PROBLEM POSING AS A MEANS FOR DEVELOPING MATHEMATICAL KNOWLEDGE OF PROSPECTIVE TEACHERS

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Abstract

In the present study we aim at exploring the development of mathematical knowledge and problem solving skills of prospective teachers as result of their engagement in problem posing activity. Data was collected through the prospective teachers' reflective portfolios and weekly class discussions. Analysis of the data shows that the prospective teachers developed their ability to examine definition and attributes of mathematical objects, connections among mathematical objects, and validity of an argument. However, they tend to focus on common posed problems, being afraid of their inability to prove their findings. This finding suggests that overemphasizing the importance of providing a formal proof prevents the development of inquiry abilities.

Introduction

Problem posing (PP) is recognized as an important component of mathematics teaching and learning (NCTM, 2000). In order that teachers will gain the knowledge and the required confidence for incorporating PP activities in their classes, they have to experience it first. While experiencing PP they will acknowledge its various benefits. Hence such an experience should start by the time these teachers are being qualified towards their profession. Therefore, while working with prospective teachers (PT) we integrate activities of PP into their method courses. In addition, accompanying the process with reflective writing might make the PT be more aware to the processes they are going through (Campbell et al, 1997), and as a result increase the plausibility that the PT will internalize the effect of the processes that are involved in PP activities. This reflective writing also enables teacher educators to evaluate the PT progress and performance (Arter & Spandel, 1991).

In the present study we aim at exploring the effects of experiencing PP on the development of PT's mathematical knowledge and problem solving skills. For that purpose we employed two evaluative tools – portfolio and class discussion.

Theoretical Background

This section includes a brief theoretical background regarding problem posing, with a special focus on the "what if not?" (WIN) strategy, and regarding the educational value of integrating PP into PT's training programs.

Problem posing. Problem posing is an important component of the mathematics curriculum, and is considered to be an essential part of mathematical doing (Brown & Walter, 1993, NCTM, 2000). PP involves generating of new problems and questions aimed at exploring a given situation as well as the reformulation of a problem during the process of solving it (Silver, 1994). Providing students with opportunities to pose their own problems can foster more diverse and flexible thinking, enhance students' problem solving skills, broaden their perception of mathematics and enrich and consolidate basic concepts (Brown & Walter, 1993, English, 1996). In addition, PP might help in reducing the dependency of students on their teachers and textbooks, and give the students the feeling of becoming more engaged in their education. Cunningham (2004) showed that providing students with the opportunity to pose problems enhanced students' reasoning and reflection. When students, rather than the teacher, formulate new problems, it can foster the sense of ownership that students need to take for constructing their own knowledge. This ownership of the problems results in a highly level of engagement and curiosity, as well as enthusiasm towards the process of learning mathematics.

The 'What If Not?' strategy. Brown & Walter (1993) suggested a new approach to problem posing and problem solving in mathematics teaching, using the 'What If Not?' (WIN) strategy. The strategy is based on the idea that modifying the attributes of a given problem could yield new and intriguing problems which eventually may result in some interesting investigations. In this problem posing approach, students are encouraged to go through three levels starting with re-examining a given problem in order to derive closely related new problems. At the first level, students are asked to make a list of the problem's attributes. At the second level they should address the "What If Not?" question and than suggest alternatives to the listed attributes. The third level is posing new questions, inspired by the alternatives. The strategy enables to move away from a rigid teaching format which makes students believe that there is only one 'right way' to refer to a given problem. The usage of this problem posing strategy provides students with the opportunity to discuss a wide range of ideas, and consider the meaning of the problem rather than merely focusing on finding its solution.

The educational value of integrating problem posing into PT's training programs. Teachers have an important role in the implementation of PP into the curriculum (Gonzales, 1996). However, although PP is recognized as an important teaching method, many students are not given the opportunity to experience PP in their study of mathematics (Silver et al., 1996). In most cases teachers tend to emphasize skills, rules and procedures, which become the essence of learning instead of instruments for developing understanding and reasoning (Ernest, 1991). Consequently, mathematics teachers miss the opportunity to help their students develop problem solving skills, as well as help them to build confidence in managing unfamiliar situations. Teachers rarely use PP because they find it difficult to implement in classrooms, and because they themselves do not possess the required skills (Leung & Silver, 1997). Therefore, PT should be taught how to integrate PP in their lessons. Southwell (1998) found that posing problems based on given problems could be a valuable strategy for developing problem solving abilities of mathematics PT. Moreover, incorporating PP activities in their lessons enables them to become better acquainted with their students' mathematical knowledge and understandings.

