WHAT IS A BEAUTIFUL PROBLEM? AN UNDERGRADUATE STUDENTS' PERSPECTIVE

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In this paper, we present an approach to exploring students' aesthetical preferences in mathematics. Based on analysis of 9 undergraduate students' responses and behaviors in two problem solving workshops, we report essential elements of a preliminary student-centred model of the notion "beautiful mathematical problem." The preliminary model includes cognitive, metacognitive and social factors and seeks to appreciate the complexity of the students' aesthetical judgements.

THEORETICAL BACKGROUND

This paper is part of a series of reports on results of a large-scale project, in progress, whose purpose is to investigate high school students, undergraduate students and mathematics teachers' beliefs and actions through the lens of mathematical aesthetics and to check the possibility of incorporating an aesthetic dimension in learning and teaching mathematics. The goal of this report is two-fold. First, we describe a research approach that seems to be useful in revealing complicated mechanisms involved in undergraduate students' aesthetical judgement of a mathematical problem. Second, we suggest and illustrate elements of a preliminary model of the notion "beautiful problem" that seems to reasonably explain the (ostensibly) controversial evidence collected at the pilot stage of the project as well as some intriguing results of past research.

The study is oriented within a theoretical framework that gradually emerges during the last 20 years from research on the role of aesthetics in doing, learning and teaching mathematics (e.g., Dreyfus & Eisenberg, 1986; 1996; Silver & Metzger, 1989; Sinclair, 2004; Koichu & Berman, 2005; Sinclair & Crespo, 2006) and seeks to contribute to that framework.

Twenty years ago, Dreyfus & Eisenberg (1986) pioneered the student-centric approach to exploration of aesthetics of mathematical thoughts. They suggested that the aesthetical considerations involve a personal internalized metric upon a solution to a particular problem. The factors contributing to the aesthetic appeal of a solution are clarity, simplicity, brevity, conciseness, structure, power, cleverness and surprise. This view fits that of many recognized mathematicians, including Hadamard (1945), Poincaré (1946) and Hardy (1956). In time, it figured out that what is viewed by the experts as "beauty of mathematics" does not look the same for high school, undergraduate and even graduate students. Dreyfus and Eisenberg explored university level students' aesthetical preferences using a set of fairly difficult mathematical problems. Each problem had more than one solution, including a solution that had been

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previously evaluated by the experts as "slickly" or "elegant" one. The problems were given in different formats to pre-service teachers of mathematics in Israel and graduate students in the USA. Not surprisingly, the students rarely came up with the elegant solutions. More surprisingly, when the elegant solution was presented to them and understood, most of the students "did not find the elegant [solution] any more attractive than the ones they had come with on their own—they failed to grasp its aesthetic superiority" (Dreyfus & Eisenberg, 1986, p. 7) even after probing. Dreyfus and Eisenberg concluded that the students showed no inner sense of feeling for the elegancy of a solution. This conclusion is in line with Krutetskii's (1976) point that the tendency to appreciate the elegancy of a mathematical problem is an attribute of only exceptional mathematical giftedness. It is also in line with even more radical opinion of Von Glasersfeld who noted that we cannot expect students to show an appreciation for the beauty of mathematics (cited in Dreyfus & Eisenberg, 1986). In what follows, we suggest reconsidering a typical student's "aesthetical blindness" on the ground of the evidence collected in the new experimental format and by calling into play more comprehensive explanations of the observed behaviors. At this point, we only point the reader's attention to the fact that past research shows that the elite world of professional mathematicians and exceptionally gifted learners, on one side, and the rest of the world, on the other, seem antagonistic with respect to appreciating the beauty of mathematics.

The situation within the world of professional mathematicians and the exceptionally gifted is also far from being simple or fully understood (see, for example, the responses to the Dreyfus and Eisenberg's paper in the same 1986 issue of "For the learning of mathematics"). Two points from the research focusing on that elite population are particularly relevant to this paper. First, Silver & Metzger (1989) found that affect and aesthetics appear to serve as a basis for linking metacognitive processes, such as planning and monitoring in mathematicians' problem solving (see also Goldin, 2002, and Sinclair, 2004, for analysis of a *generative* role of beliefs). Second, Koichu & Berman (2005) found that both social and cognitive factors interplay in effectiveness-elegancy conflict encountered by exceptionally gifted students when solving olympiad-style mathematics problems.

