INDUCTION, ANALOGY, AND IMAGERY IN GEOMETRIC REASONING[1]

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Mathematical thinking that advances into generalization through induction, analogy and imagery is an important tool with which mathematicians find mathematical principles. Mathematically gifted students, also, need to experience this thinking process. This research is focused on followings: how mathematically gifted 6th and 8th grade students utilize induction, analogy and imagery in their geometric reasoning; how the problems that were developed to give impetus to the diverse thinking of students are solved using what strategy of the gifted students in actuality; and whether they are solved through the thinking types and paths predicted by the researchers through thought experiment are observed.

INTRODUCTION

According to the studies that observed and analysed the thinking characteristics of mathematically gifted students (Heid, 1983; Presmeg, 1986; Sriraman, 2003, 2004; Lee, 2005), mathematically gifted students efficiently utilize the problem-solving strategies including generalization, simplification, visualization as necessity arise, grasp the meaning and structure of a problem in a very short time and solve it progressively. Sriraman (2003) reported that gifted students invest considerable time in understanding the meaning and condition of a problem and their thinking behaviour including creative problem solving, generalization, formalization, etc. correspond to those of mathematicians. Lee (2005) found that gifted students have the tendency to advance into the higher-level reasoning through the reflective thinking about their early reasoning.

Polya (1954, 1962) emphasized induction and analogy as very important mathematical reasoning faculties. A study on the meaning and development of induction (Holland et al., 1986) and those on the meaning and development of analogical reasoning (English, 1997; Alaxander et al., 1997), show how much important elements induction and analogy are in the development of logical thinking. Studies have been made also on the important role of the mathematical imagery in mathematical learning (Wheatley, 1991; 1997; Presmeg, 1992). Apparently, induction, analogy and imagery seem to be great tools that make one feel the beauty and strength of mathematical thinking. However, studies into which tasks and which teaching methods can stimulate such reasoning and

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those into in what way mathematically gifted students combine or uniquely utilize those reasoning elements are insufficient. The objective of this research is to obtain detailed information on the way mathematically gifted students utilize induction, analogy and imagery in their geometric reasoning. To achieve this objective, this research was conducted focusing on the following two questions:

- How do the mathematically gifted students utilize induction, analogy and imagery in their task solving process?
- What role do induction, analogy and imagery play in making mathematical discoveries?

THEORETICAL FRAMEWORK

Polya (1954, 1962) insisted that exploration into a polyhedron can be made briskly through induction by suggesting the question, "Is it generally true that the number of faces increases when the number of vertices increases?" In this research, too, to induce the students to try generalization through induction, a question of the similar structure will be used while teaching.

As assumed in the experiment of Alaxander et al. (1997), students might fail to utilize analogy in a desirable way as they pay attention to surface-level similarities. The task used in this research needs relational or structural analogy; so it is expected that students might pay attention to surface-level similarities even though the students were identified as gifted in mathematics. If so, we will closely observe in what type of similarity they take notice of and what role induction and imagery play at those times.

Presmeg (1992) argued that image exists in diverse forms including concrete, dynamic, pattern, abstract, etc., playing diverse roles. In this research, also, in anticipation that students will form diverse shapes of image in accordance with individual experience, habit and preceding knowledge, will look into how they are utilized in problem solving, particularly in the discovery of mathematical idea.

The results of the studies on induction, those on analogy and those on imagery (Holland et al., 1986; English, 1997; Wheatley, 1991; 1997) suggested specific process of mathematical thinking a learner might experience. In this research, based on the results of preceding studies, attention will be paid more to the relations between the three thinking elements.

RESEARCH METHOD

Participants

To find out how mathematically gifted students utilize induction, analogy and imagery, this research was performed following the research method that intentionally conducts the sampling of proper cases, observes and makes an in-depth analysis (Strauss & Corbin, 1990). The subjects of this research are three 6^{th} graders (age 12) (E1, E2, E3) in elementary school, three 8^{th} graders (age 14) (M1, M2, M3) – all of them are receiving education for the gifted in an academy for the gifted attached to a university.

