A PICTURE IS WORTH A 1000 WORDS – THE ROLE OF VISUALIZATION IN MATHEMATICS LEARNING

Bettina Rösken Katrin Rolka

University of Duisburg-Essen, Germany

This paper describes some important aspects concerning the role of visualization in mathematics learning. We consider an example from integral calculus which focuses on visual interpretations. The empirical study is based on four problems related to the integral concept that highlight various facets of visualization. In particular, we are interested in the visual images that students use for working on specific problems and how they deal with given visualizations. The findings show the importance as well as the difficulties of visualization for the students.

INTRODUCTION

The first part of the title is borrowed from a common proverb which highlights the importance of visualization in general. Likewise, visualization has a long tradition in mathematics and the list of famous mathematicians using or explicitly advocating visualization is large.

One prominent example is certainly the blind Euler whose restriction did not have an effect on his creative power. During the years of his blindness he was able to produce more than 355 papers – due to his visual imagination as well as his phenomenal memory (Draaisma, 2000). Hadamard (1954) pointed out the importance of visualization by referring to Einstein and Poincaré. They both emphasized using visual intuition. In Pólya's (1973) list of heuristic strategies for successful problem solving, one prominent suggestion is ”draw-a-figure” which has become a classic pedagogical advice.

However, in this paper we discuss some findings which focus on the role of visualization ranging from being useful to being an impediment.

VISUALIZATION IN MATHEMATICS LEARNING

The role of visualization in mathematics learning has been the subject of much research (e.g. Arcavi, 2003; Bishop, 1989; Eisenberg & Dreyfus, 1986; English, 1997; Kadunz & Straesser, 2004; Presmeg, 1992; Stylianou & Silver, 2004). In accordance to Zimmermann and Cunningham (1991) as well as Hershkowitz et al. (1989), Arcavi (2003, p. 217) defines visualization as follows:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

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This definition emphasizes that, in mathematics learning, visualization can be a powerful tool to explore mathematical problems and to give meaning to mathematical concepts and the relationship between them. Visualization allows for reducing complexity when dealing with a multitude of information.

However, the limitations and difficulties around visualization and even the reluctance to visualize have also been largely discussed (Arcavi, 2003; Eisenberg, 1994; Stylianou & Silver, 2004). Visual techniques which rely on "not always procedurally 'safe' routines" (Arcavi, 2003, p. 235) are considered to be cognitively more demanding than analytical techniques.

In a different context, visualization is discussed as an important part of so-called "concept images" (Tall & Vinner, 1981). The concept image includes visual images, properties and experiences concerning a particular mathematical concept. To understand a formal mathematical concept requires of the learner to generate a concept image for it. Nevertheless, Vinner (1997, p. 67) points out that "in some cases the intuitive mode of thinking just misleads us." In this paper, we focus only on the visual aspects of the concept image.

TREATMENT OF INTEGRAL CALCULUS IN SCHOOL

Many topics in mathematics have visual interpretations and the integral calculus is certainly one of those. This paper is not the place to go into detail on teaching and learning integral calculus in Germany; for this general discussion we refer to Blum and Törner (1983) and Kirsch (1976).

For the sake of brevity we limit ourselves in the following to the presentation of the major aspect relevant to our study. A classical approach to the integral in school is the area calculation problem. This problem allows for using the geometric reference for visualization. Thus, the most basic way of introducing integrals is using the close connection between the idea of an integral and the idea of an area, initially for functions with positive areas in the first quadrant. Later on, this idea is expanded by identifying the integral as sum of the oriented areas.

RESEARCH QUESTIONS

Much of the research into mathematics students' knowledge of the integral has been oriented by assumptions about what students should know. Instead, we report on some ongoing research into what students do know with a special focus on visual aspects of the integral. This paper presents some results gained within the scope of a larger study to investigate students' mental representations concerning the integral (Rösken, 2004). Our research questions in this study were:

- What visual images do students have concerning the integral?
- How do students deal with a given visualization?
- To what extent are visual images used by the students?

METHODOLOGY

The study employed qualitative methods to capture the importance of visualization in the learning of integral calculus. The observation of the lessons in question and the analysis of the teaching material led to constructing a questionnaire containing several problems related to the integral. The students worked on this questionnaire in the classroom under supervision and were allowed to use a calculator. For the purposes of this paper we focused on four problems revealing diverse aspects of the integral.

The subjects in this study were students in grade 12 of two German high-schools. The first class consisted of 24 students, 14 female and 10 male students. The second class consisted of 28 students, 6 female and 22 male students. The two classes together form a total of 52 students. For the analysis, we do not distinguish between these two classes.