The study

The study participants. 25 mathematics PT (8 male and 17 female) from an academic college participated in the course. They are in their third year of studying towards a B.A. degree in mathematics education. The students represented all talent levels. The students are graduated towards being teachers of mathematics and computer science or teachers of mathematics and physics in secondary and high school.

The course. The course in which the research was carried out is a two-semester course and is a part of the PT training in mathematics education. This course is the first mathematical method course the PT attend, and takes place in the third year of their studies (out of four). The research was carried out during the first semester. Problem posing and problem solving, based on the WIN strategy, is one of the main issues discussed in this semester. The lessons usually bear a constant format: By the end of the lesson the PT were given an assignment, which was aimed at supporting them progressing in their work (more details about the phases of the work are presented at the next section). During the week the PT had to accomplish the assignment and describe their work in a reflective manner using a written portfolio. A copy of the portfolio was sent to us by e-mail, reporting their progress, indecisions, doubts, thoughts and insights. When we got the impression that the PT were not progressing we suggested them with new view-points, which were consistent with their course of work. These portfolios served as a basis for class discussion at the successive lesson. Part of each class discussion was allocated to presentations of the PT's works.

The problem. The described process was the first time the PT experienced PP. We demonstrated its various phases using Morgan's theorem (Watanabe, Hanson & Nowosielski, 1996). Afterwards, the PT experienced it themselves, starting with the problem described at Figure 1. The reason for choosing this specific problem was its variety of attributes and its many possible directions of inquiry which might end with generalizations.

Triangle *ABC* is inscribed in circle O_1.
D is a point on circle O_1. Perpendiculars are drawn from *D* to *AB* and *AC*. *E* and *F* are the perpendiculars' intersection points with the sides, respectively. Where should *D* be located so that *EF* will be of a maximum length?

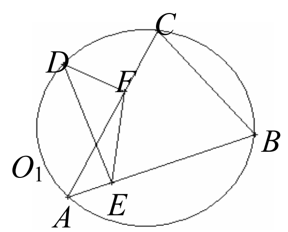

Figure 1

Employing the WIN strategy. The PT were asked to go through the following phases: (1) Solve the given geometrical problem; (2) Produce a list of attributes; (3) Negate each attribute and suggest alternatives; (4) Concentrate on one of the alternatives, formulate a new problem, and solve the new problem; (5) Raise assumptions and verify/refute them; (6) Generalize the findings and draw conclusions; (7) Repeat phases 4-6, up to the point in which the PT decide that the process had been completed.

Data Collection and analysis. In this paper we focus on data concerning the development of mathematical knowledge and inquiry abilities as a result of employing the WIN strategy, based on the above problem. Two main sources of data informed the study: The PT's portfolios, and the class discussions in which the PT presented their works and discussed various issues that were raised while reflecting on their experience. When the data collection phase was completed, we followed the process of analytic induction (Goetz & LeCompte, 1984), reviewing the entire corpus of data to identify themes and patterns and generate initial assertions regarding the effect of the PP on the PT mathematical and didactical knowledge. These research tools enabled us to study the PT's development of mathematical knowledge as well as their inquiry abilities.

RESULTS AND DISCUSSION

In this section we focus on results obtained at phases 2, 3 and 4.

Phases 2 and 3. Figure 2 summarizes the list of attributes (in bold) the PT related to the given problem; the number of students that suggested a certain attribute and in parentheses the percentage of students who suggested it, out of 25 PT. Then appear the suggested alternatives and the number of students that proposed the alternative together with percentages (out of the total students who suggested this alternative). For example, the attribute "**any triangle**" was suggested by 18 PT which is 72% of the 25 study participants. Out of the 18 PT, 9 suggested the alternative "Acute triangle". Namely, (50) designate the fact that 50% of the 18 PT suggested this alternative.