Tasks

The tasks were prepared by the researchers either by reviewing, of the existing studies on the fields of geometry, algebra, probability and statistics, those that paid attention to the improvement of mathematical thinking ability and then partially revising them or through new development. Teaching materials with which a polyhedron can be easily made were provided for the subjects to use in the task-solving process. The tasks that will be mainly analysed in this paper fall under the field of geometry and are as follows:

- [Task 1] The sum of all the interior angles of a triangle is 180°. Can you find a similar property in a tetrahedron? Make tetrahedrons with the tools in front of you as needed; observe them; and find the similar statement or property.
- [Task 2] As to a solid that has n-number of faces, can the sum of the internal angles of the polygons that compose each face be generally worked out? Why do you think so?

Both the two tasks were developed for the objective of making students discover mathematical ideas and justify them using induction and analogy. The intention was to make them experience induction and analogy while solving task 1, and apply the experience to solving task 2. So task 1 was provided with a view to have the students learn the thinking pattern required to solve task 2. In the case of imagery, since it is hard to make specific anticipation about it in advance, it will be arranged by classifying the students' responses.

Procedures

Nine units in three education programs for each field of geometry, algebra, probability and statistics were provided to them. Elementary school students and middle school students participated in the research being separated from each other; and for each student one research assistant was assigned to conduct concentrative observation and interview. Each teaching unit program lasted for three hours; all the responses of the students were audio-/video-taped; and background information including their activity record, home background, etc. was also collected.

Considering that the students' mathematical thinking cannot be completely expressed in language, data was analysed taking heed of such non-linguistic responses as facial expression, behaviour, etc. The interaction between the students was allowed only if necessary: the students were to solve most of the tasks in accordance with the individual habit or strategy. Their thinking characteristics were analysed inductively focused on the scene where induction, analogy and imagery are specifically linked to problem solving, particularly on how the three kinds of reasoning are linked to each other.

RESULTS

The mathematically gifted students were identified as very promising in mathematics by the selection process run by the university professors of the gifted centre. However their performance in this study was not so good unlike the researchers' expectation which proposed the changes in the selection process. In the participants' geometrical reasoning, how induction, analogy, and imagery were revealed and connected in some way is explained in the following table:

Task	E1	E2	E3	M1	M2	M3
1	Imagery ↓ Analogy	Induction	Imaging	Imaging	Induction	Imagery ↓ Analogy
2	Imagery ↓ Induction	Imagery ↓ Induction	Imagery ↓ Induction	Analogy ↓ Induction	Induction	Induction ↓ Imagery ↓ Analogy

Table 1: Emerged Reasoning Patterns

In case of trying just induction without utilizing analogy and imagery, appropriate geometrical reasoning was not completed. As a result, the focus of the task was missed (M2). In spite of the fact that the task was asking for induction, a student who succeeded in analogy based on suitable imagery without using induction could resolve the task (E1, M3). Compared with elementary students, middle school students try not to reason based on imagery. When imagery is combined with induction or analogy, the conversion of thinking was realized rapidly. When induction and analogy was combined or at least utilized at the same time, inductive analogy and analogical induction were not found.

Surface-level analogy and blind imaging

Task 1 is making and observing various kinds of tetrahedron and to find out its specific features on the sum of some angles. In spite of the fact that the task was asking to observe various tetrahedrons, some students just made one tetrahedron and didn't make other tetrahedrons any more. Focusing on just one tetrahedron, they changed the imagery about angle and tetrahedron. Based on such imagery, they tried to induce. The following is a part of conversation made between M1 and a researcher.

M1: Here, this angle is made of three dimensions!

Interviewer: (pointing at the angle of the picture drawn by M1) Are you talking about this interior angle?

