RF04: THEORIES OF MATHEMATICS EDUCATION

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The purpose of this Forum is to stimulate critical debate in the area of theory use and theory development, and to consider future directions for the advancement of our discipline. The Forum opens with a discussion of why theories are essential to the work of mathematics educators and addresses possible reasons for why some researchers either ignore or misunderstand/misuse theory in their work. Other issues to be addressed include the social turn in mathematics education, an evolutionary perspective on the nature of human cognition, the use of theory to advance our understanding of student cognitive development, and models and modelling perspectives. The final paper takes a critical survey of European mathematics didactics traditions, particularly those in Germany and compares these to historical trends in other parts of the world.

INTRODUCTION

Our conception and preference for a particular mathematics education theory invariably influences our choice of research questions as well as our theoretical framework in mathematics education research. Although we have made significant advances in mathematics education research, our field has been criticized in recent years for its lack of focus, its diverging theoretical perspectives, and a continued identity crisis (Steen, 1999). At the dawn of this new millennium, the time seems ripe for our community to take stock of the multiple and widely diverging mathematical theories, and chart possible courses for the future. In particular, we need to consider the important role of theory building in mathematics education research.

Issues for consideration include:

1. What is the role of theory in mathematics education research?
2. How does Stokes (1997) model of research in science apply to research in mathematics education?
3. What are the currently accepted and widely used learning theories in mathematics education research? Why have they gained eminence?
4. What is happening with constructivist theories of learning?
5. Embodied cognition has appeared on the scene in recent years. What are the implications for mathematics education research, teaching, and learning?
6. Theories of models and modelling have received considerable attention in the field in recent years. What is the impact of these theories on mathematics research, teaching, and learning?

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 170-202. Melbourne: PME.

7. Is there a relationship between researchers' beliefs about the nature of mathematics and their preference for a particular theory?
8. How do theories used in European mathematics didactics traditions compare with those used in other regions of the world? Do European traditions reveal distinct theoretical trends?

There are several plausible explanations for the presence of multiple theories of mathematical learning, including the diverging, epistemological perspectives about what constitutes mathematical knowledge. Another possible explanation is that mathematics education, unlike "pure" disciplines in the sciences, is heavily influenced by cultural, social, and political forces (e.g., D'Ambrosio, 1999; Secada, 1995; Skovsmose & Valero, 2002). As Lerman indicates in his paper, the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasized a view of mathematics as a social product. Social constructivism, which draws on the seminal work of Vygotsky and Wittgenstein (Ernest, 1994) has been a dominant research paradigm ever since. On the other hand, cognitively oriented theories have emphasized the mental structures that constitute and underlie mathematical learning, how these structures develop, and the extent to which school mathematics curricula should capture the essence of workplace mathematics (e.g., see Stevens, 2000).

Stokes (1997) suggested a new way of thinking about research efforts in science, one that moves away from the linear one-dimensional continuum of "basic, to applied, to applied development, to technology transfer." Although this one-dimensional linear approach has been effective, Stokes argued that it is too narrow and does not effectively describe what happens in scientific research. In Pasteur's Quadrant, Stokes proposed a 2-dimensional model, which he claimed offered a completely different conception of research efforts in science. If one superimposes the Cartesian co-ordinate system on Stokes' model, the Y -axis represents "pure" research (such as the work of theoretical physicists) and the X-axis represents "applied" research" (such as the work of inventors). The area between the two axes is called "Pasteur's Quadrant" because it is a combination (or an amalgam) of the two approaches. If we apply Stokes' model to mathematics education research, we need to clearly delineate what is on the Y-axis of Pasteur's quadrant if we are to call our field a science. Frank Lester elaborates further on this issue in the opening paper of this Forum. Steve Lerman extends the discussion initiated in Lester's contribution on the pivotal, albeit misunderstood role of theories in mathematics education, and presents theoretical frameworks most frequently used in PME papers during the 1990-2001 time period. Lerman's analysis reveals that a wide variety of theories are used by PME researchers with a distinct preference for social theories over cognitive theories. An interesting avenue for discussion is whether the particular social theories used in this time period reveal a distinct geographic distribution, and if so why? Luis Moreno-Armella presents an evolutionary perspective on the nature of human cognition, particularly the evolution of representations, which he aptly terms pre-theory, as it serves as a foundation for the present discussion. John Pegg and David Tall compare neo-Piagetian theories in order to use the similarities and differences among theories to address fundamental questions in learning. Lyn English and Richard Lesh present a models and modeling perspective which innovatively combines the theories of Piaget and Vygotsky to pragmatically address the development and real life use of knowledge via model construction. The Forum concludes with a review by Günter Törner and Bharath Sriraman on European theories of mathematics education, with a focus on German traditions. Eight major tendencies are highlighted in 100 years of mathematics education history in Germany; these tendencies reflect trends that have occurred internationally.

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THE PLACE OF THEORY IN MATHEMATICS EDUCATION RESEARCH

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As most, if not all, of you know, the current emphasis in the United States being placed on so called *scientific research* in education, is driven in large part by political forces. Much of the public and some of the professional conversation has begun with an assumption that the purpose of research is to determine “what works,” and the discourse has focused largely on matters of research design and methods. One consequence has been the rekindling of attention to experimental designs and quantitative methods that had faded from prominence in education research over the past two decades or so. Far less prominent in recent discussions about educational research has been the place of theory.

Scholars in other social science disciplines (e.g., anthropology, psychology, sociology) often justify their research investigations on grounds of developing understanding by building or testing theories. In contrast, the current infatuation in the U.S. with "what works" seems to leave education researchers with less latitude to conduct studies to advance theoretical goals. It is time for a serious examination of the role that theory should play in the formulation of problems, in the design and methods employed, and in the interpretation of findings in education research. In this brief presentation, I speculate about why so many researchers seem to misunderstand or misuse theory and suggest how we might think about the goals of research that might help eliminate this misunderstanding and misuse.

Why is so much of our research atheoretical?

Mathematics education research is an interesting and important area for such an examination. Although math ed research was aptly characterized less than 15 years ago by Kilpatrick (1992) and others as largely *atheoretical*, a perusal of recent articles in major MER journals reveals that theory is alive and well: indeed, Silver and Herbst (2004) have noted that expressions such as "theory-based," "theoretical framework," and "theorizing" are common. In fact, they muse, manuscripts are often rejected for being atheoretical. The same is true of proposals submitted for PME meetings. However, the concerns raised decades ago persist; too often researchers ignore, misunderstand, or misuse theory in their work.

We are our own worst enemies

In my mind there are two basic problems that must be dealt with if we are to expect theory to play a more prominent role in our research. The first has to do with the widespread misunderstanding of what it means to adopt a theoretical stance toward our work. The second is that some researchers, while acknowledging the importance of theory, do not feel qualified to engage in serious theory-based work. I attribute both of these problems to: (a) the failure of our graduate programs to properly equip novice researchers with adequate preparation in theory, and (b) the failure of our research journals to insist that authors of research reports offer serious theory-based explanations of their findings.

Writing about the state of U.S. doctoral programs, Hiebert, Kilpatrick, and Lindquist (2001) suggest that mathematics education is a complex system and that improving the process of preparing doctoral students means improving the entire system, not merely changing individual features of it. They insist that "the absence of system-wide standards for doctoral programs [in mathematics education] is, perhaps, the most serious challenge facing systemic improvement efforts. . . . Indeed, participants in the system have grown accustomed to creating their own standards at each local site" (p. 155). One consequence of the absence of commonly accepted standards is that there is a very wide range of requirements of different programs. At one end of the continuum of requirements are a few programs that focus on the preparation of researchers. At the other end are those programs that require little or no research training beyond taking a research methods course or two. In general, with few exceptions, doctoral programs are replete with courses and experiences in research methodology, but woefully lacking in courses and experiences that provide students with solid theoretical underpinnings for future research. Without solid understanding of the role of theory in conceptualizing and conducting research, there is little chance that the next generation of mathematics education researchers will have a greater appreciation for theory than is currently the case. Put another way, we must do a better job of cultivating a predilection for theory within the mathematics education research community.

