THE REFLECTIVE ABSTRACTION IN THE CONSTRUCTION OF THE CONCEPT OF THE DEFINITE INTEGRAL:
A CASE STUDY

Theodorus Paschos and Vassiliki Farmaki
Department of Mathematics, Athens University

In this paper we report the case study of Maria, a first year university student of Mathematics. By an activity and an interview, we try to analyze her mental operations. Employing the Piagetian theory of reflective abstraction we study the way in which she acts in order to calculate the distance covered in a time interval of a non-uniformly accelerating motion problem. This case study is a part of a research activity that aims at the intuitive approach and understanding of Calculus concepts, using motion problems. The focal analysis of the interview’s content allows for an investigation in depth of qualitative elements of the student’s mathematical thought.

INTRODUCTION

The transition to formal mathematical thinking is not an obvious intellectual process for the majority of students. Research in didactics of mathematics has investigated various students’ difficulties in order to understand definitions, concepts, propositions and their proofs, when we teach them in strict symbolic formulation. The intuitive approach via mathematical or real life situations, which are familiar to the students, may constitute a substantial step toward the emergence of new concepts. The students can develop mathematical models for manipulating the concepts images which may lead them to the need for formal mathematical argument. Many researchers apply a solving problem strategy that first develops new concepts which may be useful, before the appropriate definitions are constructed in order to form the basis for a formal theory (e.g. Poincaré, 1913; Hadamard, 1945). In this paper we report the case study of Maria, a first year student of Mathematics Department, aiming to interpret her mathematical activity on a non-uniformly accelerated motion problem. The graphical representation of the motion in a system of velocity-time axes leads her to the algebraic context aiming to description of a calculation method of the distance covered as an area of the region formed in the graph. In the interview she attempts to justify her initial intuitive answers revealing interesting sides of her mathematical thinking. We employed the theoretical framework of reflective abstraction (Piaget, 1980), as a general scheme able to describe the emergence of the concept of the definite integral as observed from Maria’s mental operations.

This case study is included in a wider research activity which concerns the introduction of the concept of the definite integral and the approach to the Fundamental Theorem of Calculus, to first year students of Mathematics Department. In our analysis of the interview’s content we employed the focal analysis (Sfard, 2001; Kieran & Sfard, 2001), as we describe below in the section of Methodology.
THEORETICAL FRAMEWORK

Reflective abstraction is drawn from what Piaget (1980, pp. 89-97) called the general coordinations of actions, and as such, its source is the subject and it is completely internal. This kind of abstraction leads to a generalization which is constructive and results in “new syntheses in midst of which particular laws acquire new meaning” (Piaget & Garcia, 1989, p.299). From Piaget’s psychological viewpoint, reflective abstraction is the method that “it alone supports and animates the immense edifice of logico-mathematical construction” (Piaget, 1980, p. 92). Piaget distinguishes various kinds of construction in reflective abstraction: (a) The interiorization, as a construction of internal processes, as a way of making sense out of perceived phenomena; as “translating a succession of material actions into a system of interiorized operations” (Piaget, 1980, p. 90). Dubinsky (1991, p. 107), argues that “interiorization permits one to be conscious of an action, to reflect on it and to combine it with other actions”. (b) The coordination or composition of two or more processes for the construction a new one. (c) The encapsulation or the conversion of a (dynamic) process into a (static) object, in the sense that “… actions or operations become thematized objects of thought or assimilation” (Piaget, 1985, p. 49). Piaget considered that “…mathematical entities move from one level to another, an operation on such ‘entities’ becomes in its turn an object of the theory…” (Piaget, 1972, p.70). (d) When a subject learns to apply an existing schema to a wider collection of phenomena, then we say that the schema has been generalized. Generalisation can also happen when a process is encapsulated to an object. The schema remains the same except that it now has a wider applicability. Piaget referred to all of this as a reproductive or generalizing assimilation (Piaget, 1972, p.23) and he called the generalization extensional (Piaget & Garcia, 1989, p. 299). Dubinsky (1991, p. 102) argues that the interiorization of a process, it is possible for the subject to think of it in reverse, as a means of constructing a new process which consists of reversing the original process. The case of differentiation-integration is an example.

