diagrams.

Student	Problem 1 	Problem 2 
Abby	Because that one has 1, 2, 3, 4, 5 and that one has 1, 2, 3, 4, 5. They both have 5.	1 2 3 4 5 and 1 2 3 4 5 6 8. That it not equal because one has 5 and one has 8.
Brianna	Balanced. You can easily tell by the numbers, 5 and 5.	Not balanced because 5 and 8 they are not balanced.
Ethan	Because they both have the same amount of balls. But when you look far away they	That has more than that one so that is the heaviest and that is the lightest. That one

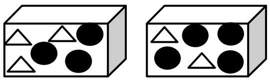
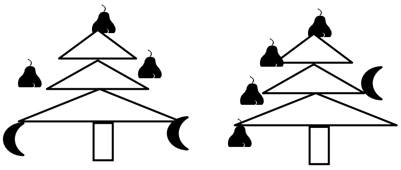
	look not balanced. A triangle can't balance things. A triangle wobbles.	should be down [gesturing to the LH pan] and that should be up.
Olivia	They look equal. They are the same height, the same level [ignoring the number of balls in each pan].	I just went like that with my finger [holding her finger in the air] and it goes down so it is not equal.

Interestingly the picture of the balance scale both assisted and detracted from reaching an understanding of equivalence. For Abby and Brianna it appeared to act as an effective analogue for equivalence, assisting them in identifying and comparing the two differing sides of the equations, and ascertaining if they were the same or different. By contrast, for Ethan and Olivia their attention turned to the icon itself and decisions were made according to if the icon looked as if it was horizontal/level rather than if the number of balls in each were the same. Even the triangular shape came into play, with Ethan commenting that *a triangle wobbles* so it cannot be balanced.

In the second interview they were again asked to explain the words equal, same and different. Their responses were similar to those offered in Interview 1. They all proffered examples utilising same and equal to describe situations where the attribute was the same and different from for situations where the attribute was different. They were then asked to examine two situations (the boxes and Christmas trees) and (a) write an equation, and (b) make up a story for each.

Table 3

Student's equations for the toy boxes and Christmas trees.

Student	Completion of Phase 2. Problem 1 	Problem 2 
Abby	[illegible]	5 + 5 = 10
Brianna	3• + 3Δ 4• 2Δ	3P + 2b = 4P + 1b
Ethan	3Δ = 3O	2[illegible] + 3[illegible] = 1[illegible] + 4[illegible]
Olivia	3 + 3 = 4 + 2	2 + 3 = 1 + 4

Students seemed to have more success in representing the Christmas trees as an equation as compared with the two toy boxes. Abby still persisted in finding an answer, tallying all the pears and bananas in both trees. Brianna and Ethan both exhibited aspects of 'fruit salad' algebra where their notation systems were short hand for objects (e.g., p for pears or a drawing of a pear). Their stories give further insights into their understandings of the two contexts. For the toy boxes,

Abby's story: Once upon a time there was two toy boxes and there were three dots in that one and four dots in that one and they had three triangles in that one and two ones in that one, two triangles in that one. There were twelve toys in each box and there was six toys and six toys in that one and then they had an equal number in each box.

Briana's story: In that box there are three dots and three triangles and in that box there are four dots and two triangles.

Ethan's story: I have never done that before.

Olivia's story: Once upon a time there were two toy boxes and in one toy box there was three circles and three triangles and in the other toy box there was four circles and two triangles and then there was an equal amount of toys in the boxes was different from…..*What do you mean by that?* The boxes look different.

Each response showed varying understandings of equals. Abby's story seemed to exhibit components of computational thinking, there were 12 toys altogether (finding an answer – the result of a sum) and components of equivalence thinking as quantitative sameness, each box had 6 toys. Even though Briana resisted finding an answer to the toy box activities she experienced difficulties in including equal in her explanation. Ethan could write equations for each of the situations exhibiting a symbolic understanding of equals as quantitative sameness, though he failed to create a story for each. By contrast, Olivia could not only write correct equations for each situation but also explain each using appropriate language. She also endeavoured to include the language of different from by focussing on the attribute of the appearance of the boxes.

DISCUSSION AND CONCLUSION

This research suggests that young students can engage in conversations about equal as quantitative sameness, each giving comparative examples such as the same height as each other then they are equal. Embedding their initial explorations in a numberless world allowed them to develop some understanding of the language commonly used to describe equivalent situations. As Davydov suggested, it also appeared to assist them in focusing on the important mathematics in this situation, that is, quantitative sameness incorporates comparing two quantities that are the same. The physical context in a numberless world appeared to assist them in developing the language of equivalence and they were capable of transferring this language to situations involving number, although success appeared to be closely related to context. For example, the students exhibited greater success when comparing the two Christmas trees than comparing the shapes in the box. Phase 2 interviews occurred just before the Christmas break where many of the students were engaged with placing ornaments on their Christmas tree. They certainly found both of these contexts simpler than the context with the balance scales.

During Phase 1 students were introduced to the notion of balance and explored whether the scales were even/level. They also engaged in whole body experiences where various objects were placed in either of their hands and they were asked to model using their arms if they were balanced or not. Initially the attribute on which they focussed was mass of different sized objects, with decisions being made on whether the pans or their arms were level or not. This resulted in some confusion especially when considering situations where there were a different number of objects in the pan with each having the same mass. Most seemed incapable of separating the attribute of number from the icon of the scales themselves. Past research has evidenced the advantages of balance scales in exploring equivalent situations as they are not directional in any way and can cope with the need to consider the equations as an entity rather than an instruction to act and achieve a result. While balance scales can act as a useful metaphor for equivalence, physical representations of the concept can in fact interfere with transfer to number situations. For these young students equivalence was judged not only on the number of objects in each scale but also as to whether the scales appeared level, a consideration that for three students took precedence. The sequence in which balance scales are presented to young students needs further research.

The results also begin to exhibit the development stages of students' own notation systems. This is especially applicable to the toy boxes in Table 3. Abby not only miscounted the number of shapes drawn on the box but was still writing the numerals as mirror images. Brianna began to incorporate the addition sign in her response. Ethan had begun to use the equal sign. All of these three students included a drawing of a triangle and circles in their equations. Their verbal response mirrored their notation system, for example, 3 triangles equals 3 circles. By contrast, Olivia could not only separate the number notation from a representation of the object but could incorporate both the addition sign and equal sign in her response. She also expressed this as 3 add 3 equals 4 + 2. Past research in the domain of patterning has evidenced that there is a strong relationship between describing a pattern and writing the rule for the pattern (e.g., Warren, 2006). This research indicates that there also appears to be strong links with verbalising an equation and writing the equation in symbolic form. The responses to the Christmas tree story also evidence the use of letters as short hand for objects, a common error that occurs with adolescent students as they transition into algebraic thinking.

Past research has presented many examples of how adolescent students hold a persistent belief that the equal sign is a syntactic indicator for a place to put the answer. Our conjecture is that this is due to the type of arithmetic activities that occur in the elementary years, especially experiences of arithmetic as a computation tool. The results of this research begin to indicate that young students can come to some understanding of equal as quantitative sameness. Whether they can continue to negotiate two different meanings of equals as they begin to experience arithmetic as a computational tool needs further investigation. Algebraic activity can occur at an earlier age than we had ever thought possible and these experiences with appropriate teacher actions may assist more students join the conversation in their adolescent years. The expansion of compare stories to include comparisons that are the same, as well as different proved to be a productive means for introducing young students to equal as quantitative sameness.

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