Observation of Figure 2 reveals the following: (a) In case the attribute includes a geometrical shape, most of the suggested alternatives were either a common geometrical shapes or a shape that belongs to the same family as the negated shape. For example, in case of negating the triangle (attribute no. 1), all PT suggested the common shape - quadrangle. Most of them (20) suggested the square. In case of negating attribute no. 5, most PT suggested as alternatives for the circle, shapes which belong same family, namely – square, quadrangle, rectangle and trapezoid. Only minority of them suggested polygons with more than four sides. Moreover, only two of them referred to a three dimensional shape. (b) In case the attribute includes a numerical value, most of the PT suggested as an alternative to this attribute another specific numerical value. For example, 11 PT listed attribute no. 7. All of them suggested as an alternative "Three segments are drawn from the point". These findings are consistent with Lavy & Bershadsky (2003), who found that while PT are posing problems on the basis of spatial geometrical problem, they tend to replace numerical values with other numerical values, and geometrical shapes with shapes that belong to the same family. (c) Part of the alternatives given by the PT includes generalization. Though only minority of the alternatives was formed as generalization, two types of generalization can be observed: generalization of a numerical value of an attribute and generalization of a geometrical shape. As to the former, a specific numerical value was replaced by *n*-value. For example, a generalization of attribute no. 7 is 7.2, and of attribute no. 8 is 8.2. As to the latter, the generalization of a geometrical shape can be divided into two sub-categories: generalization of the number of the shape's sides (e.g. attribute no. 1 is generalized by "*n*-sided polygon"), and generalization of shape's dimensions – a shift from a planar into a spatial shape. In attribute 5, for example, the planar shape "circle" was replaced by "sphere" up to "any spatial body".

1. **Triangle** 25(100) *alternatives*: (1.1) Quadrangle 25 (100); (1.2) Square 20 (80); (1.3) Pentagon 14 (56); (1.4) *n*-sided polygon 4 (16). 2. **Placing a point on the circle perimeter** 25(100) *alternatives*: (2.1) Placing a point inside the circle 25 (100); (2.2) Placing a point outside the circle 25 (100). 3. **Two heights are drawn from the point to the sides of the triangle** 25(100); *alternatives*: (3.1) Two angle bisectors are drawn 25 (100); (3.2) Two medians are drawn 25 (100); (3.3) Two perpendicular bisectors are drawn 10 (40); (3.4) One height and one median are drawn 3 (12). 4. **Looking for a location to point D in order for EF to be a maximum** 25(100);) *alternatives*: (4.1)Looking for a location to point D in order for EF to be a minimum 25 (100); (4.2) Looking for a location to point E in order for EF to be a maximum 17 (68); (4.3) Looking for a location to point E in order for EF to be a minimum 17 (68); (4.4) Looking for a location to point F in order for EF to be a maximum 17 (68); (4.5)Looking for a location to point F in order for EF to be a minimum 17 (68); (4.6) Looking for a location to point D in order for ratio between the area of ABC and DEF to be maximal/minimal 5 (20); (4.7) Looking for a location to point D in order for ratio between the perimeter of ABC and DEF to be maximal/minimal 5 (20); (4.8) Looking for a location to point D in order for ABC and DEF to similar triangles 4 (16); (4.9) Looking for a location to point D in order for DEF to be isosceles/equilateral/right/ acute/obtuse triangle 2 (8). **5. Triangle inscribed in a circle** 25(100) *alternatives*: (5.1)Triangle inscribed in a square25 (100); (5.2) Square inscribed in a circle25 (100); (5.3) Rectangle inscribed in a circle22 (88); (5.4) Triangle inscribed in a quadrangle21 (84); (5.5) Triangle inscribed in a rectangle 16 (64); (5.6)Triangle inscribed in a trapezoid 14 (56); (5.7) Pentagon inscribed in a circle12 (48); (5.8) Triangle inscribed in a pentagon 9 (36); (5.9) Triangle inscribed in a polygon 4 (16); (5.10) Triangle inscribed in a sphere 2(8); (5.11) Triangle inscribed in a cube 2(8); (5.12) Triangle inscribed in a pyramid 2(8); (5.13) Triangle inscribed in any spatial body 2(8**). 6. Any triangle** 18(72) *alternatives*: (6.1) Isosceles triangle 18 (100); (6.2) Equilateral triangle 18 (100); (6.3) Right triangle 18 (100); (6.4) Acute triangle 9 (50); (6.5) Obtuse triangle 9 (50).**7. Two segments are drawn from the point** 11 (44) *alternatives*: (7.1) Three segments are drawn from the point 11 (100); (7.2) *n* segments are drawn from the point 2 (18.18). **8. The segments that are drawn from the point are perpendicular to the sides of the triangle** 11 (44) *alternatives*: (8.1) The segments bisect the sides 11 (100); (8.2) The segments divide each side into *n* equal parts 2 (18.18). **9. Inscribed triangle** 7 (28) *alternatives*: (9.1) Circumscribed triangle 7 (100). **10. Circumcircle** 7 (28) *alternatives*: (10.1) Quadrangle inscribed in a triangle 7 (100); (10.2) Triangle inscribed in a triangle 3 (42.85). **11. Polygon** 6 (24) *alternatives*: (11.1) Circle 6 (100); (11.2) Parabola 1 (16.6)