EMPIRICAL RESULTS

This is not the place to give a detailed analysis of the observed lessons. The main approach to the integral discussed above emerged in both classes. In this section, we restrict ourselves to the presentation of the problems, the underlying mathematical aspects and the students' answers.

Problem 1:

Draw a figure to illustrate the geometric definition of the integral.

The geometric definition refers to the area concept as already mentioned. We were interested in the visual representations that students associate with this aspect of the integral. The following table shows the distribution of the students' solutions:

Table 1: Students' answers to problem 1

Positive area	Positive and negative areas	No answer
77%	13%	10%

90% of the students were able to illustrate the geometric definition of the integral. However, it is remarkable that 77% of the students disposed of an image that is limited to a positive area. Figure 1 shows an example of such visualization which represents merely one aspect of the integral concept. This restricted visualization will turn out to be an obstacle for working on the other problems. Figure 2 shows an example for a more adequate visualization which was only used by 13% of the students.

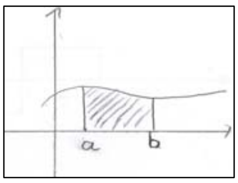


Figure 1

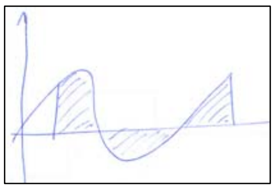
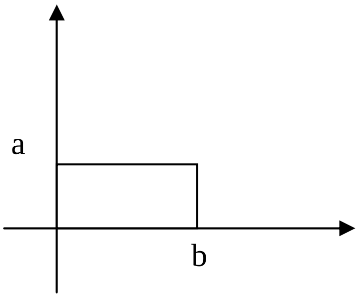
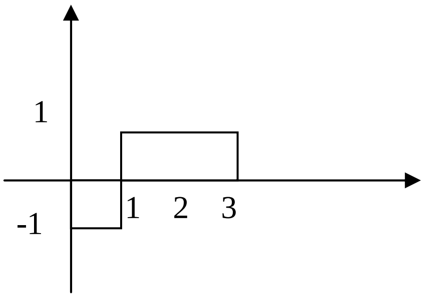


Figure 2

Problem 2:

Find a formula for the area by using integration.

a) 

b) 

In contrast to problem 1, the students were given a concrete visualization and were asked to find the integrand as well as the limits of integration. In problem 2b, the students additionally had to consider the orientation of the area.

Table 2 shows the distribution of the students' answers to problem 2a:

Table 2: Students' answers to problem 2a

Correct answer	Incorrect answer	No answer
50%	40%	10%

It is notable that the given visualization of this problem differed only slightly from the visualization the students chose in problem 1. However, half of the students were not able to give a correct answer. Among the incorrect answers, the following terms can be found:

$$\int_a^b kdx \quad \int_a^b adx \quad \int_a^b a(x)dx \quad \int_a^b ada \quad \int_a^b f(a)dx$$

One difficulty for the students was to name the limits of integration. It is evident that finding the integral for the given image conflicts with the standard notation: $\int_a^b f(x)dx$

Furthermore, the students had major problems to recognize the given constant function as a possible integrand. Obviously, they were missing an x-term. One student gave the correct answer but stated the following:

$\int_0^b f(x)dx$: Not possible, because this is a constant function and there is no x in it and that's why it is not possible to put in the limits.

Two students solved this conflict by drawing a supporting straight line as shown in figure 3. They obtained the answer to this problem in a creative though complicated way.

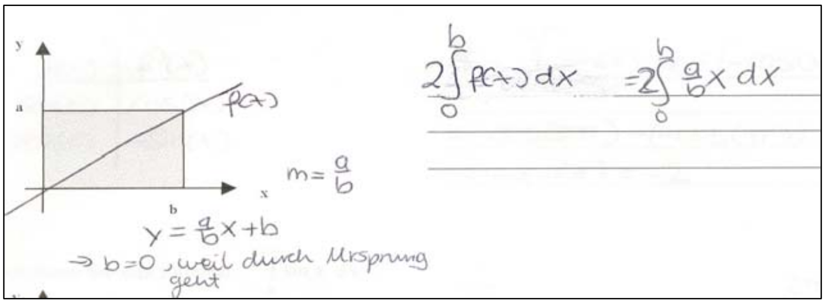


Figure 3

Table 3 shows the distribution of the students' answers to problem 2b:

Table 3: Students' answers to problem 2b

Correct answer	Incorrect answer	No answer
40%	54%	6%

While the difficulties to find the integrand remained, the problem to name the limits of integration minimized due to the concrete numbers provided in the illustration. However, a new obstacle emerged because of the orientation of the area. Instead of the area, the students calculated the integral. This was found in more than half of the incorrect answers. Some students solved this conflict by shifting the square above the x-axis.