During my term as editor of the *Journal for Research in Mathematics Education* in the early1990s, I found the failure of authors of research reports to pay serious attention to explaining the results of their studies one of the most serious shortcomings. A simple example from the expert-novice problem solver research illustrates what I mean. It is not enough simply to report: Experts *do X when they solve problems and novices do Y.* Were the researcher guided by theory, a natural question would be to ask WHY? Having some theoretical perspective guiding the research provides a framework within which to attempt to answer *Why* questions. Without a theoretical orientation, the researcher can speculate at best or offer no explanation at all.

Many mathematics educators hold misconceptions about the role of theory

Time constraints prevent me from providing a detailed discussion of what I see as the most common misconceptions about theory, so I will simply list four and say a few words about them.

1. Theory-based explanation given by "decree" rather than evidence. Some researchers (e.g., Eisenhart, 1991) insist that educational theorists prefer to address and explain the results of their research by "theoretical decree" rather than with solid evidence to support their claims. That is to say, there is a belief among some researchers that theorists make their data fit their theory.

2. Data have to "travel." Sociologist and ethnographer, John Van Maanen (1988), has observed that data collected under the auspices of a theory has to "travel" in the sense that (in his view) data too often must be stripped of context and local meaning in order to serve the theory.

3. Standard for discourse not helpful in day-to-day practice. Related to the previous concern, is the observation that researchers tend to use a theory to set a standard for scholarly discourse that is not functional outside the academic discipline. Conclusions produced by the logic of theoretical discourse too often are not at all helpful in day-to-day practice. Researchers don't speak to practitioners! The theory is irrelevant to the experience of practitioners (cf., Lester & Wiliam, 2002).

4. No triangulation. Sociologist, Norman Denzin (1978) has discussed the importance of theoretical triangulation, by which he means the process of compiling currently relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses, and appropriateness, and using some subset of these perspectives and explanations as the focus of empirical investigation. By using a single theoretical perspective to frame one's research, such triangulation does not happen.

There is no doubt that rigid, uncritical adherence to a theoretical perspective can lead to these sorts of offenses. However, I know of no good researchers who are guilty of such crimes. Instead, more compelling arguments can be marshaled in support of using theory.

Why theory is essential

Again, time constraints for this presentation prevent me from elaborating on the reasons why theory should play an indispensable role in our research. Let me mention a few of the most evident. (In the following brief discussion I borrow heavily from an important paper written about 15 years ago by Andy diSessa [1991])

1. There are no data without theory. We have all heard the claim, "The data speak for themselves!" Dylan Wiliam and I have argued elsewhere that actually data have nothing to say. Whether or not a set of data can count as evidence of something is determined by the researcher's assumptions and beliefs as well as the context in which it was gathered (Lester & Wiliam, 2000). One important aspect of a researcher's beliefs is the theoretical perspective he or she is using; this perspective makes it possible to make sense of a set of data.

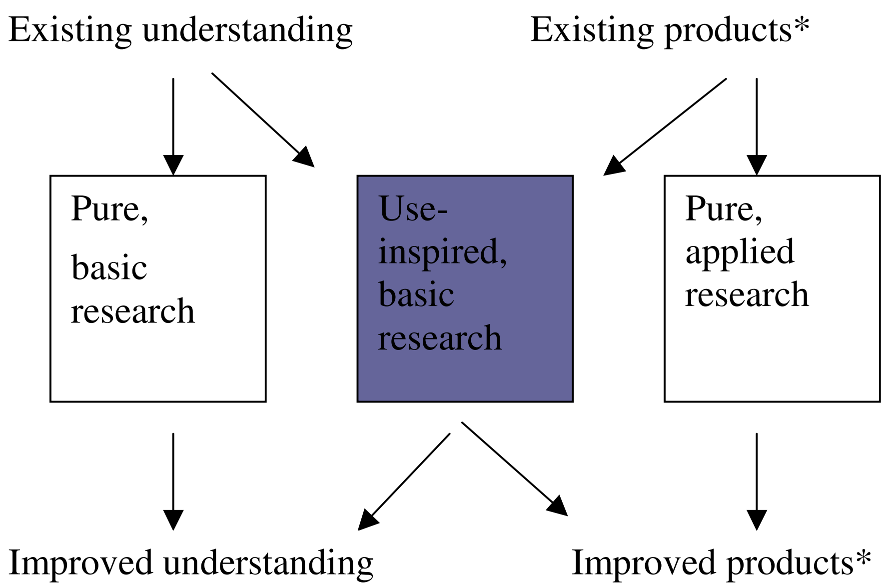
2. Good theory transcends common sense. In the paper mentioned above, diSessa (1991) argues that theoretical advancement is the linchpin in spurring practical progress. He notes that, sure, you don't need theory for many everyday problems—purely empirical approaches often are enough. But often things aren't so easy. Deep understanding that comes from concern for theory building is often essential to deal with truly important problems.

3. Need for deep understanding, not just "for this" understanding. Related to the above, is the need we have to deeply understand some things—the important, big questions (e.g., What does it mean to be intelligent? What does it mean to understand something?)—not simply find solutions to immediate problems and dilemmas. Theory helps us develop deep understanding. (I say more about understanding in the next section.)

A different way to think about the goals of research and the place of theory

In his book, *Pasteur's Quadrant: Basic Science and Technological Innovation,* Donald Stokes (1997) presents a new way to think about scientific and technological research and their purposes. Stokes begins with a detailed discussion of the history of development of the current U.S. policy for supporting advanced scientific study (I suspect similar policies exist in other industrialized countries). He notes that from the beginning of the development of this policy shortly after World War II there has been an inherent tension between the pursuit of *fundamental understanding* and *considerations of use*. This tension is manifest in the, often radical, separation between basic and applied science. He argues that prior to the latter part of the 19^{th} Century, scientific research was conducted largely in pursuit of deep understanding of the world. But, the rise of microbiology in the late 19th Century brought with it a concern for putting scientific understanding to practical use. He illustrates this concern with the work of Louis Pasteur. Of course, Pasteur working in his laboratory wanted to understand the process of disease at the most basic level, but he wanted that understanding to be applicable to dealing with silk worms, anthrax in sheep, cholera in chickens, spoilage in milk, and rabies in people. The work of Pasteur suggests that one could not understand his science without knowing the extent to which he had considerations of use in mind as well as fundamental understanding. Stokes proposed a model for thinking about scientific research that blends the two motives: the *quest for fundamental understanding* and *considerations of use*.

Adapting Stokes's model to educational research in general, and mathematics education research in particular, I have come up with a slightly different model (see Figure 1). In the final section of this short paper, I describe the relationship between my model and the place of theory in mathematics education research.



* "Products" include such things as instructional materials, professional development programs, and district educational policies.

Figure 1. Adaptation of Stokes's model to educational research

A *bricolage* approach to theory in mathematics education research

Even if there is no need to make a case for the importance of theory in our research, there is a need to suggest how researchers, especially novices, can deal with the almost mystifying range of theories and theoretical perspectives that are being used. In a chapter to appear in a forthcoming handbook of research in mathematics education, Cobb (in press) considers how mathematics education researchers might cope with the multiple and frequently conflicting theoretical perspectives that currently exist. He observes:

The theoretical perspectives currently on offer include radical constructivism, sociocultural theory, symbolic interactionism, distributed cognition, information-processing psychology, situated cognition, critical theory, critical race theory, and discourse theory. To add to the mix, experimental psychology has emerged with a renewed vigor in the last few years. . . . In the face of this sometimes bewildering array of theoretical alternatives, the issue . . . is that of how we might make and justify our decision to adopt one theoretical perspective rather than another.[1]

Cobb goes on to question the repeated (mostly unsuccessful) attempts that have been made in mathematics education to derive instructional prescriptions directly from background theoretical perspectives. He insists that it is more productive to compare and contrast various theoretical perspectives in terms of the manner in which they orient and constrain the types of questions that are asked about the learning and teaching of mathematics, the nature of the phenomena that are investigated, and the forms of knowledge that are produced. To his recommendation, I would add that comparing and contrasting various perspectives would have the added benefit of both enhancing our understanding of important phenomena and increasing the usefulness of our investigations (c.f., Lester & Wiliam, 2002).