Figure 2: Attributes and alternatives suggested by the PT

Although Phases 2 and 3 appear to require merely a technical work, in order to perform it well, the PT had to demonstrate mathematical knowledge concerning the formal definitions and characteristics of the negated attributes. For example, only six PT related to the attribute "polygon". This fact implies that most of the PT did not consider the formal definition of triangle, namely, "a 3-sided polygon". Referring to this issue during the class discussion, we realized that the PT related primarily to the visual aspects of triangle and not to its definition. This finding is consistent with Tall & Vinner (1981) regarding concept image and concept definition, and with the prototype phenomenon described by Hershkowitz (1989). From the PT's portfolios we realized that this phase enabled them to rethink geometrical objects, their definition and attributes. As can be seen from some of the PT's reflections: Anna (end of Phase 2): "*Analyzing the attributes helped me realize that there is much more in a problem then merely givens. Discussing each data component and its definition enables to rethink of definitions of mathematical objects and some interconnections between them".* Roy (end of Phase 2): "*The class discussion made me realize that there are so many attributes in one problem. Indeed I listed most of the attributes, but it is those which I didn't list that made me appreciate the richness that one can find in any mathematical problem*".

Phases 4. In this phase the PT had to concentrate on one of the alternatives, formulate a new problem, and solve it. Examination of the PT's portfolios reveals that 16 PT (64%) chose to focus on alternatives to attribute no. 5 (among them 14 chose "square inscribed in a circle"), 4 PT (16%) chose to focus on alternatives to attribute no. 3 (2 PT chose "two medians are drawn" and 2 chose "one height and one median are drawn"), 3 PT (12%) chose to focus on alternatives to attribute no. 7 (2 PT chose "Three segments are drawn from the point" and 1 on "*n* segments are drawn from the point"), 2 PT (8%) chose to focus on alternatives to attribute no. 4 (both replaced "maximum" by "minimum"). This finding implies that most of the PT chose as alternatives **common geometrical objects,** as square, median and height. Wondering why the PT demonstrated such a 'conservatives' behavior, we found the answer in their portfolios: Noa (beginning of phase 4): "*At the beginning Shir and I thought of choosing the alternative of triangle inscribed in a sphere [5.10], and later perhaps even a tetrahedron inscribed in a sphere. But then we realized that perhaps you expect us to prove what we are about to discover and whatever we will formulate as a conjecture. We were afraid that we will discover something that we will not be able to prove, and then all our efforts would be worthless*". From this excerpt it can be seen that teachers tend to overemphasize the importance of proof. As a result, the students feel insecure, and they do not dare to relate to mathematical inquiry as an 'adventurous' process. Rather, they tend to approach it in a hesitancy manner, believing that proof is the most important aspect of the mathematical doing. This attitude prevents the development of inquiry abilities.

Additional aspects on which the PT reflected concerned the validity of an argument and the meaning of definition. Gil (during Phase 4): "*At the beginning of the inquiry process we felt like we must first rethink of the mathematical meaning of the concepts* *involved. We asked ourselves whether the new problem situation is mathematically valid and everything is well defined. The truth is that we had never done this before. We were used to solve problems given by our teachers or the textbooks. In these cases there was no need to check the validity of any argument or to probe into the definitions of the objects involved. It was certain that everything is valid, well defined and solvable".*

To sum, reflective portfolios and class discussions turned out to be a useful tool for reflecting on processes, and tracing the PT's development of mathematical knowledge. We found that involvement in PP has the potential to develop the mathematical knowledge, and consolidate basic concepts, as suggested by Brown & Walter (1993) and English (1996). This development of knowledge came to fruition especially in the ability to examine definitions and attributes of mathematical objects, connections among mathematical objects, the validity of an argument, and appreciation of the richness that underlines mathematical problems. However, one major weak point was discovered. The PT tended to attach to familiar objects, and were not 'daring'. This tendency actually prohibits the development of problem solving skills and inquiry abilities instead of developing it. We found that this tendency can be explained by the redundant emphasize of the importance of providing formal proof. Teacher educators need to identify ways for reducing PT's fears from handling formal proves, and remove the focus to the analysis of a given situation, connections among mathematical objects and looking for generalization. This, in turn will develop their problem solving skills and their insights as regards to mathematical objects.

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