On one hand, these findings point to complexity of the ways by which expert problem solvers call into play aesthetic considerations. On the other hand, they give a clue that metacognitive and social factors should be taken into account when exploring aesthetical judgements of different categories of problem solvers. This idea works well in research on problem solving strategies (e.g., Schoenfeld, 1987). Therefore, it can work also in research on aesthetical aspects of problem solving. The latter implication has driven our study and, specifically, our thinking of the following question: Which factors affect the undergraduate students' conception of the notion "beautiful mathematical problem"?

THE METHOD

The research setting and participants

The data presented in the paper were collected during two consecutive workshops in the context of the undergraduate course “Selected problems in mathematics.” The notion “beautiful problem” had not been deliberately discussed in the course till the lessons described below. The participants were 9 third-year undergraduate students from the department of education, the department of mathematics and several engineering departments at the Technion. All the participants had solid background in formal mathematics. The authors of the paper designed the workshops, and the first author served as an instructor. The workshops were videotaped with the camera trained at the participants; all written work of the students was collected.

The first workshop

The first workshop was designed to explore which characteristics of the given problems appear in the students’ aesthetical judgements. At the beginning of the workshop, the students were given 3 problems, each one consisting of three parts:

Pr. 1. In the letters shop, one can buy letters. The cost of the letters needed to write the word ONE is \$6. The cost of TWO is \$9 and the cost of ELEVEN is \$15.

a) What is the cost of the word TWELVE?
b) What is the cost of THIRTEEN?
c) What is the cost of TEN?

Pr. 2. A series of numbers is formed in the following way: The first number is 1, and then every number is obtained from the previous one according to the rule described below. Danny computed the first 2007 numbers in each of the three cases. How many (in each case) are divisible by 5?

a) A number is obtained from the preceding number by adding 2.
b) A number is obtained by multiplying the preceding one by 2 and adding 1.
c) A number is obtained by multiplying the preceding one by 2.

Pr. 3. A pedestrian and a bicyclist left Haifa and Atlit at 7:00 moving towards each other along the beach. The pedestrian walked from Haifa to Atlit while the bicyclist rode from Atlit to Haifa. Both of them moved in a constant velocity. When did the pedestrian reach Atlit in each of the following three cases?

a) The rider's velocity is 3 times the walker’s one and they passed each other at 8:15.
b) At 8:00 the walker was in the middle between the rider and Haifa and the two passed each other at 8:15.
c) The rider reached Haifa at 8:40.

All the three problems are formulated in different styles to which we refer as “unconventional story”, “no story” and “conventional story”, respectively. Besides, they were carefully designed by the authors to meet several conditions: each problem contains items having more than one solution, one of which is shorter and more “slickly” than others; the items in each problem look similarly, but the solutions and ways of finding them are different with respect to their difficulty and heuristic arsenals involved. To illustrate these conditions, consider Pr. 1 in some detail. The "unconventional story" can be represented as a system of 3 linear equations with 7 variables. The "slickly" part follows. In item (a), there is no need for solving the system fully – it is enough to find the cost of the sum T+W+2E+L+V, which can be done very quickly; the only answer is "$18". The answer to item (b) depends on the letters I and R, which are not mentioned among the givens. Thus, the answer "the cost is not determined by the givens" can be deduced with no technical effort. In item (c), the letters T, E and N appear among the givens, but the cost of their sum T+E+N cannot be found straightforwardly. The answer to (c) is the same as to (b), but to obtain it one should use apparatus learned in the first-year linear algebra course.

The students were given about 40 minutes to individually approach/solve the problems. Afterwards, the students filled in the questionnaire, in which they were asked to individually evaluate *difficulty*, *challenge* and *beauty* of the problems using 1 to 10 scales. The three features were evaluated separately for each item, thus, each student indicated $9 \times 3 = 27$ numerical responses. The students were also encouraged to briefly explain the responses. When the questionnaires were completed, each student explained orally his or her opinion about the problems to the classmates and the instructor, and then the whole class discussion emerged. It was focused on the relationship among *difficulty*, *challenge* and *beauty* of the given problems. Finally, the students were asked to come back to their questionnaires and indicate whether or not they reconsider their previous responses.