METHODOLOGY

The data that we will present is part of a qualitative action research aiming at the investigation of how the students shift from the intuitive to the formal mathematical knowledge. Initially, the aim of the experimental instructive approach was to introduce the first year students to the definite integral. In an interactive milieu the students worked in pairs (activities on work sheets) and discussed about the solution of various motion problems. A group of students participated in individual interviews which fully transcribed. We applied the focal analysis in order to analyse in depth the data from the transcripts.

According to Sfard (2001), focal analysis is a methodological tool for investigation of communication effectiveness between individuals. The communication will not be regarded as effective unless, at any given moment, all the participants seem to know what they are talking about and feel confident that all the parties involved refer to the same things when using the same words. The word focus is interpreted as the
expression used by an interlocutor to identify the object of her or his attention. Sfard considers two focal ingredients: pronounced and attended. The pronounced focus concerns the key-words or phrases that imply the attended object. However, there is more to communication than the pronounced and attended aspects. Whatever is pronounced or seen evokes a whole cluster of experiences, and relates the person to an assortment of statements he or she is now able to make on the entity identified by the pronounced focus. This collection of experiences and discursive potentials is called intended focus. The intended focus, which seems to be the crux of the matter, is an essentially private dynamic entity that changes from one utterance to another. In our report we employed focal analysis as a tool for research, interpretation and understanding the mental trajectory of the student interviewed. The attended focus in our analysis is presented in the form of explicit or implicit operations of the interviewee as these are presented by the pronounced focus (Famaki & Paschos, 2005).

**THE CASE STUDY**

The aim of the activity, in which Maria worked, was: (a) to connect the distance covered during a time interval with the area of the region between the velocity graph and the time axis; and, (b) to determine a unit of measurement of area on the given graph and calculate the area of the region by approximation. In the previous activities the students worked on uniform and uniform accelerated motion problems in which they calculated the distance covered as rectilinear figures’ area on the (u-t) graph.

The worksheet of Maria (fig. 1):

Consider that the following u-t graph represents the movement of a material point. Calculate by approximation the distance covered during the first second of the motion.

![Figure 1: Maria wrote without reasoning that \( U \cong t^2 \). She mentioned: “I partition the interval [0, 1] in k equal time sub-intervals in which I consider that the velocity variation is constant”. She also wrote the formula \( \Delta S = U \Delta t + (1/2) \Delta U \Delta t \) in order to calculate the distance covered.](image-url)
AN EPISODE OF THE INTERVIEW. (Interviewer : I, Maria: M)

1 I: We suppose that the graph represents the movement of the point and we
2 want to calculate by approximation the distance covered during the 1st sec.
3 On the worksheet you wrote $U \approx t^2$. How is this derived?
4 M: From the graph. It looks alike, so I write ‘roughly’ [$U \approx t^2$].
5 I: Are you trying to find the right formula? What if given a different graph?
6 M: I would search for something else.
7 I: Do you mean that you would try to find a correspondence between the
8 graph and a particular function?
9 M: May be, taking into account the parts of the graph.
10 I: Then you would find different formulas for different parts of the graph?
11 M: May be.
12 I: What would your next step be?
13 M: I partition the time interval in $k$ equal sub-intervals and I assume that in
14 each of them the velocity’s variation is constant, ..., aha !!, ... the crucial
15 observation is that the velocity’s variation is constant, hence the velocity
16 looks like this [$U \approx t^2$]. Since the variation of the velocity is constant, using
17 the knowledge concerning the derivative and that the velocity is 1 at the
18 time instant 1, I come to the conclusion that the velocity looks like this.
19 I: Here, you use the formula $\Delta S = U_{in} \Delta t + (1/2)\Delta U \Delta t$.
20 M: $U_{in}$, aha!, I mean $U_{in}$ at a time sub-interval $\Delta t$. I take the rectangle and
21 the triangle above it (she shows on the graph).
22 I: Do you consider the elementary arc as a line segment?
23 M: Yes, exactly.
24 I: And you do so in order to find an approximation to the area of the region?
25 M: I was trying to work with $k$, so that when $k$ increases the line segment
26 approaches continuously to the curve. OK, the area which I will find will
27 be always bigger than the area of the curvilinear region, but it will
28 approach it continuously as $\Delta t$ decreases.
29 I: Which is the area that you will find?
30 M: The area which results if we add all the elementary $\Delta S$.
31 I: How do you know this method for calculating the distance covered?
32 M: I am using previous knowledge obtained in High school.
## FOCAL ANALYSIS OF MARIA’S INTERVIEW

