TEACHER ACTIONS THAT ASSIST YOUNG STUDENTS WRITE GENERALIZATIONS IN WORDS AND IN SYMBOLS

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Past research has indicated that there is evidence of a relationship between adolescent students' ability to express generalisations as written descriptions and in symbolic forms. This research investigates whether this relationship occurs with younger students and identifies teacher actions that begin to assist the development of this relationship. A teaching experiment was conducted with twenty seven students with an average age of 10 years. From the results it appears that after a short intervention period, one fifth of these students could express generalizations in both written and symbolic form. It also indicated that particular teacher actions assisted students to identify the generalization and to express it both as a rule and as an equation.

Generalising as an algebraic activity has long been widely accepted as an important approach to introducing algebraic concepts to young adolescents (12-13 year olds) (e.g., Mason, Pimms, Graham, & Gower, 1985). Such approaches build on the explorations of visual growing patterns, using the patterns to generate algebraic expressions. One purpose for setting pattern tasks within pictorial and practical contexts is to endeavour to provide an alternative format to lists of numbers (Orton, Orton & Roper, 1999). A patterning approach to algebra allows students to experience unknowns as variables, and provides students with opportunities to observe, verbalise and symbolize generalizations (English & Warren, 1998). For the purpose of this research we define growing patterns as having a discernable unit that grows by a constant amount and thus the position (step) in the pattern depends upon the previous position and the numerical value of its position.

Young students are capable of articulating the generality of the growing pattern in terms of its position in the pattern (Warren, 2005) and teacher actions that assisted the development of this capability were conjectured to be (a) the use of concrete materials, (b) patterns where the relationship between the pattern and position were explicit, and (c) explicit questioning that linked the position to the pattern. This paper builds on the research reported in Warren (2005) by examining teacher actions that assist elementary students to begin to express their written description in symbols, especially in abstract symbol systems including the use of notation systems for the unknown.

Approaches used to find the general rule, that is, defining the growing pattern in relation to its position in the pattern appear to fall into three broad categories (Redden, 1996). These are, (a) using one example to predict the relationship between uncountable examples (e.g., if the 5th step has 11 tiles then the 10th step would have 22 tiles), (b) the additive strategy where connections among consecutive elements (e.g., for each step you add 2 tiles), and (c) functional strategy where a relationship is formed between the two data sets (e.g., the number of tiles is the step number multiplied by 2 add 1). These strategies tend to be hierarchical (Redden, 1996; Stacey, 1989; Warren, 1996). Once students perceive a pattern in a certain way, it is difficult for them to abandon their initial perception (English & Warren, 1998; Lee, 1996).

While past research has reported a tension between natural language and the impossibility of using it for the construction of symbolic representations, little research has occurred focusing on teacher actions that assist in making these links. For example, the results of Redden's (1996) longitudinal study with 26 students (aged 13) indicated that on the whole, it appeared that using natural language to describe the generality of number patterns is a necessary prerequisite for the emergence of algebraic notation. Stacey and Macgregor (1995) conjectured that correct verbal descriptions are more likely to lead to correct algebraic rules and students who could find the correct functional relationship could usually articulate this relationship verbally. Both studies simply administered tests to large groups of adolescent students and analyzed the results to identify relationships. They did not attempt to ascertain what particular actions assist in establishing these important links. The research reported in this paper takes this discussion to the next level, determining whether elementary students can effectively engage in these conversations and identifying teacher actions that assist in establishing the links between verbal and symbolic descriptions of generalizations. The specific aims of the study were to (a) identify the relationship between writing a generalization in words and in symbols, and (b) determine particular teacher actions that assist in forging this relationship.

