

LINEAR FUNCTIONS AND A TRIPLE INFLUENCE OF TEACHING ON THE DEVELOPMENT OF STUDENTS' ALGEBRAIC EXPECTATION

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The study of linear functions is important as it provides students with their first experience of identifying and interpreting the relationship between two dependent variables. This paper, which builds on previous research, reports a study undertaken with 64, year 9 students from two Australian schools. Linear functions were introduced to these students through a graphics calculator supported, functional approach to modelling contextual problems. The teaching was generally successful. Scrutiny of pre- and post-tests highlights the triple influence of the teaching on their progress in each element of Algebraic Expectation relevant to this stage.

Linear functions provide many students with their first experience of working with two related variables and so this is a significant point of transition in their mathematical development. The typical approach to this topic in Australian text books is to provide an abstract graphical introduction to a general rule, perhaps in the form $y = ax + b$, with attention given to the effect of each parameter on the graph of the function. In an earlier paper (Bardini, Pierce & Stacey, 2004) the researchers describe the mathematical development of a class of students at this point whose initial teaching followed a functional-modelling approach using context problems common to the students' everyday world, supported by the use of graphics calculators. Those students' ability to write and apply this level of algebra was closely monitored and examination of the data revealed that

three features of the program exerted a 'triple influence' on students' use and understanding of algebraic symbols. Students' concern to express features of the context was evident in some responses, as was the influence of particular contexts selected. Use of graphics calculators affected some students' choice of letters. The functional approach was evident in the meanings ascribed to letters and rules" (Bardini et al, in press).

This paper reports a further implementation of this teaching program, this time with three year 9 classes (15 year olds) at two different schools. The pre- and post-test results for these students are scrutinised for evidence of this 'triple influence' and its impact on students developing Algebraic Expectation (Pierce & Stacey, 2001, 2002).

THE 'TRIPLE INFLUENCE': THE TEACHING

Instead of following their usual textbook the teacher and students were guided in their approach by the linear functions chapter from Asp, Dowsey, Stacey and Tynan's

(1998) *Graphic Algebra*, a book that arose from research conducted during the Technology-Enriched Algebra Project of the University of Melbourne. The class teachers (all partner teachers in the RITEMATHS project, HREF1), were aware of the results of the first study. They were encouraged to have students work through the material at their own pace but punctuate this by teacher intervention and whole class discussion. These would be used to teach calculator skills and emphasise important features of the algebra, for example: the meaning of letters as variables, writing algebraic rules, function notation, and transformation of linear function graphs.

The first influence: Real world context problems

The teaching was almost all set in the context of real world problems which were familiar to the students: an approach supported by Freudenthal (1991) who argues that mathematics starts within commonsense and that students' mathematical ideas develop by starting with such experientially real situations. Students in the earlier study commented that working with context problems helped them to 'relate things and to produce answers which made sense'. We may expect to see the imprint of this 'sense' in the students' choice and ordering of symbols in their written algebra.

The unit, which the teachers were asked to follow, begins with a story about a girl selling homemade lemonade. The profit she makes is set up as a function of how much lemonade she sells, first in a table and then on a graph. Students graph the function and read various information related to the problem setting, from the graph. Later, the story introduces other drink sellers with different prices and ingredient costs. These corresponding functions are graphed and the graphs and functions are compared. Points of intersection, slopes, intercepts and intervals are interpreted in context. The program then introduces other real world problems. For example, a comparison of mobile phone charges drew out the significance of slope and intercepts in terms of both the original problem and related algebraic equations, while an investigation of the relationship between height and arm span, led students to model the relationship by drawing a line of best fit by eye through data, writing the rule and then exploring the consequences of varying the parameters.

The second influence: a functional approach

In addition to the modelling approach as described above using real contexts, the perspective taken in this teaching has elements of a functional approach to algebra. While there was no formal teaching of definition of function as a single valued mapping over a domain, function notation is used and the notion of a relationship between a dependent and an independent variable is emphasised, for example, as described in the lemonade problem. Algebraic letters stand primarily for variables. This approach is expected to influence students' meaning of letters, identification of structure and interpretation of features of functions and graphs.

