INFINITY OF NUMBERS: HOW STUDENTS UNDERSTAND IT

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Some results of the research project ‘Development of Understanding and Self-confidence in Mathematics’, implemented at the University of Turku (Finland) during the academic years 2001–03, are reported. The project was funded by the Academy of Finland (project #51019). It was a two-year study for grades 5-6 and 7-8. The study included a quantitative survey for approximately 150 Finnish mathematics classes out of which 10 classes were selected to a longitudinal part of the study. This paper is based on the survey results, and will focus on students’ understanding of infinity and the development of that understanding. The results show that most of the students did not have a proper view of infinity but that the share of able students grew, as the students got older.

Most primary children are very interested in infinity, and they enjoy discussing the concept, if the teacher is only ready for it. On one hand they have a concrete view on the world around and mathematics, and on the other hand they are ready to play with numbers. Thus, questions on infinity may also come into light. Infinity awakes curiosity in children already before they enter school: “*preschool and young elementary school children show intuitions of infinity*” (Wheeler, 1987). However, this early interest is not often met by school mathematics curriculum, and infinity remains mysterious for most students throughout school years.

INFINITY IN MATHEMATICS

Actual and potential infinity

Consider the sequence of natural numbers 1, 2, 3, … and think of continuing it on and on. There is no limit to the process of counting; it has no endpoint. Such ongoing processes without an end are usually the first examples of infinity for children; such processes are called *potentially infinite.*

In mathematics, such unlimited processes are quite common. Consider, for example, drawing regular polygons with more and more sides inside a circle, or counting more and more decimals of □. However, the interesting cases in mathematics are, when infinity is conceptualised as a realised “thing” – the so-called *actual infinity*. The set of all natural numbers is an example of actual infinity, because it requires us to conceptualise the potentially infinite process of counting more and more numbers as if it was somehow finished. (Lakoff & Núñez 2000)

The question of infinity has its roots already in the mathematics of ancient Greece, for example, the famous paradox of Zenon (cf. Boyer 1985). However, the transition from potential to actual infinity includes a transition from (an irreversible) process to a mathematical object. This step the Greek mathematicians were unable to

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accomplish (Moreno & Waldegg, 1991). In the history of mathematics, the exact definition of and dealing with infinity is something more than one hundred years old. The foundation of infinity as modern mathematics sees it was laid when Dedekind and Cantor solved the problem of potential infinity at the end of the 19. century, and Cantor developed his theory of cardinal numbers. (e.g. Boyer 1985, Moreno & Waldegg, 1991)

We may distinguish different kinds of infinities in mathematical objects. For example, the set of natural numbers has infinitely many elements, and it has no upper bound. Therefore, the numbers may become bigger and bigger. But every bounded subset of natural numbers is automatically finite, whereas the same is not valid for rational numbers. For example, the set of rational numbers between zero and one has infinitely many elements, but it is bounded. Furthermore, between any two rational numbers there are infinitely many rational numbers. This property of rational numbers is called *density*, whereas no set of natural numbers is dense.

Tsamir & Dreyfus (2002) summarise the problems mathematicians have had with actual infinity, as follows:

> Actual infinity, a central concept in philosophy and mathematics, has profoundly contributed to the foundation of mathematics and to the theoretical basis of various mathematical systems. It has long history and persistently been rejected by mathematicians and philosophers alike, and was highly controversial even in the last century in spite of the comprehensive framework provided for it by Cantorian set theory.

Hence, although the concept of infinity as a potentiality is relatively easy for mathematicians, the concept of actual infinity is counterintuitive and difficult.

Students' conceptions of infinity

Infinity has been an inspiring, but difficult concept for mathematicians. It is no wonder, that also students have had difficulties with it, although they might be fascinated about it. Previous research has identified typical problems and constructive teaching approaches to cardinality of infinite sets. Students use intuitively the same methods for the comparison of infinite sets as they use for the comparison of finite sets. Although students have no special tendency to use 'correct' Cantorian method of "one-to-one correspondence," they are prone to visual cues that highlight the correspondence. For example, students tend to match set {1, 2, 3…} more easily with the set $\{1^2, 2^2, 3^2 \ldots\}$ than with the set {1, 4, 9 …}. (Tsamir & Dreyfus, 2002)

Fishbein, Tirosh and Hess inquired students' view of infinite partitioning through using successive halvings of a number segment (Fishbein & al. 1979). They concluded that students on grades 5–9 seem to have a finitist rather than a nonfinitist or an infinitist point of view in questions of infinity.