I propose to view the theoretical perspectives we adopt for our research as sources of ideas that we can appropriate and modify for our purposes as mathematics educators. This process of developing tools for our research is quite similar to that of instructional design as described by Gravemeijer (1994). He suggests that instructional design resembles the thinking process characterized by the French word *bricolage,* a notion borrowed from Claude Levi–Strauss. A *bricoleur* is a handyman who invents pragmatic solutions in practical situations and is adept at using whatever is available. Similarly, I suggest, as do Cobb and Gravemeijer, that rather than adhering to one particular theoretical perspective, we act as *bricoleurs* by adapting ideas from a range of theoretical sources to suit our goals—goals that should aim not only to deepen our fundamental understanding of mathematics learning and teaching, but also to aid us in providing practical wisdom about problems practitioners care about. If we begin to pay serious attention to these goals, the problem of theory is likely to be resolved.

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[1] Because Cobb's paper is currently in draft form and is not yet publicly available, no page numbers are provided.

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THEORIES OF MATHEMATICS EDUCATION: A PROBLEM OF PLURALITY?

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Today, in many countries around the world, constraints on the funding of Universities are leading to restrictions on educational research. In some countries national policy is also placing constraints on the kinds of research that will be funded (e.g. the effects of *No Child Left Behind* policy in the USA). At the same time we see research in mathematics education proliferating, not just in quantity but also, as in the concerns of this Research Forum, in the range of theories that are drawn upon in our research. In my contribution I want to ask: is this surprising, or unusual, and is it necessarily a hindrance to the effectiveness of educational research in mathematics?

In discussing this, I would argue that we need a specific language that enables an analysis of intellectual fields and their growth, a language that will not be provided by mathematics or by psychology. I will draw on some of the later work of the sociologist of education, Basil Bernstein, in particular his 1999 paper on research discourses (Bernstein, 1999). Following that, I will make some remarks about the use of theory.

A Language of Research Fields

Bernstein draws on two notions: hierarchy and verticality. Discourses are described as hierarchical where knowledge in the domain is a process of gradual distancing, or abstraction, from everyday concepts. Hierarchical discourses require an apprenticeship; they position people as initiated or apprenticed. Clearly academic and indeed school mathematics are examples of hierarchical discourses. Research (Cooper & Dunne, 2000) shows that setting mathematics tasks in everyday contexts can mislead some students, namely those from low socio-economic background, into privileging the everyday context and the meanings carried in them over the abstract or esoteric meanings of the discourse of academic mathematics.

His second notion, verticality, describes the extent to which a discourse grows by the progressive integration of previous theories, what he calls a vertical knowledge structure, or by the insertion of a new discourse alongside existing discourses and, to some extent, incommensurable with them. He calls these horizontal knowledge structures. Bernstein offers science as an example of a vertical knowledge structure and, interestingly, both mathematics and education (and sociology) as examples of horizontal knowledge structures. He uses a further distinction that enables us to separate mathematics from education: the former has a strong grammar, the latter a weak grammar, that is, with a conceptual syntax not capable of generating unambiguous empirical descriptions. Both are examples of hierarchical discourses in that one needs to learn the language of linear algebra or string theory just as one needs to learn the language of radical constructivism or embodied cognition. It will be obvious that linear algebra and string theory have much tighter and specific concepts and hierarchies of concepts than radical constructivism or embodied cognition. Adler and Davis (forthcoming) point out that a major obstacle in the development of accepted knowledge in mathematics for teaching may well be the strength of the grammar of the former and the weakness of the latter. Where we can specify accepted knowledge in mathematics, knowledge about teaching is always disputed.

As a horizontal knowledge structure, then, it is typical that mathematics education knowledge will grow both within discourses and by the insertion of new discourses in parallel with existing ones. Thus we can find many examples in the literature of work that elaborates the functioning of the process of reflective abstraction, as an instance of the development of knowledge within a discourse. But the entry of Vygotsky's work into the field in the mid-1980s (Lerman, 2000) with concepts that differed from Piaget's did not lead to the replacement of Piaget's theory (as the proposal of the existence of oxygen replaced the phlogiston theory). Nor did it lead to the incorporation of Piaget's theory into an expanded theory (as in the case of non-Euclidean geometries). Indeed it seems absurd to think that either of these would occur precisely because we are dealing with a social science, that is, we are in the business of interpretation of human behaviour. Whilst all research, including scientific research, is a process of interpretation, in the social sciences, such as education, there is a double hermeneutic (Giddens, 1976) since the 'objects' whose behaviour we are interpreting are themselves trying to make sense of the world.

Education, then, is a social science, not a science. Sociologists of scientific knowledge (Kuhn, Latour) might well argue that science is more of a social science than most of us imagine, but social sciences certainly grow both by hierarchical development but especially by the insertion of new theoretical discourses alongside existing ones. Constructivism grows, and its adherents continue to produce novel and important work; models and modelling may be new to the field but already there are novel and important findings emerging from that orientation.

I referred above to the incommensurability, in principle, of these parallel discourses. Where a constructivist might interpret a classroom transcript in terms of the possible knowledge construction of the individual participants, viewing the researcher's account as itself a construction (Steffe & Thompson, 2000), someone using socio-cultural theory might draw on notions of a zone of proximal development. Constructivists might find that describing learning as an induction into mathematics, as taking on board concepts that are on the intersubjective plane, incoherent in terms of the theory they are using (and a similar description of the reverse can of course be given). In this sense, these parallel discourses are incommensurable.

There is an apparent contradiction between the final sentences of the last two paragraphs. If I am claiming that there is important work emerging in different discourses of mathematics education research, but I also claim that discourses are incommensurable, within which discourse am I positioning myself to write these sentences? Is there a meta-discourse of mathematics education in which we can look across these theories? I will make some remarks about this position in the next section.

Theories in Use in Mathematics Education

First I will make some remarks drawn from a recent research project on the use of theories in mathematics education. Briefly we (Tsatsaroni, Lerman & Xu, 2003) examined a systematic sample of the research publications of the mathematics education research community between 1990 and 2001, using a tool that categorised research in many ways. I will only refer here to our findings concerning how researchers use theories in their work as published in PME Proceedings.

Our analysis showed that just over 85% of all papers in the proceedings had an orientation towards the empirical, with a further 5% moving from the theoretical to the empirical, and this has changed little over the years. A little more than three-quarters are explicit about the theories they are using in the research reported in the article. Again this has not varied across the years. The theories that are used have changed, however. We can notice an expanding range of theories being used and an increase in the use of social theories, based on the explicit references of authors, in some cases by referring to a named authority. These fields or names represent theories used, not the frequency of their occurrence in papers.

Year	Theoretical fields other than educational psychology and/or mathematics
1990	Brousseau
1991	Philosophy of mathematics
1992	Vygotsky
1993	Vygotsky
1994	Brousseau, Chevellard, Poststructuralism
1995	Embodied cognition, Educational research
1996	Vygotsky, Situated cognition, Philosophy of mathematics
1997	Situated cognition, Vygotsky, Philosophy of mathematics
1998	Situated cognition, Vygotsky, Philosophy of mathematics
1999	Socio-historical practice
2000	Chevellard
2001	Semiotics, Bourdieu, Vygotsky, Philosophy

Table 1: Theoretical fields

We might suggest that there is a connection here with creating identities, making a unique space from which to speak in novel ways, but we would need another study to substantiate and instantiate this claim.

We can say that there has been a substantial increase in the number of fields from 1994, although it is too early to say whether this trend will continue, as 1999 and 2000 showed a dropping off. What is clear is that the range of intellectual resources, including sociology, philosophy, semiotics, anthropology, etc., is very broad.

In our analysis of how authors have used theories we have looked at whether, after the research, they have revisited the theory and modified it, expressed dissatisfaction with the theory, or expressed support for the theory as it stands. Alternatively, authors may not revisit the theory at all, content to apply it in their study. We have found that more than three-quarters fall into this last category, just over 10% revisit and support the theory, whilst four percent propose modifications. Two authors in our sample ended by opposing theory. This pattern has not changed over the years. Further findings can be found in Tsatsaroni, Lerman and Xu (2003).