The second workshop

The second workshop was designed to explore to which extent knowing the expert-provided "elegant" solution to a problem affects the students' aesthetical judgement. Three problems from Pólya and Kilpatrick's (1974) "*The Stanford mathematics problem book with hints and solutions*" were used for this purpose (Pr. 58-1, 58-2 and 58-3, p. 17). There is no space to present the problems here. We only note that: the problems included an "unconventional story", "no story" and "conventional story" in the meaning explained above; each problem had several solutions, including the "slickly" one; the problems were more difficult than those of the first workshop.

The students were asked to read the problems, and, based on the first impression only, evaluate each problem's *beauty* using 1 to 10 scales. They were also encouraged to explain their numerical responses. Then the students were given the written solutions to the problems from the Pólya and Kilpatrick's book. When the solutions were fully understood (this was evident from the brief discussions of the solutions), the students were asked to consider whether or not they want to reconsider their initial evaluation of the *beauty* of the problems. This was followed by a whole group discussion, in which the students' expressed their beliefs about what a beautiful problem is.

ANALYSIS

The data analysed consist of the students' written responses to the questionnaires, transcripts of the videotaped workshops and notes the students made. Because of the small number of the participants, we treated the data chiefly as a set of individual cases, in which we looked for patterns particularly interesting with respect to the research question. Following Pierce, Clement (2000) refers to such a method of analysis as *abduction* – a process of producing a model that, if it were true, would account for the observed phenomena. Thus, the concern about viability rather than validity of the findings is relevant in our research.

In addition, correlation analysis was conducted to explore the relationships among the students' numerical evaluation of *difficulty*, *challenge*, and *beauty* of the problems given in the first workshop. To avoid overestimation of small fluctuations in the students' responses, we converted the responses from 1 to 10 into 1 to 3 scales. Namely, numerical responses 1-3 to the questions "To which extent the problem is difficult/challenging/beautiful?" were interpreted as "the problem is easy/not challenging/not beautiful" and re-denoted "1". Responses 4-7 were interpreted as "the problem is fairly difficult/challenging/beautiful" and re-denoted "2". Responses 8-10 were interpreted as "the problem is very difficult/challenging/beautiful" and re-denoted "3". The quantitative results below concern the converted responses.

RESULTS

The students expressed controversial opinions about *beauty* of the problems given at the first workshop. Correlations between *difficulty* and *beauty* as well as *challenge* and *beauty* were close to 0 for all the problems. Styles of the problems' formulation did not show themself as a relevant factor either. The students alluded rather to *novelty* and *unexpectedness* as associates of *beauty*. Consider, for instance, the students' responses and statements concerning Pr. 1. One student indicated that Pr. 1(a) is "very beautiful", 3 – that it is "fairly beautiful", and 5 – that it is "not beautiful;" nobody mentioned the "unconventional story" as a factor affecting the judgement. Three students found a "slickly" solution to Problem 1a (see the previous section), and the rest solved it by more than one page long manipulations of the initial system of equations. The student that evaluated Pr. 1(a) as "very beautiful" solved it in the "slickly" way and explained: "I like it as I've not met such problems before." Two others, who found the same short solution to the problem, did not find it beautiful for two reasons. First, "It was clear what to do," and, second, "The problem is a technical one anyway". These arguments were shared by most of the students. For example, one of the students, who solved the problem in a long way, reflected on his solution as follows: "It is not difficult. You just try different combinations, sums and differences [of the equations]. You are not looking for a new idea…"

A remarkable discussion emerged from the students' reflection on Pr. 1(b) and 1(c).

> Alex: You can see immediately that there is no specific solution to 1(b), so it is not too beautiful. But when you solve 1(c), you just work and work, and think that you

have a way, the same one as in 1(a), and finally you haven't, and must think why… This is nicer than in a problem that can be solved in a regular way.

Instructor: You gave us an excellent explanation about the difference between a beautiful problem and a difficult one. But, perhaps, the "beauty" [of a problem] equals to [its] "challenge"?

Alex: Perhaps…

Baruch: No, not equals. If a problem is beautiful, it is also challenging, but if a problem is challenging, it is not necessarily beautiful.

Eli: I disagree, not every beautiful problem is challenging. There are some geometry problems…very beautiful and not challenging… Or number theory problems…They can be very challenging, but are not really beautiful. I think there is no connection…

Gila: Yes, beauty is not a challenge and not a difficulty; it is more than that…

Interestingly, all the students showed keen interest in the whole group discussion, but nobody changed his or her aesthetical judgement of Pr. 1(a) and 1(b) by the end of the discussion; only two students changed their opinions regarding 1(c) from "not beautiful" and "fairly beautiful" to "very beautiful".

