GRASSMANNIANS VIA PROJECTION OPERATORS AND SOME OF THEIR SPECIAL SUBMANIFOLDS *

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We study submanifolds of a general Grassmann manifold which are of 1-type in a suitably defined Euclidean space of $\mathbb{F}-$Hermitian matrices (such a submanifold is minimal in some hypersphere of that space and, apart from a translation, the immersion is built using eigenfunctions from a single eigenspace of the Laplacian). We show that a minimal 1-type hypersurface of a Grassmannian is mass-symmetric and has the type-eigenvalue which is a specific number in each dimension. We derive some conditions on the mean curvature of a 1-type hypersurface of a Grassmannian and show that such a hypersurface with a nonzero constant mean curvature has at most three constant principal curvatures. At the end, we give some estimates of the first nonzero eigenvalue of the Laplacian on a compact submanifold of a Grassmannian.

1. Introduction

The Grassmann manifold of $k-$dimensional subspaces of the vector space $\mathbb{F}^m$ over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ can be conveniently defined as the base manifold of the principal fibre bundle of the corresponding Stiefel manifold of $\mathbb{F}-$unitary frames, where a frame is projected onto the $\mathbb{F}-$plane spanned by it. As observed already in 1927 by Cartan, all Grassmannians are symmetric spaces and they have been studied in such context ever since, using the root systems and the Lie algebra techniques. However, the submanifold geometry in Grassmannians of higher rank is much less developed than the corresponding geometry in rank-1 symmetric spaces, the notable exceptions being the classification of the maximal totally geodesic submanifolds (the well known results of J.A. Wolf and of Chen and Nagano) and the

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study of the faces of Grassmannians in the context of the theory of calibrations. See however [1], [2], [3], and [14] for some more recent advances regarding submanifolds of the Grassmannians. In [10] the author used a special embedding of a general Grassmannian into certain Euclidean space of $\mathbb{F}-$Hermitian matrices by identifying an $\mathbb{F}-$plane with the orthogonal projection onto it, thus considering points in a Grassmannian as projection operators with the trace equal to the dimension of the planes. This approach deserves more attention because of its potential application to tomography and affine shape analysis [17], and seems, moreover, to be more amenable to the study of submanifolds of Grassmannians.

In this paper we concentrate on the study of 1-type submanifolds of Grassmannians, and in particular on their hypersurfaces, viewed as submanifolds of a naturally defined Euclidean space E^N of Hermitian matrices. This means that the immersion vector of each such submanifold into that space allows a decomposition as

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_t, \text{ with } \tilde{x}_0 = \text{const}, \ \tilde{x}_t \neq \text{ const}, \text{ and } \Delta\tilde{x}_t = \lambda\tilde{x}_t.$$

Here Δ denotes the Laplacian on the submanifold considered (i.e. with respect to the induced metric), applied to vector-valued functions componentwise, and $\lambda \in \mathbb{R}$ is the corresponding eigenvalue [7]. Invoking a well-known result of Takahashi, this is equivalent to saying that the submanifold is minimal in the ambient Euclidean space of Hermitian matrices or in some hypersphere of that space. We give a characterization of 1-type submanifolds and produce certain partial differential equations involving the mean curvature that a 1-type real hypersurface must satisfy. In some cases the mean curvature is constant, and we further prove that a 1-type real hypersurface of a Grassmannian with constant nonzero mean curvature α can have at most three distinct principal curvatures, all of them constant, whose values are computed in terms of α, the type-eigenvalue λ, and the dimension (Theorem 3.5). The minimal hypersurfaces of type 1 are mass-symmetric. They exist in the real projective space as the projective spaces of dimension one less (the totally geodesic case) or, otherwise, the type-eigenvalue is fixed in each Grassmannian. The situation in rank-1 Grassmannians (projective spaces) is well understood and 1-type hypersurfaces in these spaces were classified in [9], [11], [12]. What these hypersurfaces are in Grassmannians of rank ≥ 2 is presently unclear, but the investigations in this paper set the stage for their eventual classification. At the end of the article we give several sharp upper bounds on the first nonzero eigenvalue of certain compact submanifolds of Grassmannians.