The development and application of an analytical tool in a systematic way, paying attention to the need to make explicit and open to inspection the ways in which decisions on placing articles in one category or another, enables one to make all sorts of evidence-based claims. In particular, I would argue that one can observe and record development within discourses and the development of new parallel discourses because of the adoption of a sociological discourse as a language for describing the internal structure of our intellectual field, mathematics education research.

Conclusion

Finally, I will comment on concerns about the effectiveness of educational research in a time of multiple and sometimes competing paradigms, described here as discourses. 'Effectiveness' is a problematic notion, although one that certainly figures highly in current discourses of accountability. It arises because by its nature education is a research field with a face towards theory and a face towards practice. This contrasts with fields such as psychology in which theories and findings can be applied, but practice is not part of the characteristic of research in that field. Research in education, in contrast, draws its problems from practice and expects its outcomes to have applicability or at least significance in practice. Medicine and computing are similar intellectual fields in this respect.

However, what constitutes knowledge is accepted or rejected by the criteria of the social field of mathematics education research. Typically, we might say necessarily, research has to take a step away from practice to be able to say something about it. Taking the results of research into the classroom calls for a process of recontextualisation, a shift from one practice into another in which a selection must take place, allowing the play of ideology. To look for a simple criterion for acceptable research in terms of 'effectiveness' is to enter into a complex set of issues.

Indeed 'effectiveness' itself presupposes aims and goals for, in our case, mathematics education. To ignore the complexity is to lose the possibility of critique and hence I am not surprised by the multiplicity of theories in our field and the debates about their relative merits, nor do I see it as a hindrance.

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THE ARTICULATION OF SYMBOL AND MEDIATION IN MATHEMATICS EDUCATION

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I describe some basic elements of a *pre-theory* of Mathematics Education. Our field is at the crossroad of a science, mathematics, and a community of practice, education. The interests of this community include the people whose learning takes place at schools and the corresponding intellectual offer from the institutional sides. But as soon as we enter the space of mathematics, we discover a different discipline from the natural sciences. It is the strictly *symbolic nature* of mathematics that makes a big difference and gives to mathematics education, as a research field, its characteristic features that distinguishe it from similar endeavours with respect to other scientific fields, such as biology for instance. I am not implying, of course, that there is no abstraction or concept development involved in those other fields.

More recently, the presence of computers has introduced a new way of looking at symbols and mathematical cognition and has offered the potentiality to re-shape the goals of our whole research field. The urgency to take care of teaching and learning from the research activities has resulted in practices without corresponding theories. Again, I must make clear I am not dismissing the considerable and important results this community has produced. I simply want to underline that institutional pressures can result more frequently than desirable, in losing track of research goals. Perhaps this is a motive to re-consider the need to enter a more organized level of reflection in our community. There is nothing bad in having the chance to look at educational phenomena from different viewpoints but it is better if we can generate a synergy between those viewpoints that, eventually, has as its output a new and stronger theory. Nevertheless the tension between the local and the global also comes to existence here. Being an interested observer and modest participant in the field, I have come to think that only local explanations are possible in our field. *Local theories* might be the answer to the plethora of explanations we encounter around us. But even if local, a mathematics education theory must be developed from scaffolding that eventually crystallizes in the theory. In our case, part of that scaffolding is constituted by mathematics itself, and by a community of practice, as already mentioned.

What sort of machine is the human brain, that it can give birth to mathematics? – an old question that Stanislas Dehaene has aptly posed anew in his book *The Number Sense* (1997). This is the kind of question that, in the long run, must be answered in order to improve the understanding of our field. Nevertheless, trying to answer it will demand an interdisciplinary and longitudinal effort. At the end of the day, we will need to understand why we are able to create symbolic worlds (mathematics, for instance) and why our minds are essentially incomplete outside the co-development with material and symbolic technologies. Our symbolic and mediated nature comes to the front as soon as we try to characterize our intellectual nature. Evolution and culture have left its traits in our cognition, in particular, in our capacity to duplicate the world at the level of symbols.

Diverging epistemological perspectives about what constitutes mathematical knowledge modulate multiple conceptions of learning and the present theories of what constitutes mathematical education as a research discipline. Today, however, there is substantial evidence that the encounter between the conscious mind and distributed cultural systems has altered human cognition and has changed the tools with which we think. The origins of writing and how writing as a technology changed cognition is key from this perspective (Ong, 1988). These examples suggest the importance of studying the evolution of mathematical systems of representation as a vehicle to develop a proper epistemological perspective for mathematics education.

Human evolution is coextensive with tool development. In a certain sense, human evolution has been an *artificial* process as tools were always designed with the explicit purpose of transforming the environment. And so, since about 1.5 millions years ago, our ancestor Homo Erectus designed the first stone tools and took profit from his/her voluntary memory and gesture capacities (Donald, 2001) to evolve a pervasive technology used to consolidate their early social structures. The increasing complexity of tools demanded optimal coherence in the use of memory and in the transmission, by means of articulate gestures, of the building techniques. We witness here what is perhaps the first example of deliberate teaching. Voluntary memory enabled our ancestors to engender a mental template of their tools. Templates lived in their minds, resulting from activity, granting an objective existence as abstract objects even before they were *extracted* from the stone. Thus, tool production was not only important for plain survival, but also for broadening the mental world of our ancestors –introducing a higher level of objectivity.

The actions of our ancestors were producing a *symbolic* version of the world: A world of intentions and anticipations they could imagine and *crystallize* in their tools. What their tools meant was the same as what they *intended* to do with them. They could *refer* to their tools to *indicate* their *shared* intentions and, after becoming familiar with those tools, they were looked as *crystallized* images of all the activity that was embedded in them.

We suggest that the synchronic analysis of our relationship with technology, no matter how deep, hides profound meanings of this relationship that coheres with the co-evolution of man and his tools. It is then, unavoidable, to revisit our technological past if we want to have an understanding of the present. Let us present a substantial example.

Arithmetic: Ancient Counting Technologies

Evidence of the construction of one–to–one correspondences between arbitrary collections of concrete objects and a *model set* (a template) can already be found between 40000 and 10000 B.C. For instance, hunter-gatherers used bones with marks (tallies). In 1937, a wolf bone dated to about 30000 B.C. was found in Moravia (Flegg, 1983). This reckoning technique (using a one-to-one correspondence) reflects a deeply rooted trait of human cognition. Having a set of stone bits or the marks on a bone as a *modeling set* constitutes, up to our knowledge, the oldest counting technique humans have designed. The modeling set plays, in all cases, an instrumental role for the whole process. In fact, something is crystallized by marking a bone: The *intentional* activity of finding the size of a set of hunted pieces, for instance, or as some authors have argued, the intentional activity of computing time.

The modeling set of marks, plays a role similar to the role played by a stone tool as both mediate an activity, finding the size, and both crystallize that activity. Between 10000 and 8000, B.C. in Mesopotamia, people used sets of pebbles (clay bits) as modeling sets. This technique was inherently limited. If, for instance, we had a collection of twenty pebbles as modeling set then, it would be possible to estimate the size of collections of twenty or less elements. Nevertheless, to deal with larger collections (for instance, of a hundred or more elements), we would need increasingly larger models with evident problems of manipulation and maintenance. And so, the embodiment of the one-to-one technique in the set of pebbles inhibits the extension of it to further realms of experience. It is very plausible that being conscious of these difficulties, humans looked for alternative strategies that led them to the brink of a new technique: the idea that emerged was to replace the elements of the model set with clay pieces of diverse shapes and sizes, *whose numerical value were* *conventional*. Each piece *compacted* the information of a whole former set of simple pebbles —according to its shape and size. The pieces of clay can be seen as embodiments of pre-mathematical symbols. Yet, they lacked rules of transformation that allowed them to constitute a genuine mathematical system.