The additional phenomena deduced from the analysis of the first workshop include:

- Six students did not change any of their opinions about the beauty of any problem during the workshop.

- Pr. 2(b) was considered the most beautiful one (mean=2.88, SD=0.35), but when a simple solution was presented, 3 students changed their opinion from "very beautiful" to "fairly beautiful" and "not beautiful" (mean=2.22, SD=0.83)

- The students' opinions about beauty of Pr. 2 and Pr. 3 varied, but 7 of them gave the same rates to items (a), (b) and (c) of these problems, even when knowing that the solutions are very different. It seemed that the mathematical affiliations of the problem (e.g., a series problem and a word problem) affected the students' aesthetical judgments more than inter-item differences.

Additional phenomena were observed in the second workshop. Namely, the students were able to evaluate to which extent the given problems are beautiful based on their first impression. All the students, but two, did not change their opinions after understanding the expert-provided solutions. Pre-post explanations of those who did not change their opinions include:

Hava: [Pre:] It looks like a tricky question. [Post:] I was right, it is a tricky question.

Tamar: [Pre:] I've solved a similar problem in the past, that's why the level of beauty is not high. [Post:] The way of the solution is as I expected.

Baruch: [Pre:] It looks like a challenging problem. [Post:] I did not appreciate the solution as I could discover it myself.

Two students, who essentially changed their opinions, explained:

Uri: [Pre:] It is not nice and unsolvable. [Post:] When I looked at the solution, I realized that there was a nice solution.

Eli: [Pre:] It is an interesting problem, it says something general about triangles. [Post:] It is even more beautiful than I thought as it is very general, but relies on simple and basic geometry facts.

In the follow-up discussion, the students elaborated their written responses. A brief summary of the discussion is this: A student's aesthetical judgment of a problem is based mostly on the first impression and cannot be easily changed. The changes, if any, are based on acknowledgement that an idea of an expert-provided solution could be discovered by the student independently, but had not come to his or her mind when reading the problem.

DISCUSSION AND CONCLUDING REMARKS

The research question under exploration was: Which factors affect the undergraduate students' conception of the notion "beautiful mathematical problem"? To address this question, we designed a research setting, in which undergraduate students' aesthetical preferences in mathematical problem solving could be evoked. We hope that the setting can be used in the growing body of research on mathematical aesthetics.

In response to the research question, we suggest: (1) a student's perception of a notion "beautiful problem" is deeply individual and involves more sophisticated considerations than difficulty, challenge or a style of a problem's formulation; (2) from a student's perspective, a problem can be *beautiful* if it is characterized by the following traits: it has a mathematical affiliation associated with a high level of aesthetic value (e.g., geometry for one student and number theory for another); it looks new; its solution is accessible, but includes elements of surprise, for instance, it is easier or based on more elementary mathematical tools than it was expected by the student when reading the problem.

These suggestions are in line with those by Koichu & Berman (2005), who utilized the *principle of parsimony* to explain the conflict between the mathematically gifted students' conceptions of elegancy and effectiveness in problem solving. They are also in a good agreement with Silver & Metzger's (1989) point about the role of aesthetics in metacognitive processes, such as planning and monitoring, of mathematicians. Thus, the presented findings may imply that the gap, with respect to mathematical aesthetics, between mathematicians and the gifted, on one side, and university level students, on the other, is smaller than it seems.

The presented findings may also lead to reconsideration of the "aesthetical blindness" of university level students indicated in past research. On one hand, in our research there were participants who refused to acknowledge "the aesthetic superiority" of the "slickly" solutions suggested by the experts (see discussion of Dreyfus & Eisenberg, 1986, in *Theoretical Background* section). On the other hand, the findings points to the importance of socially-based factors like self-esteem of the students as problem solvers, which has not been taken into account in past research. We suggest that the latter factor could in part explain why, in the Dreyfus & Eisenberg's (1986) study, the students' aesthetical judgments were not apparent.

In closing, let us note that we are fully aware of the limitations of the implemented method and the preliminary character of the findings. We hope that in the near future the viability of the presented elements of the student-centered model of the notion "beautiful mathematical problem" will be tested by additional observations.

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