2. Preliminaries

Let $\mathbb{F}$ denote one of the fields $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (algebra of quaternions) and let $d = \dim_R \mathbb{F}$. For any integers $m \geq 3$ and $1 \leq k \leq m-1$ let $l = m-k$ and let $U_{\mathbb{F}}(m)$ denote the $\mathbb{F}-$unitary group, i.e. the group of matrices preserving the standard $\mathbb{F}-$Hermitian scalar product $\langle z, w\rangle = \sum_{k=1}^{m} \bar{z}_k w_k$ in $\mathbb{F}^m$ (thus one of $O(m), U(m)$, or $Sp(m)$, corresponding to the three cases $\mathbb{R}, \mathbb{C}, \mathbb{H}$). In our paper [10] we considered the embeddings of real, complex, and quaternion Grassmannians $G_k\mathbb{F}^m = U_{\mathbb{F}}(m)/U_{\mathbb{F}}(k) \times U_{\mathbb{F}}(l)$ of $k-$dimensional $\mathbb{F}-$planes into a suitable Euclidean space $\mathbf{H}(m)$ of $\mathbb{F}$-Hermitian matrices by identifying an $\mathbb{F}$-plane with the projection operator onto it. The standard Euclidean metric in $\mathbf{H}(m)$ is $\langle A, B\rangle = \frac{1}{2} Re\, \mathrm{tr}_{\mathbb{F}}(AB)$, which turns it into the Euclidean space E^N of dimension $N = m + d\binom{m}{2}$. Note that '*Re*' is actually needed only in the quaternion case because of the absence of commutativity in the skew-field $\mathbb{H}$. The Grassmannian is a smooth symmetric Einstein space of real dimension $g = dkl$. We denote by Φ this embedding which is $U_{\mathbb{F}}(m)-$equivariant, where the action of $U_{\mathbb{F}}(m)$ in $\mathbf{H}(m)$ is by conjugation. The image of the Grassmannian is given by

$$\Phi(G_k\mathbb{F}^m) = \{P \in \mathbf{H}(m) |\, P^2 = P,\ \mathrm{tr}_{\mathbb{F}} P = k\},$$

which sits minimally in the hypersphere $S_C^{N-1}(R)$ whose center is $C = kI/m$ and radius $R = \sqrt{kl/2m}$. It also belongs to the hypersphere centered at the origin (i.e. zero matrix) of radius $\sqrt{k/2}$ (since $\langle P, P\rangle = \frac{1}{2} Re\, \mathrm{tr} P^2 = k/2$) and lies in the hyperplane $\{\mathrm{tr}\, A = k\}$, which has I as its normal vector and is at a distance $k/\sqrt{2m}$ from the origin.

The tangent and the normal space of the Grassmannian at an arbitrary point P, together with the representative vectors at $P_0 := \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$, the point we will refer to as the 'origin', are given respectively by

$$T_P G = \{X \in \mathbf{H}(m) \,|XP + PX = X\}, \quad \begin{pmatrix} O & B \\ B^* & O \end{pmatrix},\ B \in M_{k\times l}(\mathbb{F})$$

$$T_P^{\perp}(G) = \{Z \in \mathbf{H}(m) \,|ZP = PZ\}, \quad \begin{pmatrix} C & O \\ O & D \end{pmatrix},\ C \in \mathbf{H}(k),\ D \in \mathbf{H}(l)$$

As we know, the second fundamental form σ and the shape operator $\bar{A}_Z$ have the following expressions [10]:

$$\sigma(X, Y) = (XY + YX)(I - 2P)\,,$$
$$\bar{A}_Z X = (XZ - ZX)(I - 2P)\,,$$

for vector fields X, Y tangent to the Grassmannian and a vector field Z normal to it. Moreover σ is parallel, i.e. $\bar{\nabla}\sigma = 0$, and the vectors P and I are normal to the Grassmannian. Thus

$$\langle \sigma(X,Y), P\rangle = -\langle Y, \tilde{\nabla}_X P\rangle = -\langle X, Y\rangle \quad \text{and} \quad \langle \sigma(X,Y), I\rangle = 0, \qquad (2.1)$$

the connection being that of the ambient Euclidean space.

Since $X(I-2P) = -(I-2P)X$ and $(I-2P)^2 = I$ one immediately gets

$$\bar{A}_P = -I \qquad \text{and} \qquad \bar{A}_I = 0. \qquad (2.2)$$

With the induced metric from $\mathbf{H}(m)$, a Grassmannian has the sectional curvature $\bar{K}$ given by $\bar{K}(X,Y) = \frac{1}{2}\, Re\, \mathrm{tr}\,[X^2Y^2 + YX^2Y - 2(XY)^2]$ for an orthonormal pair of tangent vectors$\{X, Y\}$, which in the case of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ reduces to $\bar{K}(X,Y) = \mathrm{tr}\,[X^2Y^2 - (XY)^2]$. Since for Hermitian matrices $\mathrm{tr}\,(X^2Y^2) \geq \mathrm{tr}\,(XY)^2$ with equality if and only if X and Y commute (proof by diagonalizing X; a similar inequality holds in the quaternionic case), the sectional curvature of the Grassmannian is nonnegative. In rank-1 Grassmannians $\bar{K}$ is strictly positive. For the range of possible values of $\bar{K}$ see [19] and [1].