Much later, the consolidation of the urbanization process (about 4000 B.C.) demanded, accordingly, more complex symbol systems. In fact, the history of complex arithmetic signifiers is almost determined by the occurrence of bullae. These clay envelopes appeared around 3500-3200 B.C. The need to record commercial and astronomic data led to the creation of symbol systems among which mathematical systems seem to be one of the first. The counters that represented different amounts and sorts —according to shape, size, and number— of commodities were put into a bulla which was later sealed. And so, to secure the information contained in a bulla, the shapes of the counters were printed on the bulla outer surface. Along with the merchandise, producers would send a bulla with the counters inside, describing the goods sent. When receiving the shipment, the merchant could verify the integrity of it.

A counter in a bulla *represents* a *contextual* number — for example, the number of sheep in a herd; not an abstract number: there is five of something, but never *just five*. The shape of the counter is impressed in the outer surface of the bulla. The mark on the surface of the bulla *indicates* the counter inside. That is, the mark on the surface keeps an *indexical relation* with the counter inside as its referent. And the counter inside has a *conventional* meaning with respect to amounts and commodities. It must have been evident, after a while, that *counters inside were no longer needed;* impressing them in the outside of the bulla was enough. That decision altered the semiotic status of those external inscriptions. Afterward, instead of impressing the counters against the clay, scribes began using sharp styluses that served *to draw* on the clay the shapes of former counters. From this moment on, the symbolic expression of numerical quantities acquired an infra-structural support that, at its time, led to a new epistemological stage of society. Yet the semiotic contextual constraints, made evident by the simultaneous presence of diverse numerical systems, was an epistemological barrier for the mathematical evolution of the *numerical ideographs*. Eventually, the collection of numerical (and contextual) systems was replaced by one system (Goldstein, La naissance du nombre en Mesopotamie. La Recherche, *L'Univers des Nombres* (hors de serie),1999). That system was the sexagesimal system that also incorporated a new symbolic technique: numerical value according to position. In other words, it was a *positional* system. There is still an obstacle to have a complete numerical system: the presence of zero that is of primordial importance in a positional system to eliminate representational ambiguities. For instance, without zero, how can we distinguish between 12 and 102? We would still need to look for the help of context.

Mathematical objects result from a sequence of crystallization processes that, at a certain level of evolution, has an ostensible social and cultural dimension. As the levels of reference are hierarchical the crystallization process is a kind of recursive process that allows us to state:

Mathematical symbols co-evolve with their mathematical referents and the induced semiotic objectivity makes possible for them to be taken as shared in a community of practice.

In what follows, we should try to articulate some reflections regarding the presence of the computational technologies in mathematical thinking. It is interesting to notice that even if the new technologies are not yet fully integrated within the mathematical universe, their presence will eventually erode the mathematical way of thinking. The blending of mathematical symbol and computers has given way to an *internal mathematical universe* that works as the reference fields to the mathematical signifiers living in the screens of computers. This takes abstraction a large step further.

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USING THEORY TO ADVANCE OUR UNDERSTANDINGS OF STUDENT COGNITIVE DEVELOPMENT

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INTRODUCTION

Over recent years, various theories have arisen to explain and predict cognitive development in mathematics education. Our focus is to raise the debate beyond a simple comparison of detail in different theories to move to use the similarities and differences among theories to address fundamental questions in learning. In particular, a focus of research on fundamental learning cycles provides an empirical basis from which important questions concerning the learning of mathematics can and should be addressed.To assist us with this focus we identify two kinds of theory of cognitive growth:

- **global theories of long-term growth** of the individual, such as the stage-theory of Piaget (e.g., Piaget & Garcia, 1983).
- **local theories of conceptual growth** such as the action-process-object-schema theory of Dubinsky (Czarnocha et al., 1999) or the unistructural-multistructural-relational-extended abstract sequence of SOLO Model (**S**tructure of **O**bserved **L**earning **O**utcomes, Biggs & Collis, 1982, 1991; Pegg, 2003).

Some theories (such as that of Piaget, the SOLO Model, or more broadly, the enactive-iconic-symbolic theory of Bruner, 1966) incorporate both aspects. Others, such as the embodied theory of Lakoff and Nunez (2000) or the situated learning of Lave and Wenger (1990) paint in broader brush-strokes, featuring the underlying biological or social structures involved. A range of global longitudinal theories each begin with physical interaction with the world and, through the use of language and symbols, become increasingly abstract. Table 1 shows four of these theoretical developments.

Piaget Stages	van Hiele Levels (Hoffer,1981)		SOLO Modes	Bruner Modes
Sensori Motor	I	Recognition	Sensori Motor	Enactive
Preoperational	II	Analysis	Ikonic	Iconic
Concrete Operational	III	Ordering	Concrete	Symbolic
Formal Operational	IV	Deduction	Symbolic	
	V	Rigour	Formal	
			Post-formal	

Table 1: Global stages of cognitive development

What stands out from such 'global' perspectives is the gradual biological development of the individual, growing from dependence on sensory perception through physical interaction and on, through the use of language and symbols, to increasingly sophisticated modes of thought. SOLO offers a valuable viewpoint as it explicitly nests each mode within the next, so that an increasing repertoire of more sophisticated modes of operation become available to the learner. At the same time, all modes attained remain available to be used as appropriate. As we go on to discuss fundamental cycles in conceptual learning, we therefore need to take account of the development of modes of thinking available to the individual.

LOCAL CYCLES

Our current focus is on 'local' theories, formulated within a 'global' framework whereby the cycle of learning in a specific conceptual area is related to the overall cognitive structures available to the individual. A recurring theme identified in these theories is a fundamental cycle of growth in the learning of specific concepts, which we frame within broader global theories of individual cognitive growth.

One formulation is found in SOLO. This framework can be considered under the broad descriptor of neo-Piagetian models. It evolved as reaction to observed inadequacies in Piaget's formulations and shares much in common with the ideas of such theorists as Case, Fischer, and Halford.

In particular, SOLO focuses attention upon students' responses rather than their level of thinking or stage of development. It arose, in part, because of the substantial décalage problem associated with Piaget's work when applied to the school learning context, and the identification of a consistency in the structure of responses from large numbers of students across a variety of learning environments in a number of subject and topic areas. While SOLO has its roots in Piaget's epistomelogical tradition, it is based strongly on information-processing theories and the importance of working memory capacity. In addition, familiarity with content and context invariably plays an influential role in determining the response category.

At the 'local' focus SOLO comprises a recurring cycle of three levels referred to as *unistructural, multistructural, and relational* (a UMR cycle). The application of SOLO takes a multiple-cycle form of at least two UMR cycles in each mode where the R level response in one cycle evolves to a new U level response in the next cycle. This not only provides a basis to explore how basic concepts are acquired, but it also provides us with a description of how students react to reality as it presents itself to them. The second cycle then offers the type of development that is most evident and a major focus of primary and secondary education.

Another formulation concerns various theories of process-object encapsulation, in which processes become interiorised and then conceived as mental concepts, which has been variously described as *action*, *process*, *object* (Dubinsky), *interiorization*, *condensation*, *reification* (Sfard) or *procedure*, *process*, *concept* (Gray & Tall).

Theories of 'process-object encapsulation' were formulated at the outset to describe a sequence of cognitive growth. Each of these theories, founded essentially on the ideas of Piaget, saw cognitive growth through actions on existing objects that become interiorized into processes and then encapsulated as mental objects.

Dubinsky described this cycle as part of his APOS theory (action-process-object-schema), although he later asserted that objects could also be formed by encapsulation of schemas as well as encapsulation of processes. Sfard (1991) proposed an operational growth through a cycle she termed interiorization-condensation-reification, which she complemented by a 'structural' growth that focuses on the properties of the reified objects formed in an operational cycle.

Gray and Tall (1994) focused more on the role of *symbols* acting as a pivot, switching from a process (such as addition of two numbers, say 3+4) to a concept (the sum 3+4, which is 7). The entity formed by a symbol and its pivotal link to *pro*cess or con*cept* they named a *procept*. They observed that the growth of procepts occurred often (but not always) through a sequence that they termed procedure-process-procept. In this model a procedure is a sequence of steps carried out by the individual, a process is where a number of procedures (≥0) giving the same input-output are regarded as the same process, and the symbol shared by both becomes process or concept.