Even at the university level, the concept of infinity of real numbers is not clear for all students (cf. Merenluoto & Pehkonen 2002). For example, Wheeler (1987) points out that university students distinguished between 0.999… and 1, because "*the three dots tell you the first number is an infinite decimal*".

Focus of the paper

We want to find out what is the level of students' understanding on infinity in Finnish comprehensive school, and how this understanding develops from grade 5 to grade 7.

We will distinguish three levels of students understanding of infinity. The lowest level is when they do not understand infinity, but use only finite numbers. In the intermediate level, the students understand potential infinity, and use processes that have no end. Those students, who have reached the third level, are able to conceptualise actual infinity and the final resultant state of the infinite process.

METHODS

The paper describes some partial results of the research project "Development of Understanding and Self-confidence in Mathematics", implemented at the University of Turku (Finland) and financially supported by the Academy of Finland. The project was a two-year longitudinal investigation on grades 5–8. More results of the project are to be found in the papers Hannula & al. (2004), Hannula & al. (2005), Maijala (2005) and Hannula & al. (2006).

In order to measure the level of students' self-confidence and understanding of number concept in grades 5 and 7 of the Finnish comprehensive school, we designed a survey. The representative random sample of Finnish students consisted of 1154 fifth-graders (11 to 12 years of age) and 1902 seventh-graders (13 to 14 years of age). The response rate of schools was 72 %. The questionnaire consisted of five parts: student background information, 19 mathematics tasks, success expectation for each task, solution confidence for each task, and a mathematical belief scale. It was administered by teachers during a normal 45-minute lesson in the fall 2001.

We focus here on mathematics tasks: In the 19 mathematical questions, there were three that measured students' understanding of infinity (tasks 5, 7 and 8). Task 5 measured understanding of infinitely large natural numbers. The two other tasks measured understanding of the density of the rational numbers.

> Task 5. Write the largest number that exists. How do you know that it is the largest?
>
> Task 7. How many numbers are there between numbers 0.8 and 1.1?
>
> Task 8. Which is the largest of numbers still smaller than one? How much does it differ from one?

In this paper we will concentrate on the results of these three infinity tasks.

RESULTS

Survey results of competence

We categorized student responses to the infinity tasks according to how proper we deemed answers to be. In each question, we can find answers that remain on the level of finite numbers, answers that describe processes that do not end (potential infinity) as well as some answers that indicate that the student has an understanding of the final state of the infinite process (actual infinity).

In the following, there are the answer categories and scoring for Task 5 given.

Task 5. *Write the largest number that exists. How do you know that it is the largest?*

Answer categories (and scoring):

- Actual infinity 2: There is no largest number (4 points)

- Actual infinity 1: Infinity, ∞ (3 points)

- Potential infinity: Unending number, e.g. 9999… (2 points)

- Finite: A number larger than one million, e.g. 99999999999999999, centillion (1 point)

To give a general description of the development from fifth grade to seventh grade we compared the answer distributions in each item. In figures 1–3 we can see, that tasks were demanding and most students scored only zero or one point per task (maximum being 4–5 points). As expected, seventh graders gave better answers.

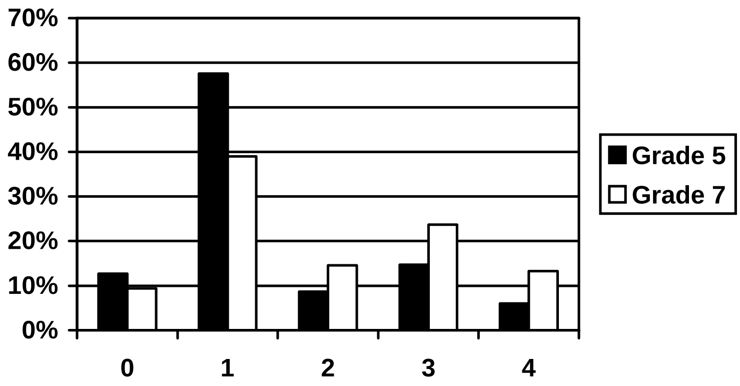

Fig. 1. Students' scoring for task 5.