The Grassmannian $G_k\mathbb{F}^m$, which we will often denote simply by G, when the rank and the dimensions are understood or not at issue, is a smooth symmetric Einstein space of real dimension $g = dkl$. It is simply-connected when $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ and doubly covered by the Grassmannian of oriented planes when $\mathbb{F} = \mathbb{R}$. By choosing a suitable orthonormal basis $\{\epsilon_i\} = \{E^u_{ar}\}$ of $T_{P_o}G$ as in [10] we compute

$$\sum_i (X\epsilon_i^2 + \epsilon_i^2 X) = mdX \quad \text{and} \quad \sum_i \epsilon_i X \epsilon_i = (2-d)X,$$

so that the Ricci tensor of the Grassmannian equals

$$\bar{Q}(X) = \sum_i (X\epsilon_i^2 + \epsilon_i^2 X - 2\epsilon_i X \epsilon_i) = (md + 2d - 4)X, \qquad (2.3)$$

and the (non-normalized) scalar curvature is constant,

$$\bar{\tau} = \mathrm{tr}\,\bar{Q} = (md + 2d - 4)g.$$

3. Submanifolds of the Grassmannians of type 1

Let us consider an isometric immersion $x : M^n \to G_k\mathbb{F}^m$ of a connected Riemannian $n-$manifold into a Grassmannian and the associated immersion $\tilde{x} = \Phi \circ x : M^n \to \mathbf{H}(m)$ into the Euclidean matrix space. The notation will be as follows: we use ∇, A_ξ, D, h, H, respectively, for the Levi-Civita connection, the shape operator in the direction of a normal vector ξ, the connection in the normal bundle, the second fundamental form, and the mean curvature vector related to the submanifold M and the immersion x. The same symbols with bar will denote the corresponding objects related to G and the embedding Φ, whereas those symbols with tilde will denote the corresponding objects related to the immersion $\tilde{x}$ and the ambient matrix space E^N. As usual, we denote by σ the second fundamental form of G in E^N. Since all immersions are isometric we use $\langle\ ,\ \rangle$ for the metric throughout. Typically $\{e_i\}$ and $\{e_r\}$ denote, respectively, orthonormal bases of tangent and normal spaces of M in G at a general point. For such a basis the shape operator A_{e_r} is abbreviated to A_r. $\Gamma(\mathcal{B})$ denotes the set of all (local) smooth sections of a bundle $\mathcal{B}$. We work in the smooth category, i.e. all tensors, vector fields, immersions, etc. are assumed smooth and all manifolds connected.

According to [7], the immersion $\tilde{x}$ above is said to be of finite type if the immersion vector can be decomposed into a sum of finitely many vector-eigenfunctions of the Laplacian on M. In particular, $\tilde{x}$ is of 1-type if and only if

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_t, \quad \tilde{x}_0 = \text{const}, \quad \Delta\tilde{x}_t = \lambda\tilde{x}_t, \tag{3.1}$$

where $\tilde{x}_t \neq \text{const}$ and $\lambda \in \mathbb{R}$. For a compact submanifold, $\tilde{x}_0$ is the center of mass and the number t is called the order of the immersion if λ is the $t-$th nonzero eigenvalue. A compact submanifold lying in a certain hypersphere is said to be mass-symmetric in that hypersphere if its center of mass coincides with the center of the hypersphere. Even for noncompact 1-type submanifold, it is easy to see that the 1-type decomposition (3.1) is unique, with the exception of null 1-type submanifolds (those satisfying $\Delta\tilde{x}_t = 0$), for which $\tilde{x}_0$ and $\tilde{x}_t$ are not uniquely determined. Since a spherical submanifold cannot be null, we say that such a submanifold is mass-symmetric when $\tilde{x}_0$ is exactly the center of the hypersphere.

The study of finite-type (specifically 1-type) submanifolds of projective spaces $\mathbb{F}P^{m-1} = G_1\mathbb{F}^m$ has been very fruitful (see [11], [12]), and it seems interesting (albeit more challenging) to extend this study to Grassmannians of arbitrary rank. The main difficulty here, in this more general setting, is that the embedding Φ is not constant isotropic, which precludes using some nice formula for the shape operator $\bar{A}_{\sigma(X,Y)}$ in the direction of the second fundamental form, such as the one available for the projective spaces [12]. As we know, the embedding Φ is not constant isotropic in the sense that $\|\sigma(X,X)\|$ has the same value for all unit tangent vectors $X \in T_pG$ and all points $p \in G$. However, a vestige of isotropy is retained in the following lemma.

Lemma 3.1 *Let X be a unit vector field parallel along a geodesic of G. Then $\|\sigma(X,X)\|$ is constant along that geodesic.*

Proof. Let Y denote the unit tangent of a geodesic in question. Since

$$\langle \sigma(X,X), \sigma(X,X) \rangle = \frac{1}{2} Re\, \mathrm{tr}\, [\sigma(X,X)]^2 = 2 Re\, \mathrm{tr}\, X^4,$$

we get

$$\begin{aligned} Y(\|\sigma(X,X)\|^2) &= 2Re\, \mathrm{tr}\, [\tilde{\nabla}_Y X^4] = 8Re\, \mathrm{tr}\, [(\tilde{\nabla}_Y X) X^3] \\ &= 8Re\, \mathrm{tr}\, [\sigma(Y,X)X^3] = 16 \langle \sigma(Y,X), X^3 \rangle \ = 0, \end{aligned}$$

since X^3 is again tangent (cf. [10, Prop. 5]), proving the claim. □

Thus, going along radial geodesics we can extend X at a point to a local field parallel along these geodesics. Unfortunately, $\|\sigma(X,X)\|$ varies depending on X.