The various process-object theories have a spectrum of development from process to object. The process-object theories of Dubinsky and Sfard were mainly based on experiences of students doing more advanced mathematical thinking in late secondary school and at university. For this reason their emphasis is on formal development rather than on earlier acquired forms of thinking such as associated with Piaget's sensori-motor or pre-operational stages. Note too that Sfard's first state is referred to as an 'interiorized process', which is the same name given in Dubinsky's second, however, both see the same main components of the second stage:– that the process is seen as a whole without needing to perform the individual steps.

We now turn to the cycles of development that occur within a range of different theories. These have been developed for differing purposes. The SOLO Model, for instance, is concerned with assessment of performance through observed learning outcomes. Other theories, such as those of Davis (1984), Dubinsky (Czarnocha et al., 1999), Sfard (1991), and Gray and Tall (1994) are concerned with the sequence in which the concepts are constructed by the individual).

SOLO of Biggs & Collis	Davis	APOS of Dubinsky	Gray & Tall
Unistructural Multistructural Relational Unistructural	Procedure (VMS) Integrated Process Entity	Action Process Object Schema	[Base Objects] Procedure Process Procept

Table 2: Local cycles of cognitive development

As can be seen from table 2, there are strong family resemblances between these cycles of development. Note that Davis used the term 'visually moderated sequence' for a step-by-step procedure. Although a deeper analysis of the work of individual authors will reveal discrepancies in detail, there are also insights that arise as a result of comparing one theory with another as assembled in table 3.

<table>
<tr><th>SOLO</th><th>Davis</th><th>APOS</th><th>Gray & Tall</th></tr>
<tr><td rowspan="2">Unistructural</td><td rowspan="3">VMS
Procedure</td><td></td><td>Base Object(s)</td></tr>
<tr><td rowspan="2">Action</td><td rowspan="2">Procedure
[Multi-Procedure]</td></tr>
<tr><td>Multistructural</td></tr>
<tr><td>Relational</td><td>Process</td><td>Process</td><td>Process</td></tr>
<tr><td>Unistructural
(Extended
Abstract)</td><td>Entity</td><td>Object
Schema</td><td>Procept</td></tr>
</table>

Table 3: The fundamental cycle of conceptual construction

CONCLUSION

Our purpose in this brief paper is not so much to attempt to produce a unified theory incorporating these perspectives. Instead, it is to advocate an approach that seeks to understand the meanings implicit in each broad theory and to see where each may shed light on the other, leading to theoretical correspondences and dissonances.

While at first glance there may appear to be irreconcilable differences between the theoretical stances (e.g., van Hiele is concerned with underlying thinking skills and SOLO with observable behaviours), a closer examination can reveal there is much to consider. A synthesis provides a fresh perspective in considering student growth in understanding.

A primary goal of teaching should be to stimulate cognitive development in students. Such development as described by these fundamental learning cycles is not inevitable. Ways to stimulate growth, to assist with the reorganisation of earlier levels need to be explored. Important questions about strategies appropriate for different levels or even if it is true that *all* students pass through all levels in sequence. Research into such questions is sparse. Nevertheless, the notion of fundamental cycles of learning does provide intriguing potential for research.

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TRENDS IN THE EVOLUTION OF MODELS & MODELING PERSPECTIVES ON MATHEMATICAL LEARNING AND PROBLEM SOLVING

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Models and modeling (M&M) research often investigates the nature of understandings and abilities that are needed in order for students to be able to use what they have (presumably) learned in the classroom in "real life" situations beyond school. Nonetheless, M&M perspectives evolved out of research on concept development more than research on problem solving; and, rather than being preoccupied with the kind of word problems emphasized in textbooks and standardized tests, we focus on (simulations of) problem solving "in the wild." Also, we give special attention to the fact that, in a technology-based *age of information*, significant changes are occurring in the kinds of "mathematical thinking" that is coming to be needed in the everyday lives of ordinary people in the 21st century – as well as in the lives of productive people in future-oriented fields that are heavy users of mathematics, science, and technology.

In modern knowledge economies, systems – ranging from communication systems to economic or accounting systems - are among the most important "things" that impact the lives of ordinary people. Some of these systems occur naturally, while others are created by humans. But, in any case, mathematics is useful for making (or making sense of) such systems precisely because mathematics is the study of structure. That is, it is the study of systemic properties of structurally interesting systems.

In future-oriented fields that range from design sciences to life sciences, industry advisors to university programs consistently emphasize that:

> The kind of people we most want to hire are those who are proficient at (a) making sense of complex systems, (b) working within teams of diverse specialists, (c) adapting rapidly to a variety of rapidly evolving conceptual tools, (d) working on multi-staged projects that require planning and collaboration among many levels and types of participants, and (e) developing sharable and re-useable conceptual tools that usually need to draw on a variety of disciplines – and textbook topic areas.

Both of the preceding trends shift attention beyond *mathematics as computation* toward *mathematics as conceptualization, description, and explanation*. But, they also raise the following kinds of questions that lie at the heart of M&M research in mathematics education.

- What is the nature of the most important classes of problem-solving situations where mathematics, science, and technology are needed for success in real life situations beyond school?
- What mathematical constructs or conceptual systems provide the best foundations for success in these situations?
- What does it mean to “understand” these constructs and conceptual systems?
- How do these understandings develop?
- What kinds of experiences facilitate (or retard) development?
- How can people be identified whose exceptional abilities do not fit the narrow and shallow band of abilities emphasized on standardized tests – or even school work?

Related questions are: (a) Why do students who have histories of getting A’s on tests and coursework often do not do well beyond school? (b) What is the relationship between the learning of “basic skills” and a variety of different kinds of deeper or higher-order understandings or abilities? (c) Why do problem solving situations that involve collaborators and conceptual tools tend to create as many conceptual difficulties as they eliminate? (d) In what ways is “mathematical thinking” becoming more multi-media - and more contextualized (in the sense that knowledge and abilities are organized around experience as much as around abstractions, and in the sense that relevant ways of thinking usually need to draw on ways for thinking that seldom fall within the scope of a single discipline or textbook topic area). (e) How can instruction and assessment be changed to reflect the fact that, when you recognize the importance of a broader range of understandings and abilities, a broader range of people often emerge as having exceptional potential?

M&M perspectives assume that such questions should be investigated through research, not simply resolved though political processes - such as those that are emphasized when “blue ribbon” panels of experts develop curriculum standards for teaching or testing. Furthermore, we believe that such questions are not likely to be answered through content-independent investigations about *how people learn* or *how people solve problems*, and they are only indirectly about the nature (and/or the development) of humans - or the functioning of human brains. This is because they are about the nature of mathematical and scientific knowledge, and they are about the ways this knowledge is useful in “real life” situations. So, researchers with broad and deep expertise in mathematics and science should play significant roles in collaborating with experts in the learning and cognitive sciences.

Theoretical perspectives for M&M research trace their lineage to modern descendents of Piaget and Vygotsky - but also (and just as significantly) to *American Pragmatists* such as William James, Charles Sanders Peirce, Oliver Wendell Holmes, George Herbert Mead, and John Dewey. And partly for this reason, M&M perspectives reflect “blue collar” approaches to research. That is, we focus on the development of knowledge (and conceptual tools) to inform “real life” decision-making issues – where (a) the criteria for success are not contained within any preconceived theory, (b) productive ways of thinking usually need to draw on more than a single theory, and (c) useful knowledge usually needs to be expressed in the context of conceptual tools that are powerful (for some specific purpose), sharable (with other people), and re-useable (beyond the context in which they were developed). Thus, M&M research often focuses on model-development rather than proceeding too quickly to theory development and hypothesis testing; and, before rushing ahead to try to teach or test various mathematical concepts, processes, beliefs, habits of mind, or components of a productive problem solving personae, we conduct developmental investigations about the nature of what it means to “understand” them.