The third influence: the use of graphics calculators

The teaching emphasised a graphic approach to linear functions. Students solved some equations both graphically and symbolically but graphical solution methods predominated. There is strong evidence (see for example Dreyfus, 1991) that students build strong conceptual schema by moving between representations and we expect that understanding gained in the graphical representation will enhance students' ability to identify the structure of two dependent variables and identify and interpret the key features of linear functions, that is the role of the constant and coefficient.

Students' work in the graphical mode was supported by each having a TI-83+ graphics calculator for all lessons. The graphic facility of these calculators allows flexibility of scaling and, in particular, allows students to move easily between different views of a graph by zooming in and out. Even beginning students find little difficulty in entering data, viewing scatterplots and testing the validity of conjectures for a line of best fit by changing the values of parameters.

The use of this technology forms the third influence of the teaching and while we expected the focus on graphs to be a positive influence there was some concern as to whether the use of graphics calculator technology with its own peculiar symbols might impact on students' by-hand algebra. The influence of these three facets of the teaching on students' Algebraic Expectation will be explored as we consider students' written responses on both the pre- and post-tests.

LINEAR FUNCTIONS AND ALGEBRAIC EXPECTATION

Algebraic Expectation was first defined for undergraduate students studying a functions and calculus course with a Computer Algebra System (CAS) available (Pierce & Stacey, 2001). The context of learning environments where sophisticated technology is increasingly on hand, especially in the form of function graphers or CAS, has challenged mathematics educators to reconsider our focus in teaching algebra. Fey (1990) and Arcavi (1994) put forward notions of 'symbol sense' to parallel 'number sense' (see for example McIntosh, 1993). Pierce & Stacey (2001), after considering the full process of mathematical modelling, suggest that the key impact of such technology is in the process of finding a mathematical solution to a mathematically formulated problem. We summarised the 'symbol sense' thinking needed to exploit and monitor work with mathematical analysis tools within the symbolic representation as 'Algebraic Expectation'. The section below considers the key 'common instances' of Algebraic Expectation which we expect to be observable when students learn to write and use linear functions:

- recognition of conventions and basic processes
- identification of structure
- identification of key features.

Creating, interpreting and working with an algebraic (symbolic) rule for a linear function is not trivial for a novice and the foundations of Algebraic Expectation

established in this context will be widely applicable. Consider a general rule for a linear function, say $y = mx + c$ or $f(x) = ax + b$. A student demonstrates Algebraic Expectation by identifying the structure of two related variables, both of degree 1, and hence recognising the rule for a linear function. To create or interpret such a rule the student must understand that x may vary in the values which it represents and that the value of y or $f(x)$ will depend on the value of x . The student needs also to come to see that the choice of letter to represent a variable is arbitrary but that common mathematical convention suggests the use of either a letter related to the context or a letter from the latter part of the alphabet. In developing Algebraic Expectation related to variables, it is also important that the student learns that the variable x , in our general rule, may stand for a variety of numerical values or in fact another object or expression which can replace x . In the rule that describes a relationship, units of measurement need not be included and multiplication is, conventionally, implicit.

For any linear function students should be able to identify the constant term and interpret this as the value of y when x is zero or an ‘initial value’. Similarly it is important to identify the coefficient of the variable and interpret this in terms of ‘rate of change’: the change in y when x changes by 1. The teaching stressed this interpretation from both graphical and real world viewpoints.

This section has briefly noted instances which would show students’ competence in Algebraic Expectation. This paper concentrates on understanding in the symbolic representation: other important understandings not considered here relate to other representations. In the next section we scrutinise pre- and post-tests of the students who participated in this study for evidence of the ‘triple influence’ of the teaching on this aspect of students’ mathematical progress.

THE EVIDENCE

The findings discussed in this paper are based on students’ scripts from pre- or post-tests completed by sixty-four students from three year 9 classes (approx 15 year olds) at two co-educational secondary schools. One of these three teachers had previously taught the class described in Bardini et al. (in press). The classes each followed the teaching program outlined above over a period of about 4 weeks.