METHOD

Four lessons were conducted in one Year 5 classroom from a middle socio-economic elementary schools from an inner suburb of a major city. The sample comprised 27 students (average age of 10 years), the classroom teacher and 2 researchers. The actions from the four lessons reported in this paper were those conducted by one of the researchers (teacher/researcher). The lessons were of approximately one hour's duration and exploratory in nature. The tasks chosen were context free. This decision was based on the concern that while context related tasks, such as the different arrangements of tables and chairs, may be seen as purposeful, there is a danger that the context itself may inhibit negotiation of the boundary between the mathematical and the real (Bills, Ainley & Wilson, 2003). The lessons consisted of five main dimensions (a) using concrete materials to represent various growing patterns (b) translating the pattern into a table to further draw out the functional relationship inherent in the pattern (c) developing specific language to support the development of the concept of the relationship between pattern position and number of tiles (d) sharing different ways of describing the generalizations in everyday language and symbols, and (e) encouraging students to justify their generalizations. The patterns chosen were linear and the representations were arranged so that the links between the visual, position and pattern rule were explicit. The generalization for each lesson were: Lesson 1 *2n+1*, Lesson 2 *2n+2* and *2n+1,* Lesson 3 *2n+1* and *3n+1*, and Lesson 4 *3n+1, 2n-1 and 2n+2.*

Data gathering techniques and procedures.

The methodology adopted for the Teaching Experiments was the conjecture driven approach of Confrey and Lachance (2000). The conjecture consists of two dimensions, mathematical content and pedagogy linked to the content. The design aimed to produce both theoretical analyses and instructional innovations (Cobb, Yackel, & McClain 2000). During and in between each lesson hypotheses were conceived ‘on the fly’ and were responsive to the teacher-researcher and the students. During the teaching phases, the researcher and classroom teacher acted as participant observers, recording field notes of significant events. All lessons were videotaped using two video cameras, one on the teacher and one on the students, focusing on the students that actively participated in the discussion The video-tapes were transcribed and worksheets collected.

At the beginning of the first lesson students were asked to complete a simple activity involving constructing growing patterns with tiles, continuing this pattern, giving a verbal description of the pattern and expressing this generalization in algebraic notation. Following reflection with the other researcher and teacher, field notes and the evidence of students’ worksheets it appeared that these students had had little experience with growing patterns and exhibited difficulties in describing these patterns in everyday language let alone using symbolic notation systems.

At the completion of the teaching phase a test comprising of three questions was administered. The questions reflected the types of activities that occurred within the lessons. Figure 1 summarises the three patterns that formed the basis for the Questions.

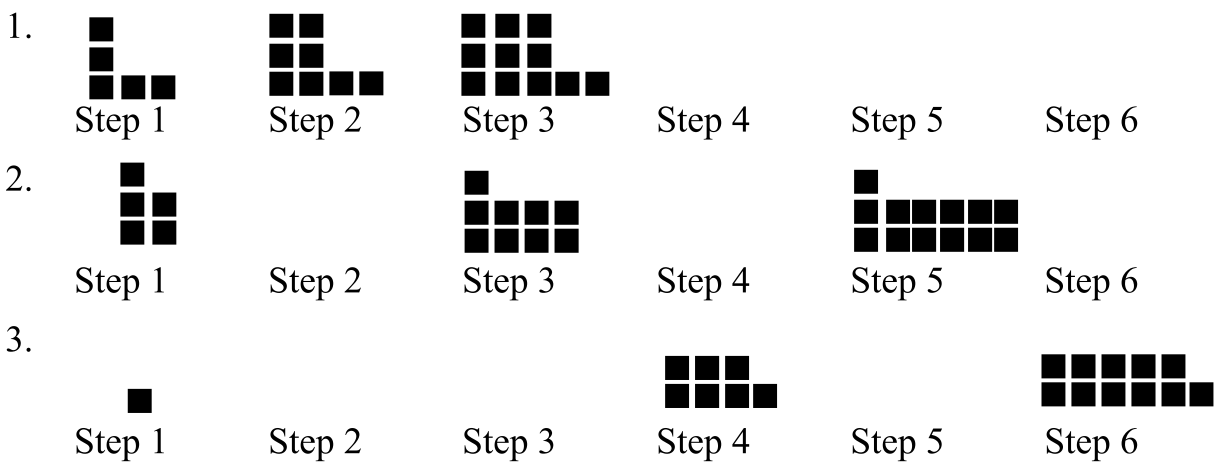

Figure 1 The patterns presented in the post test.