<table>
<thead>
<tr>
<th>Utterances</th>
<th>Pronounced focus</th>
<th>Attended focus</th>
<th>Intended focus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3/ 4 8 / 9</td>
<td>“From the graph. It looks alike… [U ≅ t^2]”. “…taking into account the parts of the graph”.</td>
<td>From the graph to the function’s formula: 1. Find a formula corresponding to graph. 2. Find the function’s formula in every case.</td>
<td>The function formula</td>
</tr>
<tr>
<td>12 / 13-14</td>
<td>“I partition the time interval in k equal sub-intervals,… in each of them the velocity’s variation is constant”.</td>
<td>Focusing on the partition of the time interval and the function’s variation in every time sub-interval.</td>
<td>The time interval partition in order to focus on the function</td>
</tr>
<tr>
<td>13-14 16-18</td>
<td>“I assume that in each of them the velocity’s variation is constant”. “Since the variation… is constant, using the knowledge concerning the derivative… the velocity is… [U ≅ t^2]”.</td>
<td>Reasoning for choosing the formula U ≅ t^2: 1. Assume that the velocity’s variation is [approximately] constant. 2. The constancy of the [rate] of the velocity’s variation leads to the initial function U ≅ t^2.</td>
<td>The function formula</td>
</tr>
<tr>
<td>19 / 20-21</td>
<td>“…at a time sub-interval Δt. I take the rectangle and the triangle above it”.</td>
<td>Focusing on the graph The elementary distance covered is represented approximately by the rectilinear figure.</td>
<td>Approximation to the curvilinear figure</td>
</tr>
<tr>
<td>22-24 / 25-26 26-28</td>
<td>“when k increases the line segment approaches continuously to the curve” “the area…will be always bigger than the area of the… region, but it will approaches it continuously as Δt decreases”. “The area which results if we add all the elementary ΔS”.</td>
<td>Area’s approximation 1. As k increases, the line segment converges to the curve. 2. The convergence of the elementary rectilinear figure area to the corresponding curvilinear region area. 3. The area of the whole figure is obtained by adding the elementary areas.</td>
<td>The distance covered is calculated as an area by approximation and addition</td>
</tr>
</tbody>
</table>
INTERVIEW’S CONTENT ANALYSIS - OBSERVATIONS

1. As we observe on the worksheet, the symbols $\Delta S$, $\Delta t$, $U_{in}$, $t_1$, $t_2$, used by M. do not appear in the graph. It seems that $\Delta S$ is the area of the region corresponding to $\Delta t$, in some of the ‘$k$ equal time intervals [\(\Delta t\)]’ in which she divided \([0, 1]\). M. interprets intuitively the graph and decides that the formula of the velocity function is $U \cong t^2$. The reservation that she expresses by writing $U \cong t^2$ is not an obstacle for her actions. Decoding her worksheet we observe that she chooses a trapezium whose area she determines as $\Delta S = U_{in} \Delta t + (1/2) \Delta U \Delta t$. The trapezium is determined by $\Delta t = t_2 - t_1$, the line segments corresponding to the values $U_1$ and $U_2$ of the velocity on the corresponding time $t_1$ and $t_2$, and the chord on the curve determined by the points $(t_1, U_1)$ and $(t_1, U_1)$. The $\Delta U$ corresponds to $U_2 - U_1$ and $U_{in}$ to $U_1$. We believe that this mental image is guiding her actions.

2. Let us try to relate what we just mentioned with the dialogue in the episode. At first sight it seems that for M. a function is adequately defined only if given by a formula. Although she observes the graph, she is searching insistently for the formula because she obviously considers that only by knowing the formula she will be able to act (lines 1-11, 14-18). However, if we connect this attempt with what she says at line 32, it becomes clear that she relies in knowledge acquired in High school mathematics, according to which the area is the limiting value of the sum of elementary areas determined by the partition of the region formed between the graph of a continuous function and the x-axis in a closed interval of the domain of the definition of the function. Thus Maria assumes that the calculation of the area presupposes the knowledge of the formula for the velocity function.