Problem 3:

a) Find the area bounded between the function f(x)=sinx and x-axis over [π,2π].

b) Calculate the integral: $\int_{-\pi}^{2\pi} \sin x dx$

For the answer to problem 3a the students had to calculate an area of negative orientation while in problem 3b the same function was given but this time they were asked to calculate the integral. Even though the limits of integration changed, the answer to problem 3b could be immediately given by visualizing the graph of the function and considering problem 3a. This problem demanded that students distinguish clearly between the area and the integral concept.

Table 4 shows the distribution of the students' answers to problem 3a:

Table 4: Students' answers to problem 3a

Correct answer	Incorrect answer	No answer
27%	67%	6%

Some of the students had difficulties to put in the limits or to give the correct antiderivative. More interestingly, 77% of the incorrect answers resulted in giving a negative value as area of the function. On the one hand, these students did not use visualization to approach the problem. On the other hand, they did not scrutinize the negative value of their result. Only 8% of all students sketched the graph.

Table 5 shows the distribution of the students' answers to problem 3b:

Table 5: Students' answers to problem 3b

Correct answer	Incorrect answer	No answer
23%	73%	4%

The distribution of the answers to problem 3b is similar to 3a and the same mistakes emerged. Remarkably, in 47% of the incorrect answers a positive value was given. The students continued calculating the area as required in 3a instead of the integral, some of them even mentioned explicitly "A=2". Only 8% of all students visualized the graph, 6% referred to their solution of problem 3a.

Problem 4

How would you proceed to calculate $\int_{-1}^{1} \sin(2x^3)dx$ *?*

This problem can be easily solved by visualization. The function is odd so that on the given interval [-1,1] the integral equals zero. Only 4% of all students took into account these considerations. 8% of the students did not answer at all while the other students proposed to work out the integral by substitution (42%), by finding the antiderivative (29%) or by integration by parts (17%). To summarize, the solutions to this task showed an explicit bias towards an algorithmic approach even though the visual one would have been significantly easier.

DISCUSSION AND CONCLUSIONS

The selected problems emphasize convincingly some important aspects inherent to visualization. On the one hand, visualization proves to be a useful tool for working on the problems and the common proverb mentioned in the title seems to be appropriate. For example, some students use visualization in a creative way by modifying the given task (problem 2; figure 3). This approach enables them to avoid the difficulties with the given visualization and thereby sheds light on the underlying obstacles concerning this task. Another interesting point is that even students that do not show visualization on their paper were able to solve problem 3 correctly. This highlights once again the importance of pictures in the mind (Presmeg, 1986).

On the other hand, visualization raises some difficulties which lead us to modify the common proverb mentioned in the title. A picture is worth a 1000 words – only if one is aware of its scope: The students in this study largely demonstrated their ability in visualizing the geometric definition of the integral (problem 1; figure 1). Nevertheless, their chosen visualization only reflects one particular aspect of the integral concept. This entails some important consequences for working on the other problems. For example, the restricted visualization proves to be a hindrance for the solution of problem 3. The connection of the integral with the area misleads the students not to distinguish clearly between the two concepts. Basically, both concepts are different from each other, but at the same time, they have a certain though marginal intersection which predominates the students' thinking.

Another interesting aspect leads us to change the proverb as follows. A picture is worth a 1000 words – only if one is able to use it flexibly: First, even if students use visualization to solve the problems 2 and 3, this does not mean that they are able to solve the problems correctly. They do not dispose of the cognitive flexibility to use both visual and algorithmic techniques (Arcavi, 2003). Second, the students usually did not choose to visualize in problem 4 but proposed an algorithmic approach instead. They are cognitively fixed on algorithms and procedures instead of recognizing the advantages of visualizing this problem – a phenomenon which Eisenberg (1994) describes as reluctance to visualize. Third, the visualization given in problem 2 differs only slightly from the visualizations the students gave in problem 1. However, most of the students were not able to deal with this given visualization and to adequately interpret the information given in this problem.

These aspects highlight the ambivalence of visualization as Tall (1994, p. 37) points out: "It is this quality of using images *without being enslaved by them* which gives the professional mathematician an advantage but can cause so much difficulty for the learner." Hence, the importance of visualization for mathematics learning and teaching is constituted in being aware of the fact that visualization never represents an isomorphism of mathematical concepts and their relationships. Therefore, visualization should be accompanied by reflective thinking to avoid being enslaved by it.

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