One way that mathematics educators have investigated questions about what is needed for success beyond school is by observing people “thinking mathematically” in everyday situations. Sometimes, such studies compare “experts” with “novices” who are working in fields such as engineering, agriculture, medicine, or business management - where “mathematical thinking” often is critical for success. Such ethnographic investigations often have been exceedingly productive and illuminating. Nonetheless, from the perspectives of M&M research, they also tend to have some significant shortcomings. For example, we must be skeptical of observations which depend heavily on preconceived notions about <u>where</u> to observe (in grocery stores? carpentry shops? car dealerships? engineering firms? Internet cafés?), <u>whom</u> to observe (street vendors? shoppers? farmers? cooks? engineers? baseball fans?), <u>when</u> to observe (when they’re estimating sizes? calculating with numbers? minimizing routes? describing, explaining, or predicting the behaviors of complex systems?), and <u>what</u> to count as “mathematical thinking” (e.g., planning, monitoring, assessing, explaining, justifying steps during multi-step projects, or deciding what information to collect about specific decision-making issues). Consequently, in simple observational studies, close examinations of underlying assumptions often expose unwarranted prejudices about what it means to “think mathematically” - and about the nature of “real life” situations in which mathematics is useful.

A second way to investigate *what’s needed for success beyond school* is to use *multi-tier design experiments* (Lesh, 2002) in which (a) students develop models for making senses of mathematical problem solving situations, (b) teachers develop models for creating (and making sense of) students' modeling activities, and (c) researchers develop models for creating (or making sense of) interactions among students, teachers, and relevant learning environments. We sometimes refer to such studies as *evolving expert studies* (Lesh, Kelly & Yoon, in press) because the final products that are produced tend to represent significant extensions or revisions in the thinking of each of the participants who were involved. Such methodologies respect the opinions of diverse groups of stake holders whose opinions should be considered. On the one hand, nobody is considered to have privileged access to the truth – including, in particular, the researchers. All participants (from students to teachers to researchers) are considered to be in the model development business; and, similar principles are assumed to apply to "scientific inquiry" at all levels. So, everybody's ways of thinking are subjected to examination and possible revision.

For the preceding kind of *three-tiered design experiments*, each tier can be thought of as *a longitudinal development study in a conceptually enriched environment*. That is, a goal is to go beyond studies of <u>typical</u> development in <u>natural</u> environments to also focus on <u>induced</u> development within <u>carefully controlled</u> environments. Finally, because the goal of M&M research is to investigate the nature and development of constructs or conceptual systems (rather than investigating and making claims students per se), we often investigate how understandings evolve in the thinking of "problem solvers" who are in fact teams (or other learning communities) rather than being isolated individuals. So, we often compare individuals with groups in somewhat the same manner that other styles of research might compare experts and novices, or gifted students and average ability students.

Investigations from an M&M perspective have led to the growing realization that, in a technology-based age of information, even the everyday lives of ordinary people are increasingly impacted by systems that are complex, dynamic, and continually adapting; and, this is even more true for people in fields that are heavy users of mathematics and technology. Such fields include design sciences such as engineering or architecture, social sciences such as economics or business management, or life sciences such as new hyphenated fields involving bio-technologies or nano-technologies. In such fields, many of the systems that are most important to understand and explain are dynamic (living), self-organizing, and continually adapting.

M&M research is showing that it is possible for average ability students to develop powerful models for describing complex systems that depend on only new uses of elementary mathematical concepts that are accessible to middle school students. However, when we ask *What kind of mathematical understandings and abilities should students master?* attention should shift beyond asking *What kind of computations can they execute correctly?* to also ask *What kind of situations can they describe productively?* ... This observation is the heart of M&M perspectives on learning and problem solving.

Traditionally, problem solving in mathematics education has been defined as getting from givens to goals when the path is not obvious. But, according to M&M perspectives, goal directed activities only become problematic when the "problem solver" (which may consist of more than an isolated individual) needs to develop a more productive way of thinking about the situation (given, goals, and possible solution processes). So, solutions to non-trivial problems tend to involve a series of modeling cycles in which current ways of thinking are iteratively expressed, tested, and revised; and, each modeling cycle tends to involve somewhat different interpretations of givens, goals, and possible solution steps.

Results from M&M research make it clear that average ability students are indeed capable of developing powerful mathematical models and that the constructs and conceptual systems that underlie these models often are more sophisticated than anything that anybody has tried to teach the relevant students in school.

However, the most significant conceptual developments tend to occur when students are challenged to repeatedly express, test, and revise their own current ways thinking - not because they were guided along a narrow conceptual trajectory toward (idealized versions of) their teachers ways of thinking (Lesh & Yoon, 2004). That is, development looks less like progress along a path; and, it looks more like an inverted genetic inheritance tree - where great grandchildren trace their evolution from multiple lineages which develop simultaneously and interactively.

In general, when knowledge develops through modeling processes, the knowledge and conceptual tools that develop are instances of situated cognition. Models are always molded and shaped by the situations in which they are created or modified; and, the understandings that evolve are organized around experience as much as around abstractions. Yet, the models and underlying conceptual systems that evolve often represent generalizable ways of thinking. That is, they are not simply situation-specific knowledge which does not transfer. This is because models (and other conceptual tools) are seldom worthwhile to develop unless they are intended to by powerful (for a specific purpose in a specific situation), re-useable (in other situations), and sharable (with other people).

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ISSUES AND TENDENCIES IN GERMAN MATHEMATICS-DIDACTICS: AN EPOCHAL PERSPECTIVE

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It is a positive sign that an international discussion on theories of mathematics education is taking place especially in the wake of TIMMS and PISA. It is laudable of PME to take the initiative to closely examine specific geographic trends in mathematics education research in comparison with trends that are concurrently occurring (or occurred) elsewhere (as reported in English et al., 2002; Schoenfeld, 1999, 2002). In doing so we can reflect and hypothesize on why certain trends seem to re-occur, sometimes invariantly across time and geographic location. Numerous reviews about the state of German mathematics didactics are available in German (see [1], Hefendehl et al., 2004; Vollrath et al., 2004). However there are no extant attempts to trace and analyze the last hundred years of "mathematics didactic" trends in Germany in comparison to what is happening internationally. This is our modest attempt to fill this void.

Some preliminary remarks on terminology and history: It has become standard practice for researchers writing in English to use the term "Mathematikdidaktik" when referring to mathematics education in Germany. However, there is no real comprehensive English equivalent for the term "Mathematikdidaktik". Neither "didactics" nor "math-education" describes the full flavor and the historical nuances associated with this German word. Even the adjective "German" is imprecise since educational research approaches in Germany splintered in the aftermath of World War II, with different philosophical schools of thought developing in the former East (GDR) and the west (FRG) on research priorities for university educators, until the reunification which occurred in 1990. Currently the 16 states in Germany reveal a rich heterogeneity in the landscape of mathematics teaching, teacher training and research methods, which manifests itself to insiders who microscopically examine the TIMSS- and PISA-results. However the reasons for this heterogeneity remain a mystery to outsiders. Given the page limits we outline in macroscopic terms the historical reasons for this heterogeneity. In doing so we do not differentiate explicitly between the alignment (or misalignment!) of theories preferred by university educators in comparison to practices of mathematics instruction in schools. The mutual dependencies between the two is certainly an interesting research question which brings into focus the system wide effectiveness (or ineffectiveness) of educational research (see for example Burkhardt & Schoenfeld, 2003).

1. **The Pedagogical tradition of mathematics teaching-Mathematics as Educational Value:** Reflections on the processes of mathematics teaching and learning have been a long-standing tradition in Germany. The early proponents of these theories of teaching and learning are recognizable names even for current

researchers. Chief among these early theorists was Adam Reise "the arithmetician" who stressed hand computation as a foundational learning process in mathematics. This emphasis is found in the pedagogical classics of the 19th century written by Johann Friedrich Herbart (1776-1841), Hugo Gaudig (1860-1923), Georg Kerschensteiner (1854-1932) (see Jahnke, 1990; Führer, 1997; Huster, 1981). The influence of this approach echoed itself until the 1960's in the so-called didactics of mathematics teaching in elementary schools to serve as a learning pre-requisite for mathematics in the secondary schools.