This paper will consider students’ responses to three multiple-part items which had equivalent forms on both tests. For the first problem, on the pre-test, students were provided with a graphical representation of the costs of hiring either Jack or Jill’s truck and asked to read information and describe the rule verbally and symbolically. The post-test question paralleled this with a graph showing the alternative costs of hiring a plumber, either Bob or Chris. The second question on the pre-test provided a scenario about the cost of a vacuum cleaners with additional dust bags and the post-test described alternative costs for fun park entrance with varying numbers of rides. The third parallel question pair required finding a rule to match a table of values, complementing the previous questions requiring rules from graphs or descriptions of

real situations. Examples in the paper are all given from the first item pair, but the percentage changes report on data from all three item pairs.

First it was clear that the teaching was generally successful. We see significant improvement from the pre-test when 50% of students either made no attempt to write algebraic rules or were incorrect to the post-test when only 2% of the students made no attempt to write rules and 18% did not write at least one correct rule. In addition, when writing a rule from graphical information on the pre-test 23% of students were successful and on the post-test 63%, from verbal information pre-test 14% to post-test 55% and from a table the success rate increased from 17% to 69%. Examples and changes in students' responses will provide evidence of the 'triple influence' of the teaching content on their improvement as demonstrated by Algebraic Expectation.

The triple influence on 'recognition of conventions and basic processes'

The influence of context on recognition of conventions and basic processes is seen in students' verbal responses and writing of algebraic rules. The responses of students 38, 34, and 36, below, highlight the range of level of attachment to the context, shown when students were asked to explain, in words, how to work out the cost of hiring Bob if you knew the number of hours he would be working.

Student 38 \$25 set fee plus \$50 an hour (*this is a correct interpretation of the graph*)

Student 34 Bob starts at a fixed rate of \$25 and for every hour after that there is an extra \$50.

Student 36 Ring Bob

Writing an algebraic rule to work out the cost of hiring a truck or plumber allows us insight into students' recognition of conventions, especially meaning of symbols. Some students like students 25 and 34, below, showed a progression from an inability to write in symbolic algebra to writing essentially correct rules.

Student 25 no response

Student 25 $c = \$25 + \$50x$

Student 34 no response

Student 34 $\text{Bob} = 25 + (x \times \$50) = C$

In other responses we see the influence of context fading as the student becomes more confident in their use of algebra:

Student 109 $\text{cost} = \$100 + \$50x$

Student 109 $c = 25 + 50x$

Then finally we observed some students who changed from writing their rule by following arithmetic logic, that is initial cost plus rate times hours gives the cost, to the more detached conventional order commonly adopted in algebra.

Student 102 $100 + 50D = C$

Student 102 $C = 50t + 25$

Through the experience of the context problems and the functional approach, students clearly came to recognize that a letter may stand for a quantity which varies. Many students, such as student 142, added notes which make this understanding explicit.

Student 142 $C = 25 + 50n$, $n = \text{number of hours}$

This is a vital step forward in their understanding of algebra from their earlier work on equations where they have met letters as standing for fixed unknowns. The imprint of context teaching is clearly seen in the increased incidence of students choosing letters related to the context (e.g. C , h (hours), t (time) etc) in their rules. As students became ‘socialized’ to the conventions of algebra we see an increase in the use of implicit multiplication (52% to 81% of responses) although the use of \$ symbols within the rule remained at 12% of total responses.

Little other impact of the functional aspect of the approach or the use of graphics calculators was evident in students’ choice of symbols. Despite the use of notation such as $P(x)$ in the teaching materials, only two students used this when writing rules. Similarly, graphics calculator syntax was not adopted by the students, again only one student wrote ‘*’ where multiplication could have been implicit. They then replaced this ‘*’ with the conventional ‘ \times ’. Using contextualized problems appears to have positively influenced students’ use and understanding of the meaning of symbols.

The triple influence on ‘identification of structure’

A fundamental aspect of identification of structure is to see that the situations presented in all three question pairs represent functional relationships between two variable quantities. The examples of students 38 and 106 demonstrate progress.