In task 5 (infinitely large), the development consisted mainly of the decrease of finite numbers as answers and of increase of different types of infinite answers.

In the following, there are the answer categories and scoring for Task 7 given.

Task 7. *How many numbers are there between numbers 0.8 and 1.1?*

Answer categories (and scoring):

- Actual infinity: Infinitely many (5 points)

- Potential infinity: Unending number, e.g. 9999… (4 points)

- Finite 3: A finite number larger than one million, e.g. 9999999999999 (3 points)

- Finite 2: Working with more than one decimal, a number between 20 and one million (2 points)

- Finite 1: Working on one decimal level (even incorrectly), 2, 3 or 4 (1 point)

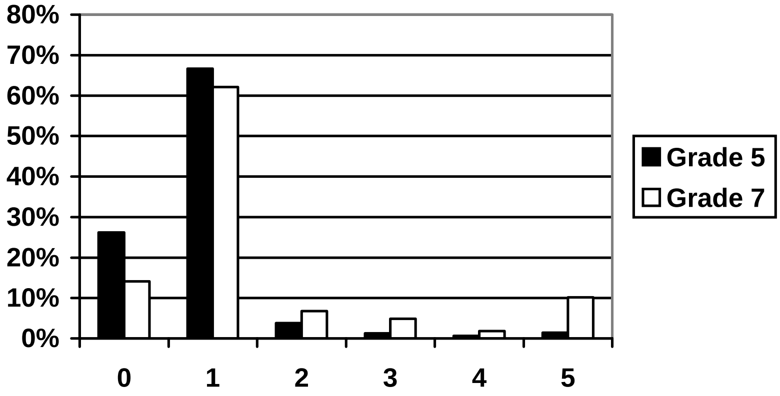

Fig. 2. Students' scoring for task 7.

In task 7 (infinitely many), the decrease was mainly in completely incorrect answers (typically 0.3) and in single decimal thinking, and the biggest increase was in correct answers (infinitely many).

In the following, there are the answer categories and scoring for Task 8 given.

Task 8. *Which is the largest of numbers still smaller than one? How much does it differ from one?*

Answer categories (and scoring):

- Actual infinity: There is no such number (5 points)
- Potential infinity 2: Such number cannot be written (4 points)
- Potential infinity 1: 0.999… (3 points)
- Finite 2: 0.999; three or more decimals (2 points)
- Finite 1: 0.9; 0.99 (1 point)

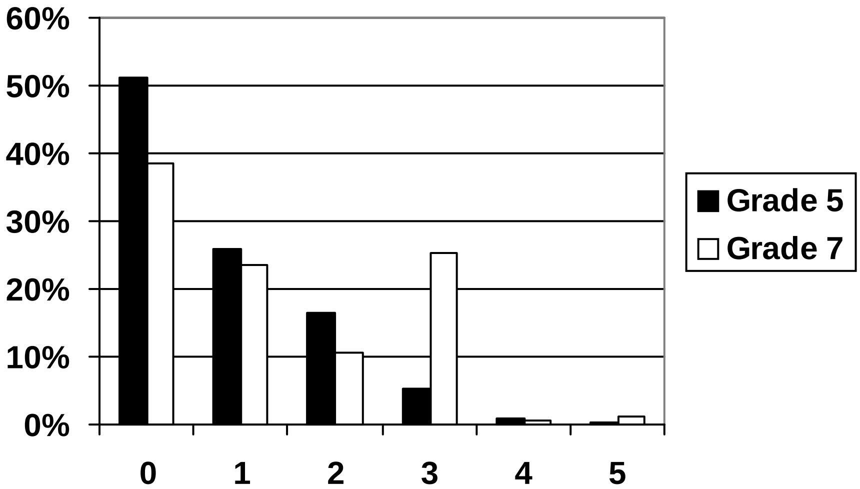

Fig. 3. Students' scoring for task 8.

In task 8 (infinitely close), the decrease was mainly in completely incorrect answers (typically 'zero' or 'minus infinity'), and a significant increase was in answers (0.999…) that require understanding of potential infinity, but not actual infinity.