3. Maria tries to answer the question ‘how is formula $U \cong t^2$ derived?’ (lines 1-11). Initially, the image of the given graph leads her to the choice of the formula $U \cong t^2$. Both this choice and the a posteriori attempt of its justification (lines 13-18) reveal interesting mental operations: (a) Maria is ‘trapped’ in the image which ‘looks like’ something familiar to her. The mental scheme she has constructed correlating function $U(t) = t^2$ with its graph, seems to constitute an obstacle in this case, exploiting only the information given that point (1,1) belongs to the graph (lines 17-18). According to Brousseau (1983), an obstacle manifests itself from non-random errors. These are rather errors related with a characteristic perception, an old ‘knowledge’ which manages to dominate in a range of actions. (b) Maria attempts to interpret her initial choice by referring to differential calculus (lines 16-18). She considers that velocity is changing at a constant rate in some $\Delta t$, so that a good approximation results in the corresponding part of the parabola (lines 13-14). However, the choice of linear function for the velocity leads her to some initial function which she knows, from differential calculus, to be of the second degree. In attempting to interpret M.’s mode of thinking, we might note that: (1) While she recognizes the distance covered as area of a region in the velocity graph (lines 20-21, 26-28), in order to justify her choice $U \cong t^2$, she does not correlate it with the initial velocity function, but with velocity itself. (2) Maria’s reference to the derivative and the way of transition from a linear to a second degree function (lines 16-18) indicate
particular mental operations on the differentiation that have been interiorized. The interiorization of the procedure of differentiation seems to guide M. to the inversion, as an action that can open the way for her mental construction of the procedure of integration.

4. In the second part of the dialogue, M. explains what she has done in order to approximate the area of the curvilinear region: (a) she “partitions the time interval in k equal time subintervals…” (lines 13-14), and (b) she chooses one of the trapezia which are formed by the partition ‘taking the rectangle and the triangle above it’ in some $\Delta t$ (lines 20-21), whose area, being equal to $\Delta S = U_{in} \Delta t + (1/2) \Delta U \Delta t$, represents approximately the distance covered. It seems that in the particular trapezium she ‘observes’ all the trapezia in general that are formed by the partition. The choice of the formula $\Delta S = U_{in} \Delta t + (1/2) \Delta U \Delta t$, implies the coordination of several processes and mental objects: on the one hand, the coordination of the function process with its graphical representation, where the value $U_{in}$ and the variation $\Delta U$ are represented as line segments’ length on the graph (here Maria coordinates–composes the geometric object (trapezium) with the velocity function graph); on the other hand, the coordination of the distance covered function with the area of the formed figure on the graph. Also, (c) she studies the generic trapezium in a dynamic manner, describing a process of approaching at the limit the curvilinear region (lines 25-28), each approximating distance covered being determined by the sum of all the elementary areas $\Delta S$ (line 30). Maria’s description shows that she has a scheme for the calculation of the curvilinear region area which, suitably extended, can lead to the formation of a general scheme. Her actions on several objects (function, graph, geometric figure, limit), the interiorization and the coordination of the processes on these objects, can lead to new processes and, finally, by encapsulation and generalization to the construction of the definite integral concept.

DISCUSSION

In the case study presented, we aim to interpret Maria’s mental operations employed for the construction of the definite integral concept; this construction involves necessarily the coordination of several mathematical objects, and possesses a complexity, typical of the process of learning in general, that does not allow for the observation of a continuous and smooth course of development. Our interpretation is based (a) on the general methodological tool of focal analysis, developed by Sfard (2001), applied to the communication between student and teacher, including the interviews’ content analysis, and (b) on the theoretical framework of reflective abstraction, developed by Piaget (1980). Through the communication with students and the analysis of the data, using (a) and (b), the stages of knowledge, the concepts images, and the mental mechanism and operations of the students are gradually revealed. Understanding this mechanism will allow us to decide and distinguish whether the students come to a true understanding of the definition of the definite integral concept, as opposed to having just an empirical perception of integration, by which they can act effectively only in a limited and particular framework.
The methodology developed here may have a wider applicability in guiding our actions to help students develop advanced mathematical thinking. With appropriate modifications (regarding the instruction designing and the development of activities) of the methodology employed here, we may well be able to develop learning processes, by which the student is enabled to construct mathematical concepts, in general, and not just for the concept of the definite integral.

References


Sfard A. (2001). There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning. Educational Studies in Mathematics Vol. 46, pp 13–57.