Student 38 $50 \times x + 100$

Student 38 $C=50 \times x + 25$, $x = \text{hours}$

Student 106 $50x + 100$

Student 106 $c=50h+25$ $c = \text{cost}$, $h = \text{hours}$

On the pre- and post-tests, two of the questions requiring students to write algebraic rules were based on context scenarios and the third on a table of values without context. On the pre-test 20% of students who responded wrote only expressions in one variable to generalise context scenarios (e.g. students 38 and 106) but interestingly most of these students wrote rules using two variables to express the relationship between variables in a table of values. In addition, on the pre-test 49% of students correctly wrote rules using two variables regardless of context or table and this proportion increased to 97% on the post-test. On the post-test, the only students writing expressions in one variable, instead of relational rules, were from among those who did not respond to these items on the pre-test.

The use of the graphics calculator supported students’ explorations of various relationships, for example, the mobile phone charges and especially in the development of a model to link arm-span and height. The repeated use of the function entry format of $y_1=$, $y_2=$ etc reinforces the structure of two dependent variables. Some students, like student 108 below, who adopted the use of y rather than a context dependent variable, or student 21 who wrote that $b(x)=y$ may have been influenced by this but no student included subscripts in their rules. There was no evidence that the availability of the graphics calculator encouraged students to solve non-graphically presented questions graphically.

Student 108 $y=50x+25$

The triple influence on ‘identification of key features’

Across all three questions most students, who responded, correctly identified and interpreted the parameters. Teaching from context problems has influenced this. Only in pre-test responses to question 1 do we see some errors like those of Student 138 and Student 143. Fourteen percent of students failed to add the constant term for the initial cost while student 138 was the only student to omit the coefficient for the cost per hour. No students who responded to these items on the post-test made such errors.

Student 138 $x+50$ (correct answer $50x + 25$)

Student 143 $x = n \times 50$ (correct answer $50x + 25$)

That students’ interpretation of these key features was strongly influenced by context was shown in their verbal descriptions (for example student 34 above) which translated into rules. The teaching approach reinforced the broader ‘rate of change’ interpretation of the coefficient rather than just ‘gradient of a line’ as emphasized by the typical abstract graphical introduction to linear functions. In other words, the coefficient was interpreted not primarily as slope, but as the change in the dependent variable corresponding to a change of 1 in the independent variable. Students apparently understood this well and used it to construct correct algebraic rules.

We expect that the inexact modelling will also have contributed to students’ strength in identifying and interpreting these key features, although there is no direct evidence in their solutions to these items. The exploration, using the graphics calculator, for a model to link arm-span and height required students to deal with a situation where the initial value was outside of the graph window which displayed the data. Students trialled various constant and coefficient values in order to find their ‘line of best fit’. This experience was intended to draw attention to the importance and meaning of these two parameters.

CONCLUSIONS

‘Linear functions’ is certainly a basic algebra topic but it is of fundamental, not trivial, importance. It marks the point at which many students decide that mathematics is meaningless and difficult. This study complements and extends the work of Bardini, Pierce & Stacey (in press). In both studies, the majority of students made successful progress in writing conventional algebraic expressions and developing Algebraic Expectation. The previous analysis of teaching identified three important features which impacted on students: working initially with modelling contextual problems, following a functional approach and using a graphics calculator.

In this study we have scrutinised students’ responses to pre- and post-test items in order to seek evidence of the ‘triple influence’ of these three features of the teaching on their developing Algebraic Expectation as demonstrated in their writing and interpretation of algebraic rules. When marking students’ test responses a teacher will appropriately mark a variety of answers as incorrect or as correct and allocate a grade by which the student will judge their progress. In this paper it has been shown that

the variety of answers is also revealing because within both ‘correct’ and ‘incorrect’ responses can be seen a range in the students’ understandings and their progress towards working competently with de-contextualised symbols in a conventional way.

While the functional approach is demonstrably appropriate for applying algebraic techniques to real world problems and the strategic use of graphics calculators supports this approach, evidence of their direct influence on students’ test responses was limited. In contrast, the influence of modelling contextual problems was clear, especially in the work of those students who did not write correct rules on the pre-test. The link to context assisted students in understanding the meaning of symbols and identifying both the structure and key features of linear functions. There was also evidence that students, who could already link symbols to the context, progressed to writing more conventional de-contextual algebraic rules.

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