Regarding 1-type submanifolds of G we have the following characterization:

Theorem 3.1 *Let $x : M^n \to G$ be an isometric immersion of a connected Riemannian $n-$manifold into a Grassmannian. Then the associated immersion $\tilde{x} : M^n \to \mathbf{H}(m)$ is of 1-type if and only if there exists a nonzero constant λ such that for every tangent vector $X \in \Gamma(TM)$ we have:*

(i) $nA_H X - nD_X H + \sum_i \bar{A}_{\sigma(e_i,e_i)} X = \lambda X$
(ii) $\sum_r \sigma(2A_r X + (tr A_r)X,\ e_r) = 0.$

Proof. Using (3.1) we get

$$\Delta \tilde{x} = -nH - \sum_i \sigma(e_i, e_i) = \lambda(\tilde{x} - \tilde{x}_0). \tag{3.2}$$

Since $\langle \sigma(e_i,e_i), \tilde{x} \rangle = -1$, we see that $\lambda \neq 0$. Differentiating (3.2) with respect to an arbitrary tangent vector field $X \in \Gamma(TM)$ one gets

$$nA_H X - nD_X H - n\sigma(X,H) + \sum_i \bar{A}_{\sigma(e_i,e_i)} X - \sum_i \bar{D}_X \sigma(e_i,e_i) = \lambda X. \tag{3.3}$$

Since σ is parallel we have $\sum_i \bar{D}_X \sigma(e_i,e_i) = 2\sum_r \sigma(A_r X, e_r)$ and also $H = \frac{1}{n} \sum_r (\mathrm{tr}\, A_r) e_r$.Thus, by separating parts tangent to G and normal to

G in (3.3) we get (i) and (ii). Conversely, let (i) and (ii) hold for some $\lambda \neq 0$ and define $\tilde{x}_0 = \tilde{x} - (1/\lambda)\Delta\tilde{x}$. Then one easily verifies that

$$\tilde{\nabla}_X \tilde{x}_0 = X + \frac{1}{\lambda}\tilde{\nabla}_X [nH + \sum_i \sigma(e_i, e_i)] \ = 0,$$

so $\tilde{x}_0$ is constant and $\Delta\tilde{x}_t = \lambda\tilde{x}_t$, where $\tilde{x}_t = \tilde{x} - \tilde{x}_0$, which is precisely the 1-type condition (3.1) above. □

Let $\{\epsilon_j\}_{j=1}^{g}$ be an orthonormal basis of G. In [10] we computed the mean curvature vector of the embedding Φ to be

$$\bar{H}_P = \frac{2m}{kl}\left(\frac{k}{m}I - P\right) = -\frac{2m}{kl}\overrightarrow{CP}. \qquad (3.4)$$

where the vector $\overrightarrow{CP}$ joins the center $C = \frac{k}{m}I$ of the sphere in which G is minimal and a point P. Hence, the mean curvature of G in $H(m)$ is $\sqrt{2m/kl}$. Moreover, from the above expression we get

$$I = \frac{m}{k}P + \frac{1}{2dk}\sum_{j=1}^{g}\sigma(\epsilon_j, \epsilon_j), \qquad (3.5)$$

where $\{\epsilon_i\}$ is an arbitrary orthonormal basis of $T_P G$.

Analyzing the conditions (i) and (ii) of Theorem 3.1 for the benefit of extracting some information about the geometry of a 1-type submanifold is essential, but highly non-trivial. We shall first give some examples.

Examples

1. According to the result of Brada and Niglio ([4], [5]), compact minimal 1-type submanifolds of the real Grassmannian $G(k,l) := G_k\mathbb{R}^m$ (where $l := m - k$) are isometric to the products $G(k_1, l_1) \times \cdots \times G(k_s, l_s)$, with $k_1 + l_1 = \cdots = k_s + l_s$ and $\sum_{i=1}^{s} k_i \leq k$, $\sum_{i=1}^{s} l_i \leq l$, embedded in a natural way. Here, compactness is not essential and the same classification (and the proof) holds when a submanifold is assumed merely complete, as well as the corresponding local version in general. Note, however, that their embedding into the Euclidean space differs from ours by an isometry composed with a homothety.