In each instance students were instructed to complete the pattern, write the position rule for the pattern and write the rule using symbols. The questions mirrored the types of activities that occurred during the teaching phase.

RESULTS

The responses to Question 1, 2 and 3 for the sections relating to writing the general rule and expressing the rule in symbols were categorized. The responses to each component fell into 5 broad categories ranging from descriptions that gave no indication of the relationship between the pattern and its position to responses that specifically related the pattern to its position. The next section describes each category for the written responses with a typical response for each.

Category 1. No response.

Category 2. Nonsense response. (*The patterns keep on growing and growing)*

Category 3. Quantifying the growth rule. *(Goes up by two. One more on each end)*

Category 4. Relationship between position and pattern.(*Each step number x2=number of tiles* or *It's double the number of the step)*

Category 5. Relating between pattern and position with visual. (*The top and bottom row of the stars is the same number as the step)*

Both descriptions for Category 4 and 5 were considered to be correct.

Table 1 summarises the frequency of responses for each level for Q1, Q2 and Q3.

Table 1 Frequency of responses to: Write the position rule for this pattern

	Category	Q 1.	Q 2.	Q 3.
1	No response	1	3	0
2.	Nonsense response (it grows or doesn't make sense)	5	7	7
3.	Quantify the growing rule (e.g., grows by 3)	11	11	13
4.	Relationship between position and pattern	6	5	5
5.	Relationship between position and pattern with visual	4	1	2

Thirty seven percent of the sample successfully wrote the position rule for the pattern for at least one of the questions (Question 1 – 37%, Question 2 – 22%, Question 3 – 26%).

Categories for symbolic descriptions

Category 1. No response.

Category 2. Nonsense response.

Category 3. Quantifying the growing rule in symbols. *(+3)*

Category 4 Quantified specific example. *(2x3+2: 3x3+2)*

Category 5 Correct symbolic relationship using unknowns. *(3 x* [?] *+ 2)*

Table 2. Frequency of responses to: Write your rule in symbols for questions 1, 2 and 3

	Category	Q 1.	Q 2.	Q 3.
1	No response	2	5	5
2.	Nonsense response	6	6	6
3.	Quantify the growing rule in symbols (e.g., +3)	13	12	11
4.	Quantified specific example	1	0	2
5.	Correct symbolic relationship with unknowns	5	4	3

Nineteen percent of the sample could successfully write their rule in a symbolic form (Question 1 – 19%, Question 2 – 15%, Question 3 – 11%). In order to ascertain the relationship between students' verbal description and symbolic descriptions of the pattern presented in Questions 1, 2 and 3, a Wilcoxon Signed Ranked test was performed.

Table 3 Results of the Wilcoxon Signed Rank test

	Question 1	Question 2	Question 3
Z score	-.996	-.436	-1.706
Significance	.319	.663	.088

As evidenced by the results of the Wilcoxon Signed Ranked test, there was no significant difference between the level of response each student proffered for the written description and the symbolic description of the generalization for each of the three Questions. The next section briefly summarises the particular teacher actions that are believed to assist students to (a) give verbal descriptions of the generalizations and (b) use these descriptions to support the development of the correct notation system.

Teaching actions

Using language and actions to relate the visual representation to the table of values.

Past research has shown that there is a propensity for students to use (a) one example to predict the relationship between uncountable examples, and (b) the additive strategy by connecting consecutive elements. Thus specific teaching strategies and language were developed to not only encourage students to examine functional relationship in the pattern but also to transfer this understanding to the table of values. As students proffered their generalizations each was classified as a Growing rule or a Position rule.

This classification system was also used to explicitly map the two ways of examining the visual pattern (i.e., looking along the visual pattern and linking the pattern to its position) and the two ways of examining the table (i.e., looking down one column and looking across the two columns). The following extract exemplifies the types of classroom conversations that ensued during the four lessons:

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TR:	What is a position pattern?
Cath:	A position pattern is where the step number has something to do with the pattern.
TR:	Yes, the step number and the pattern. So you are looking across the table that is where the pattern is.
TR:	I want you to think how you go from step 2 to get 6, from step 3 to get 8 (drawing arrows across the table linking the two columns and writing position rule)

And in another excerpt when referring to the visual pattern:-

TR:	How would you describe this pattern?
John:	You just keep adding on 2.
TR:	Is that a growing rule or a position rule? Is it telling me how the pattern is growing? (drawing an arrow along the pattern and writing growing rule)
Sally:	Yes. It is telling us how it is growing.