2. **Mathematician-Initiators of traditions in didactics research (20th Century):** In the early part of the previous century, mathematicians like Felix Klein (1849-1925) and Hans Freudenthal (1905-1990) (who was incidentally of German origin) became interested in the complexities of teaching and learning processes for mathematics in schools. The occasionally invoked words "Erlangen program" and "mathematization" are the present day legacy of the contributions of Klein and Freudenthal to mathematics education. Klein characterized geometry (and the teaching of it) by focussing on the related group of symmetries to investigate mathematical objects left invariant under this group. The present day emphasis of using functions (or functional thinking) as the conceptual building block for the teaching and learning of algebra and geometry, is reminiscent of a pre-existing (100 year old) Meraner Program. During this time period one also finds a growing mention in studying the psychological development of school children and its relationship to the principles of arithmetic (Behnke, 1950). This trend was instrumental in the shaping of German mathematics curricula in the 20th century with the goal being to expose students to mathematical analysis at the higher levels. The most notable international development in this time period was the founding of the ICMI in 1908, presided by Felix Klein. One of the founding goals of ICMI was to publish mathematics education books, which were accessible to both teachers and their students. We see this as one of the first attempts to "elementarize" (or simplify) higher level mathematics by basing it on a sound scientific (psychological) foundation. Mathematics educators like Lietzmann (1919) claimed that "didactic" principles were needed in tandem with content to offer methodological support to teachers. This approach mutated over the course of the next 50 years well into the 1970's. The over-arching metaphor for mathematics education researchers during this time period was to be a gardener, one who maintains a small mathematical garden analogous to ongoing research in a particular area of mathematics. The focus of research was on analyzing specific content and using this as a basis to elaborate on instructional design (Reichel 1995, Steiner, 1982). This approach is no longer in vogue and is instrumental in creating a schism between mathematicians and "mathematics-didakters," partly analogous to the math wars in the United States.

3. **"Genetic" Mathematics Instruction: Ineffectual Visionary Bridges (1960 – 1990):** The word "genetic" was used to exemplify an approach to mathematics instruction to prevent the danger of mathematics taught completely via procedures

(Lenné, 1969). Several theorists stressed that mathematics instruction should be focussed on the "genetic" or a natural construction of mathematical objects. This can be viewed as an earlier form of constructivism. This approach to mathematics education did not gather momentum. The word "genetisch" occurs frequently in the didactics research literature until the 1990's.

4. The New Math (1960 – 1975): Parallel to the new math movement occurring in post-Sputnik United States, an analogous reform movement took place in Germany (mostly in the West, but partly adopted by the East, see [1]). A superficial inspection seems to point to a realization of Klein's dream of teaching and learning mathematics by exposing students to its structure. This reform took on the dynamic of polarizing scientists (mathematicians) to work in and with teacher training, the resulting outcome being a lasting influence on mathematics instruction during this time period. Unlike the United States teachers were able to implement a structural approach to mathematics in the classroom. This can be attributed to the fact that during this time period there was no social upheaval in Germany, unlike the U.S where the press for social reform in the classroom (equity and individualized instruction) interfered with this approach to mathematics education. The fact that German "new math" did not survive the tide of time indicates that there was difficulty in implementing it effectively.

5. The birth of didactics as a research discipline (1975): While the new mathematics movement was subject to a host of criticisms, one positive outcome was the founding of the Gesellschaft für Didaktik der Mathematik (German Mathematics Didactics Society), which stresses that mathematics didactics was a science whose concern was to rest the mathematical thinking and learning on a sound theoretical (and empirically verifiable foundation). This was a radical step search for mathematics education research in Germany, one that consciously attempted to move away from the view of a math educator as a part-time mathematician (recall Klein's garden). Needless to say, we could easily write an entire book if we wanted to spell out the ensuing controversy over the definition of this new research discipline in Germany (see Bigalke, 1974; Dress, 1974; Freudenthal, 1974; Griesel, 1974, Laugwitz, 1974; Leuders, 2003; Otte, 1974; Tietz, 1974 Wittmann, 1974; 1992). However, the point to be taken from the founding of this society and a new scientific specialty is that the very debate we have undertaken here, that is, to globally define theories of mathematics education has in fact many localized manifestations such as in Germany.

6. Mathematical Teaching and Learning- A Socialistic and an Individualistic Process (1980 – today): One of the consequences of founding a new discipline of science was the creation of new theories to better explain the phenomenon of mathematical learning. The progress in cognitive science in tandem with interdisciplinary work with social scientists led to the creation of "partial" paradigms about how learning occurs. Bauersfeld's (1988,1995) views of mathematics and mathematical learning as a socio-cultural process within which the individual operates can be viewed as one of the major contributions to theories of mathematics education.

7. An Orientation Crisis - The Conundrums posed by new Technology (1975 – today): Weigand's (1995) work poses the rhetorical question as to whether mathematics instruction is undergoing yet another crisis. The advent of new technologies opened up a new realm of unimagined possibilities for the learner, as well as researchable topics for mathematics educators. The field of mathematics education in Germany oriented itself to address the issues of teaching and learning mathematics with the influx of technology. However the implications of redefining mathematics education, particularly the "*hows*" of mathematics teaching and learning in the face of new technology poses the conundrum of the need to continually re-orient the field, as technology continually evolves (see Noss / Hoyles (1995) for an ongoing global discussion).

8. TIMMS and PISA -The Anti-Climax (1997 – today): The results of TIMMS and PISA brought these seven aforementioned "tendencies" to a collision with mathematics educators and teachers feeling under-appreciated in the wake of the poor results. These assessments also brought mathematicians and politicians back into the debate for framing major policies, which would affect the future of mathematics education in Germany. Mathematics education is now in the midst of new crisis because the results of these assessments painted German educational standing in a poor global light. A detailed statistically sieved inspection of the results indicated that poor scores could be related to factors other than flaws in the mathematics curriculum, and/or its teaching and learning, that is to socioeconomic and cultural variables in a changing modern German society. Thus mathematics education in Germany would now have to adapt to the forces and trends creating havoc in other regions of the globe (see Burton, 2003; Steen, 2001).

Conclusions

Epochal viewpoints: The eight major tendencies that we have highlighted in the 100 years of mathematics education history in Germany reflect trends that have occurred internationally. Each epoch is characterized by an underlying metaphor that shaped the accepted theories of that time period. Felix Klein's view of a mathematics educator was that of a mathematician-gardener tending to all aspects of a specialized domain within mathematics, including its teaching and learning. This shifted to a focus on the structure of modern mathematics itself and partly to the teacher as a "transmitter" of structural mathematics in the 1960's during the New Math period. This was followed by an epoch where the science of mathematics education and the student (finally!) came into focus and brought forth attempts to delineate theories for this new science such as Bauersfeld's socio-cultural theories. New technologies shifted the focus of theories to accommodate how learning occurs in the human-machine interface. Finally TIMMS and PISA brought into focus assessment issues along with societal and political variables that are changing conceptions of mathematics education as we speak. In a sense we have come full circle because we still haven't defined what mathematics didactics is. However, in the search through history for the answer, we have understood the epochal nuances of this interesting term. Perhaps it is time we finally defined it!

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CONCLUDING POINTS

The diversity in the perspectives presented in the six contributions parallel conundrums recently elicited by Tommy Dreyfus at the 4th European Congress in Mathematics Education (Spain, February 2005). In his concluding report about the working group on mathematics education theories, Dreyfus stated that although theories were a vital aspect of mathematics education, they were much too wide of a topic. However the field can take solace from the fact that although contradictions exist, there are also connections and degrees of complementarities among theories. The coordinators of this particular Forum have reached a similar conclusion. Many of the points we make here echo the recommendations of Tommy Dreyfus. Although it is impossible to fully integrate theories, it is certainly possible to bring together researchers from different theoretical backgrounds to consider a given set of data or phenomena and examine the similarities and differences in the ensuing analysis and conclusions. The interaction of different theories can also be studied by applying them to the same empirical study and examining similarities and differences in conclusions. Last but not least, although it is impossible to expect everybody to use the mathematics education “language,” a more modest undertaking would be to encourage researchers to understand one or more perspectives different from their own. This will ensure that the discussion continues as well as creates opportunities for researchers to study fruitful interactions of seemingly different theories. We consider such work vital to help move the field forward.