M1: (pointing at the angle of the tetrahedron she made) Yes, that angle made of three dimensions is what I am talking about.

Interviewer: How do you know the degree of that angle?

M1: That is the problem. I can't find it. But I think someone can.

M1 didn't care about the relation or structure in the given statement about the sum of interior angles of a triangle. She spent most of the time thinking how to measure an interior angle of a tetrahedron that she defined as the counterpart of an interior angle of a triangle. The 'three dimension angle' within the tetrahedron is not approachable by the existing tetrahedron image that the student has. The student just noticed the new image form such as the angle that one edge meets another edge or one face meets another face. The image of angle is in the process of expanding from a plane figure to a solid figure but the student failed to exactly capture the meaning. She couldn't explain why she had to know it. Also how the change of angle is connected to the kinds of tetrahedron was not clear to her.

Surface-level analogy and blind induction

E2 tried to observe and analogize various kinds of tetrahedron including regular tetrahedron according to the guidelines of task 1. However, he couldn't find the common feature by noticing each character that various cases have. Away from the original point, 'the sum of the interior angles' that should be analogized, he showed interest in the changes of number such as an edge, a vertex, etc. So he failed to draw a meaningful conclusion on the sum of some angles, although he continuously tried induction of what he observed. In case of M2, he made various kinds of polyhedron to resolve task 2, but the analysis on such polyhedrons was not done systematically. So he couldn't draw any conclusion.

If induction for various kinds of tetrahedron and polyhedron is not combined with proper imageries, it can't be connected to the proper reasoning, losing the direction. In other words, trying induction without direction strays from the essential.

Relational analogy fuelled by proper imagery

Student E1 also focused on the triangle of each face after making just one regular tetrahedron using the teaching manipulative while resolving task 1. He also didn't show any interests in making various kinds of tetrahedron. However the proper imagery he developed led to analogy. The following is a part of conversation made between E1 and a researcher.

E1: A regular tetrahedron has 4 regular triangles. So the sum of angles times 4 is 720.

Interviewer: What about a general tetrahedron?

E1: A tetrahedron may have a triangle and a quadrangle.

Interviewer: A triangle and a quadrangle?

(After a long pause for imaging the development figure of a tetrahedron)

E1: No, no. A quadrangle can't be fit in. If a quadrangle enters, there are not enough edges. We need more edges to connect this part (draw some figures in the air). If you connect one, it already becomes 5.

The above conversation shows that the student doesn't make other kind of tetrahedron and imagine the scene making any tetrahedron to resolve the task. Especially to explain each face of the optional tetrahedron is a triangle, he suggested a reasonable logic by imaging the solid figure and the development figure of it in his thought including a quadrangle without using induction. Unlike M1, E1 is focusing on each face's polygon rather than an interior angle. This means that the development figure of the tetrahedron is being used as an image.

The crucial meaning included in the first statement of task 1 is that the interior angles of a triangle can be different, but the total of three angles is consistent. By relational or structural analogy, he discovered the fact that the kinds of triangle compose the tetrahedron can be changed, but with only 4 triangles one can compose any tetrahedron. He exactly analogized with not just isolated element about an interior angle itself but the relation or structure included in the statement. Based on such relational or structural analogy, he resolved the task. The development figure and the shape of each face in tetrahedron seem to play a key role in the relational analogy.

Relational analogy fuelled by essential induction and proper imagery

M3 also didn't feel any necessity to check various kinds of tetrahedron when he resolved task 1. Like E1, he analogized the character of the tetrahedron, using the development figure in a tetrahedron. As for task 2, he observed various kinds of polyhedron. He suggested an example that although the number of face is the same, the sum of interior angles in a polyhedron can be different. At this point, he noticed a quadrangular pyramid and a triangular prism with 5 faces. And then he checked whether he could get the general features of each pyramid and prism. He checked that the number of vertexes and faces in the n-pyramid is n+1. Also he checked that the number of vertexes in n-prism is 2n and the number of faces in n-prism is n+2. For each case, he found the formula to get the sum of interior angles of a polygon in each face.