2. More generally, consider an orthogonal decomposition $\mathbb{F}^m = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ into $\mathbb{F}-$linear subspaces of respective dimensions $m_1, m_2, \cdots, m_s$, with $m_1 + m_2 + \cdots + m_s = m$, and consider the Grassmannians $G_{k_1}V_1, G_{k_2}V_2, \cdots, G_{k_s}V_s$ of $\mathbb{F}-$planes in $V_1, V_2, \cdots, V_s$ of respective dimensions $k_1, k_2, \cdots, k_s$, such that $k_1 + k_2 + \cdots + k_s = k$. Then the image of the embedding

$$\begin{aligned} &x : G_{k_1}V_1 \times G_{k_2}V_2 \times \cdots \times G_{k_s}V_s \longrightarrow G_k\mathbb{F}^m \qquad \text{by} \\ &(\pi_1, \pi_2, ..., \pi_s) \longrightarrow \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_s \end{aligned} \tag{3.6}$$

is a submanifold of $G_k\mathbb{F}^m$ in which each $G_{k_j}V_j$, identified with a leaf of the canonical foliation is totally geodesic. Then $\tilde{x} = \mathrm{diag}\,(\Phi_1, \Phi_2, ..., \Phi_s)$, where $\Phi_j : G_{k_j}\mathbb{F}^{m_j}(V_j) \to \mathbf{H}(m)$ is the canonical 1-type embedding described before. Moreover, $\Phi_j = \frac{k_j}{m_j}I_{m_j} + \Phi_{t_j}$ with $\Delta\Phi_{t_j} = 2dm_j\Phi_{t_j}$ (cf. [10]) where the Laplacian is defined on the corresponding factor-Grassmannian. Then letting I_j stand for the identity matrix of order m_j, we have

$$\begin{aligned} \Delta(\tilde{x} - \tilde{x}_0) &= \mathrm{diag}\,(\Delta\Phi_1, ..., \Delta\Phi_s) = \mathrm{diag}\,(2dm_1\Phi_{t_1}, ..., 2dm_s\Phi_{t_s}) \\ &= 2d\,\mathrm{diag}\,(m_1(\Phi_1 - \frac{k_1}{m_1}I_1), ..., m_s(\Phi_s - \frac{k_s}{m_s}I_s)) \\ &= 2d\,\mathrm{diag}\,(m_1\Phi_1, ..., m_s\Phi_s) - 2d\,\mathrm{diag}\,(k_1I_1, ..., k_sI_s). \end{aligned}$$

If the product submanifold is to be of 1-type than this further must equal $\lambda\,(\tilde{x} - \tilde{x}_0) = \lambda\,\mathrm{diag}\,(\Phi_1, ..., \Phi_s) - \lambda\tilde{x}_0$, for some constant λ and a constant vector(matrix) $\tilde{x}_0$. This holds if and only if $m_1 = m_2 = ... = m_s =: r$ and $m = rs$. In addition, the product submanifold is mass-symmetric if

$$2d\,\mathrm{diag}\,(k_1I_1, ..., k_sI_s) = \lambda\tilde{x}_0 = 2dr\frac{k}{m}I,$$

which implies $k_1 = k_2 = ... = k_s =: q$ and $k = qs$, so that $m/r = k/q$.

3. Consider the complex Grassmannian $G_k\mathbb{C}^m$ where the complex structure at the origin is given by $J\begin{pmatrix} O & B \\ B^* & O \end{pmatrix} = \begin{pmatrix} O & \mathbf{i}B \\ (\mathbf{i}B)^* & O \end{pmatrix}$, thus equal by equivariancy to $JX = \mathbf{i}(I - 2P)X$ for a tangent vector X at a point P. As usual, $\mathbf{i} = \sqrt{-1}$ and $*$ denotes the conjugate transpose. One easily verifies that $\sigma(JX, JY) = \sigma(X, Y)$. Let M^n be a totally real minimal submanifold of $G_k\mathbb{C}^m$ of real dimension half that of the Grassmannian (that is, a minimal Lagrangian submanifold). Then such submanifold is mass-symmetric and of 1-type. Namely,

$$\Delta\left(\tilde{x} - \frac{k}{m}I\right) = \Delta\tilde{x} = -n\tilde{H} = -\sum_{i=1}^{n}\sigma(e_i, e_i) =$$

$$= -\frac{1}{2}\sum_{i=1}^{n}[\sigma(e_i,e_i)+\sigma(Je_i,Je_i)] = -n\frac{1}{2n}\sum_{i=1}^{2n}\sigma(\epsilon_i,\epsilon_i)$$
$$= -n\bar{H} = \frac{2mn}{k(m-k)}\left(\tilde{x}-\frac{k}{m}I\right).$$

Thus, if we set $\tilde{x}_t = \tilde{x} - \frac{k}{m}I$, since $n = k(m-k)$ we have $\Delta\tilde{x}_t = 2m\tilde{x}_t$, which shows that M^n is mass-symmetric and of 1-type and that $2m$ is an eigenvalue of the Laplacian. Therefore for compact minimal Lagrangian submanifold of $G_k\mathbb{C}^m$ it follows $\lambda_1 \leq 2m$.