The chi square test revealed significant gender differences in task 5 (infinitely large) on fifth grade, and in task 7 (infinitely many) and task 8 (infinitely close) on seventh grade; in both cases boys gave significantly more frequently answers of infinite nature than girls.

<ins>Summary of competence results.</ins> In the fifth grade, 20 percent of the students have some understanding of the infinity of natural numbers, but only few have any understanding of density of rational numbers. The situation is not much better in the seventh grade. Yet, there is an obvious development from grade 5 to grade 7 in student levels of answering these questions. Infinity of natural numbers is understood earlier than infinity of subsets rational numbers, and potential infinity is understood earlier than actual infinity. Boys perform much better than girls in these tasks dealing with infinity.

Survey results of confidence

According to the chi square test both the students' success expectation and solution confidence related to their answers (with an exception of the fifth grade boys' success expectation). In the tasks 5 and 8, the students' solution confidence increased, as their answers got better. In task 7 (infinitely many), however, the relationship between answer and confidence was more complex (Table 1). Students who gave 0- or 1-point answers were modestly uncertain, while solution confidence was much lower for 2-point answers. Confidence remained low for 3- and 4-point answers and was high for 5-point answers. Students who operate on one decimal level seem to be confident on their answers, while those more advanced students who move beyond that level have lower confidence. Only when they realize that there are infinitely many numbers within the given interval, they regain high confidence.

Points for task 7	N	Success expect. mean	Std. deviation	N	Solution confid. mean	Std. deviation
0	561	4.06	1.06	539	3.42	1.35
1	1933	4.15	0.94	1922	3.68	1.16
2	171	3.99	0.94	169	2.91	1.19
3	109	3.54	1.28	104	3.10	1.56
4	42	3.88	1.11	40	3.18	1.52
5	210	4.07	1.16	210	3.92	1.12
Total	3026	4.09	1.00	2984	3.58	1.24

Table 1. The means of solution confidence for responses of task 7.

The relationship between answer and success expectation was slightly different from the relationship between answer and solution confidence presented above. For task 5 (infinitely large) those students who gave 3-point answers ("infinity") had highest expectations, for task 8 (infinitely close) expectations were highest when the answer got 2 or 3 points ("0.999", three or more decimals or "0.999…", respectively). This suggests that those who gave the best answers did not know the right answer beforehand, but they had to produce it during the test. Furthermore, for task 7 (infinitely many) only those students who gave 3-point answers (a large finite number) had much lower expectations than others. Especially those students who gave a 2-point answer (20 – one million) had roughly as high expectations as others.

In all cases, the students' success expectation was higher than their solution confidence. In the result group 2 (Working with more than one decimal, a number between 20 and one million), the difference was the biggest one, and in the best answers (group 5) the smallest one.

Summary of confidence results. The students' confidence both before and after solving the task is related to the success they have. That is what we should expect to find. However, those who gave the most sophisticated answers were not the most confident in their expectations.

In the task 7 (how many numbers are there between 0.8 and 1.1), the students' confidence had even more complex relationship with success. Many of the students indicated strong false confidence in their one-decimal thinking of numbers. Furthermore, when their thinking begun to advance, their confidence dropped. Sometimes they even had an initial expectation of success before they begun to solve the task but this confidence fell after they had tried to solve the task. Confidence was reassured when they reached the level where they had an understanding of the density of rational numbers.

CONCLUSIONS

Boys give better answers than girls in tasks dealing with infinity. This finding can be understood in the light of the general conclusion made by Fennema and Hart (1994). According to them, gender differences in mathematics still remain within the most difficult topics. The test used can be regarded as an example of a very challenging one that is likely to produce large gender differences.

In most cases students who gave better answers were also more confident of their answers. This is what we would have expected. However, findings for task 7 confront this expected tendency. Also Merenluoto (2001) has found similar results. There was a general tendency for confidence to increase as the answers got better, but also some topics where this was not the case.

In another analysis of the longitudinal development of student competence in number concept, we noticed that proper understanding of fractions as numbers is an important predictor of learning the density of rational numbers (Hannula & al. 2004). This suggests that learning fractions is an important opportunity for this challenging conceptual change.

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