The development figure which was utilized in resolving task 1 played a key role in advancing into the analogy from induction upon various polyhedrons for M3 to resolve task 2. M3 drew the development figure of any polyhedron and focused on each plane figure that composes the development figure. He made efforts to get the formula to total the interior angles of a polygon in each face. He analogized the changes of each face by disassembling special cases including a regular polyhedron, a triangular prism, etc., with the development figure and each plane figure. The following is a part of conversation made between M3 and a researcher.

M3: Let me explain it with a regular tetrahedron. The total edges are 6. We have to get the angle of each face. So if we separate each face, we get 4 triangles. There are 12 edges, so it's two times of the total edges. The total of the interior angles of each face is n-2 times π , so if we put the number of edges as n.

Interviewer: Do you put the number of edges as n? Why do you think about the number of edges?

M3: Imagine that we remove all the faces.

(He tries to draw the process of dismantling faces in the air.)

M3: Then the number of edges doubles because we remove faces from here and there.

M3 found out that if we dismantle an ordinary polyhedron by polygon, the total of the interior angles for each polygon located in each face can be calculated. Also he found out that it can be expressed as one formula by connecting the total number of faces and edges. Analogizing from task 1 and systematically inducting and utilizing imagery properly played the key role in resolving the task 2.

DISCUSSION AND CONCLUSION

The suggestion of Polya (1954, 1962) is that the character of a solid figure can be found by analogizing the character of a plane figure and reach to the generalization by induction. After applying this suggestion to the mathematically gifted students, only 2 among 6 students showed the expected response (E1, M3). Some students spent most of the time finding out the meaning of the interior angle of a tetrahedron (M1), or just observing various cases without a system (E2, M2). Those students couldn't draw out any new character about a solid figure. Without relying on induction, rather than trying the deep analysis on some example or trying the inference based on image reached the successful analogy (E1, M3). Especially changing the imagery dynamically and utilizing it in induction and analogy played a key role in discovering the mathematical idea and its generalization (M3). In the study of Alaxander et al.(1997), many students could utilize analogy by choosing a very similar calculation question in terms of structure. However, this study tried the analogy between very different structures such as a solid figure and a plane figure, which made even the mathematically gifted students get lost in analogy. Thus more experience of tackling this kind of tasks is suggested to be introduced in gifted education.

Imagery seems to provide a very important base for developing structural analogy to resolve the tasks in geometry. Because the structure among objects should be grasped to enable analogy, so some specific imagery among various features of geometric objects plays a key role. Activities with imagery, such as drawing, writing down, or describing verbally the spatial imageries students used when solving tasks (Wheatley, 1991) also needs to be heavily considered in gifted education.

Induction itself seemed to be difficult to be utilized properly to resolve the tasks in this study. E2 and M2 tried to find many cases, but they had difficulties in generalizing the results. They couldn't grasp the relation between the objects that observed. Just drawing out the isolated features, they couldn't systemize them and utilize the image. As a result, they couldn't draw a desirable induction or analogy. Like the preceding studies on the role of image (Wheatley, 1997; Presmeg, 1992), some students in this study utilized or changed existing images to utilize or apply induction or analogy. However many students are needed to develop tendency or ability to use imageries.

The relation between induction and analogy is not clear. We predicted that induction about many cases could be the base of analogy, but in reality, the deep analysis on one case developed a strong tendency to lead to analogy in this research. There was a case that tried to start from analogy and reach induction (M2), but the student failed in resolving the task because his analogy and induction were both incomplete. Induction can be the first tool when solving geometric problems, but it needs other reasoning skills such as analogy or imaging simultaneously. If we analyse the responses of students about algebra, probability and statistics, the relation among induction, analogy and imagery will be clearer.

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