4. In the quaternionic Grassmannian $G_k\mathbb{H}^m$ of real dimension $4kl$ one introduces local quaternionic structures J_1, J_2 and J_3 in a similar way and shows that $\sigma(J_qX, J_qY) = \sigma(X,Y),\ q = 1,2,3$. Then a minimal totally real submanifold M^n of dimension $n = kl$ (a minimal quaternion-Lagrangian submanifold) is also of type 1 since

$$\Delta\left(\tilde{x}-\frac{k}{m}I\right) = -\frac{1}{4}\sum_{i=1}^{n}[\sigma(e_i,e_i)+\sum_{q=1}^{3}\sigma(J_qe_i,J_qe_i)]$$
$$= -n\frac{1}{4n}\sum_{i=1}^{4n}\sigma(\epsilon_i,\epsilon_i) = 2m\left(\tilde{x}-\frac{k}{m}I\right).$$

As before, for compact M we have $2m \in \mathrm{Spec}\,(M)$.

5. Let $n \equiv 0\,(\,mod\,3)$ and let M^n be a minimal anti-Lagrangian submanifold of $G_k\mathbb{H}^m$ of dimension $n = 3p = 3kl$. Here, anti-Lagrangian means that the normal space $T^{\perp}M$ of M of dimension p, spanned by an orthonormal basis $\{\xi_r\}$, is mapped by the quaternionic structure of G onto the tangent space of M. We show that these submanifolds are also mass-symmetric and of type 1. Indeed,

$$\Delta\left(\tilde{x}-\frac{k}{m}I\right) = -\sum_{i=1}^{3p}\sigma(e_i,e_i) = -\sum_{r=1}^{p}\sum_{q=1}^{3}\sigma(J_q\xi_r,J_q\xi_r)$$
$$= -\frac{3}{4}\sum_{i=1}^{m}\sigma(\epsilon_i,\epsilon_i) = -3p\bar{H} = 6m\left(\tilde{x}-\frac{k}{m}I\right).$$

6. In each of the projective spaces (rank-1 Grassmannians) $\mathbb{F}P^m = G_1\mathbb{F}^{m+1}$ the author has determined the real hypersurfaces of type 1 as follows (see [9], [11], [12]):

In $\mathbb{R}P^m$ it is a totally geodesic $\mathbb{R}P^{m-1}$; in $\mathbb{C}P^m$ it is a geodesic hypersphere (distance hypersphere) of radius $r = \cot^{-1}\left(\sqrt{\frac{1}{2m+1}}\right)$; in $\mathbb{H}P^m$ it is a geodesic hypersphere of radius $r = \cot^{-1}\left(\sqrt{\frac{3}{4m+1}}\right)$. In the last two cases these geodesic spheres posses an interesting property - they are the maximal stable geodesic hyperspheres with respect to the variations preserving the enclosed volume (see [11], [12]).

The problem we want to address in this paper is the study of 1-type submanifolds of higher-rank Grassmannians. In particular, find and study some examples of real hypersurfaces in $G_k\mathbb{F}^m$ which are of 1-type.

If we replace X in (i) of the Theorem 3.1 with $2A_rX$ and $(\operatorname{tr} A_r)X$, respectively, and take the inner product with e_r we get

$$-2n\langle D_{A_rX}H,\ e_r\rangle + \sum_i \langle \sigma(e_i, e_i), \sigma(2A_rX,\ e_r)\rangle = 0$$

and

$$-n(\operatorname{tr} A_r)\langle D_X H,\ e_r\rangle + \sum_i \langle \sigma(e_i, e_i), \sigma((\operatorname{tr} A_r)X,\ e_r)\rangle = 0.$$

Add the two formulas, sum on r, and use (ii) to get

$$n\langle D_X H,\ H\rangle + 2\sum_r \langle D_{A_rX}H,\ e_r\rangle = 0.$$

If the normal vector e_{n+1} is chosen to be parallel to H, then the mean curvature α is defined (up to a sign) by $H = \alpha e_{n+1}$. With $\nabla\alpha$ denoting the gradient of the mean curvature, the above can be written in the form

$$\begin{aligned} nX\langle H, H\rangle &= \langle 2n\alpha\nabla\alpha, X\rangle \\ &= -4\sum_i \langle D_{e_i}H, h(X, e_i)\rangle \\ &= -4\langle h(X, \nabla\alpha), e_{n+1}\rangle - 4\alpha\sum_i \langle D_{e_i}e_{n+1}, h(X, e_i)\rangle \\ &= -4\langle A_{n+1}(\nabla\alpha), X\rangle - 4\alpha\sum_r \omega^r_{n+1}(A_rX), \end{aligned}$$

where ω^r_{n+1} are the corresponding connection 1-forms. Therefore,

$$2A_{n+1}(\nabla\alpha) + n\alpha\nabla\alpha + 2\alpha \sum_{r=n+2}^{g} (\omega_{n+1}^{r} \circ A_r)^{\sharp} = 0, \tag{3.7}$$

where $\omega^{\sharp}$ represents the metric dual vector field of a 1-form ω. For a hypersurface, this implies that $\nabla\alpha$ is a principal direction.