Introducing specific language to assist in describing the visual pattern.

Specific language was introduced to assist students to describe the pattern in general terms. We conjectured that language may be another barrier to identifying co-variational thinking. It was evident that in all instances it was simpler to say "the pattern is growing by two tiles" than to say, "both rows always have the same number of tiles as the position number". Hence language such as rows, columns, double and multiply was introduced throughout the lesson sequence. By the completion of the four lessons many students were still experiencing difficulties in expressing generalizations in language, however analysis of the videos indicated a marked improvement in the manner in which they described their generalizations.

TR:	Tell me what the 4th step looks like
Mary:	Four greens and five reds.
TR	How do you know that?
Mary	The green row is always one more than the step and the red one below is one more than the green.
TR:	If I have step 10, how many tiles would there be?
Tom:	22
TR:	How did you work it out?
Cath:	Whatever the number is you double that number and add 2.

Justifying their verbal descriptions:

Explicit justification of their descriptions appeared to assist students refine their descriptions.

Nick	What I did is, plus 1 to the step number then times it by two.
TR:	So would it work, lets see. So for 5 what would I do? (Pointing to the step on the board)
Nick	Plus 1 equals 6 times 2, equals 12 .
TR:	Who has a different one?
Jill	Times the step by 2 add 4 and take 2.

The conversation then focused on Jill justifying her description and ascertaining if it worked for a wide variety of step numbers.

Explicitly translating verbal descriptions into symbolic notation systems:-

As students gave their verbal descriptions they were asked to express it in symbolic form. These conversations were ongoing throughout the four lessons. The following excerpt represents a typical conversation that occurred.

Amy:	Well the step number is 1 and you add 4 and the 2nd step you add 4 to it.
TR:	Can you give it to me in general terms?
Amy	You add 4 on every step number. Oh, I don't know.
TR:	Can someone write it as a mathematical sentence using unknown?
Sue:	Unknown plus 4 equals unknown.
TR.	How do we write this? (Sue writes ? + 4 = ? on the blackboard)

Introducing specific symbols for unknown amounts and relating this to large numbers:-

TR.	If we want to talk about any step how do we describe this?
Carol	We call it the n step where n is any number you want it to be.
TR:	How many green tiles would there be?
Carol	Depends on which number it is. Could be two billionth step. There would be Two billion reds and one left over and there would be two billion greens.
TR:	Excellent. Does anyone have another description?
Tom	The n step is anyone with one left over.
Henry	There are n green ones and n plus one red ones.

DISCUSSION AND CONCLUSIONS

This research commences to not only identify teacher actions that support examining growing patterns as functional relationships between the pattern and its position, but also delineate thinking that impacts on this process. Many of the difficulties these children experienced mirror the difficulties found in past research with young adolescents. This suggests that perhaps these difficulties are not so much developmental but experiential, as these early classroom experiences began to bridge many of these gaps.

The results from Table 1 and Table 2 indicated that after the four lessons thirty seven percent of the students could successfully write a rule relating the pattern elements to their position and nineteen percent could successfully write the rule using symbols. These results are comparable with the results from studies conducted with young adolescent students (e.g., Warren, 1996). Our conjecture is that algebraic activity can occur at an earlier age than we had ever thought possible and that these experiences with appropriate teacher actions may assist more students join the conversation in their adolescent years. Presently in many instances the transition from arithmetic to algebra appears too abrupt, with many young adolescents quickly moving from arithmetic to the introduction of the concept of a variable to symbolic manipulation (e.g., Bills, Ainley and Wilson, 2003). The impact these early conversations has on the transition to formal algebraic experiences requires further investigation.

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