We now turn to the case of real hypersurfaces. Let $x : M^n \to G$ be a real hypersurface ($n = g - 1$) of type 1 in $\mathbf{H}(m)$ via $\tilde{x}$, let ξ be a local unit normal vector field, and A the corresponding shape operator. Then by using (2.2) and (3.5) we reformulate Theorem 1 as follows

Theorem 3.2 *A real hypersurface M^n with the mean curvature function α in a Grassmannian $G_k(\mathbb{F}^m)$ is of 1-type if and only if for every $X \in \Gamma(TM)$ we have*

(i) $\bar{A}_{\sigma(\xi,\xi)}X = n\alpha AX + (2md - \lambda)X - n\langle X, \nabla\alpha\rangle\xi$

(ii) $\sigma(2AX + n\alpha X,\ \xi) = 0,$

where λ is a nonzero constant.

Setting $X = \nabla\alpha$ in (i) and using (3.7) we get

$$\bar{A}_{\sigma(\xi,\xi)}\nabla\alpha = (2md - \lambda - n^2\alpha^2/2)\nabla\alpha - n|\nabla\alpha|^2\xi, \tag{3.8}$$

$$\bar{A}_{\sigma(\xi,\xi)}X = (2md - \lambda + n\alpha\mu)X, \quad \text{for } X \perp \nabla\alpha \text{ and } X \in V_\mu, \tag{3.9}$$

where V_μ represents the eigenspace of an eigenvalue (principal curvature) μ. From (i) we get

$$\langle \bar{A}_{\sigma(\xi,\xi)}\xi, X\rangle = -n\langle\nabla\alpha, X\rangle \tag{3.10}$$

and from (i), (3.4), and (3.5)

$$\begin{aligned}(n\alpha)^2 + n(2md - \lambda) &= \sum_{i=1}^{n} \langle \sigma(\xi,\xi), \sigma(e_i, e_i)\rangle \\ &= \langle \sigma(\xi,\xi), \sum_{i=1}^{g} \sigma(\epsilon_i, \epsilon_i) - \sigma(\xi,\xi)\rangle \\ &= \langle \sigma(\xi,\xi), g\bar{H}\rangle - \|\sigma(\xi,\xi)\|^2 \\ &= 2md - \langle \bar{A}_{\sigma(\xi,\xi)}\xi, \xi\rangle.\end{aligned}$$

Therefore,

$$\bar{A}_{\sigma(\xi,\xi)}\xi = [n\lambda - 2(n-1)md - (n\alpha)^2]\,\xi - n\nabla\alpha. \tag{3.11}$$

We now exploit various components of the 1-type equation (3.2) which we rewrite as $\Delta\tilde{x} - \lambda\tilde{x} = -\lambda\tilde{x}_0$. Denote the vector field on the left hand side by L. We use (i), (2.2), and the equation $n\tilde{H} = n\alpha\xi + 2md(\frac{k}{m}I - \tilde{x}) - \sigma(\xi,\xi)$, which follows from (3.5), to get

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X L, \sigma(Y,\xi)\rangle = -\langle \tilde{\nabla}_X (n\tilde{H}), \sigma(Y,\xi)\rangle \\ &= -X\langle n\tilde{H}, \sigma(Y,\xi)\rangle + \langle n\tilde{H}, \tilde{\nabla}_X \sigma(Y,\xi)\rangle \\ &= X\langle \sigma(\xi,\xi), \sigma(Y,\xi)\rangle - \langle n\tilde{H}, \bar{A}_{\sigma(Y,\xi)}X\rangle \\ &\quad +\langle n\tilde{H}, \sigma(\nabla_X Y,\xi)\rangle + \langle AX,Y\rangle\langle n\tilde{H}, \sigma(\xi,\xi)\rangle - \langle n\tilde{H}, \sigma(Y,AX)\rangle \\ &= -n\langle Y, \nabla_X(\nabla\alpha)\rangle - n\alpha\langle \sigma(X,\xi), \sigma(Y,\xi)\rangle - \langle AX,Y\rangle\|\sigma(\xi,\xi)\|^2 \\ &\quad +n\alpha\langle AX, AY\rangle + (2md-\lambda)\langle X, AY\rangle\,. \end{aligned}$$

By using (3.11) we finally obtain

$$\begin{aligned} n\alpha\langle \sigma(X,\xi), \sigma(Y,\xi)\rangle &= [2mdn - (n+1)\lambda + (n\alpha)^2]\langle AX,Y\rangle \\ &\quad +n\alpha\langle AX,AY\rangle - n\langle Y, \nabla_X(\nabla\alpha)\rangle, \end{aligned} \tag{3.12}$$

where the last term is readily recognized as a multiple of the Hessian,

$$Hess_\alpha(X,Y) =: XY\alpha - (\nabla_X Y)\alpha.$$

Similarly, from

$$0 = \langle \tilde{\nabla}_X L, \sigma(Y,Z)\rangle = -X\langle n\tilde{H}, \sigma(Y,Z)\rangle + \langle n\tilde{H}, \tilde{\nabla}_X \sigma(Y,Z)\rangle$$

we can obtain

$$\begin{aligned} \alpha\langle \sigma(X,\xi), \sigma(Y,Z)\rangle &= \alpha\langle (\nabla_X A)Y, Z\rangle + \langle AX,Y\rangle\langle Z,\nabla\alpha\rangle \\ &\quad +\langle AY,Z\rangle\langle X,\nabla\alpha\rangle + \langle AX,Z\rangle\langle Y,\nabla\alpha\rangle. \end{aligned} \tag{3.13}$$

The other components of $\tilde{\nabla}_X L$ give no additional information.

Theorem 3.3 *For a 1-type hypersurface M^n of $G_k\mathbb{F}^m$ with the mean curvature function α, the following conditions hold:*

(i) $A(\nabla\alpha) = -\frac{n\alpha}{2}\nabla\alpha\,.$

(ii) Let $\mathcal{D} = \{\nabla\alpha\}^\perp$ be the $(n-1)-$ dimensional distribution that is perpendicular to $\nabla\alpha$ on an open set where $\nabla\alpha \neq 0$. Then $\mathcal{D}$ is completely integrable and its integral manifolds have flat normal connection as submanifolds of $G_k\mathbb{F}^m$.

(iii) $\Delta\alpha = \alpha(\lambda - trA^2 - md - 2d + 4)$.

(iv) *If M has constant nonzero mean curvature then its scalar curvature is also constant.*

(v) *If M is compact, trA^2 is constant, and α does not change sign, then $\alpha = const$.*

Proof. (i) follows from (3.7). Therefore on an open (possibly empty) set $\mathcal{U} = \{\nabla\alpha \neq 0\}$ the gradient of the mean curvature gives one of the principal directions. The hypersurfaces of Euclidean space satisfying condition (i) are called H-hypersurfaces and were studied is some detail by Hasanis and Vlachos. The property (ii) is basically proved in [13, p. 149], formulated there for hypersurfaces of Euclidean space but applicable also for Grassmannians (and in general). Namely, let $X, Y \in \Gamma(\mathcal{D})$. Differentiating $\langle X, \nabla\alpha\rangle = 0$ and $\langle Y, \nabla\alpha\rangle = 0$ respectively along Y, X one gets $\langle\nabla_Y X, \nabla\alpha\rangle + Hess_\alpha(Y, X) = 0$ and $\langle\nabla_X Y, \nabla\alpha\rangle + Hess_\alpha(X, Y) = 0$. Subtracting the two equations and using the symmetry of the Hessian we have $[X, Y] = \nabla_Y X - \nabla_X Y \in \mathcal{D}$, which means that $\mathcal{D}$ is an involutive distribution and thus, by the theorem of Frobenius, integrable. Every integral submanifold N through a point in $\mathcal{U}$ has codimension 2 in G and its normal space is spanned by ξ and $e_n := \nabla\alpha/|\nabla\alpha|$. In either case, because e_n is an eigenvector of the shape operator one proves $\nabla^\perp_X\xi = \nabla^\perp_X e_n = 0$, so that the normal connection $\nabla^\perp$ of N is flat [13]. Furthermore, from the Gauss equation

$$n\alpha\bar{R}(\xi, X, \xi, Y) = n\alpha\langle\sigma(\xi, X), \sigma(\xi, Y)\rangle - n\alpha\langle\sigma(\xi, \xi), \sigma(X, Y)\rangle$$

combining Theorem 3.2 (i) and (3.12) it follows

$$\begin{aligned} n\alpha\bar{R}(\xi, X, \xi, Y) = \ & n\alpha\langle AX, AY\rangle + [2mdn - (n+1)\lambda]\langle AX, Y\rangle \\ & - n\alpha(2md - \lambda)\langle X, Y\rangle - nHess_\alpha(X, Y). \end{aligned}$$

Then setting $X = Y = e_i$ and summing on i, using the Einstein property (2.3) and formula (3.11), we get (iii). Parts (iv) and (v) follow from (iii), the Gauss equation, and the Hopf's lemma. □

From (3.12) it follows that if $\alpha = 0$ (the minimal case) then either $A = 0$ or $\lambda = 2mdn/(n+1)$. However, the first possibility is easily resolved since there are no totally geodesic real hypersurfaces of a Grassmannian except in the case of a real projective space whose 1-type hypersurfaces are exactly the totally geodesic projective subspaces of dimension one less ([6], [9], [16], [18]). In other cases, when M is not totally geodesic, the type-eigenvalue is fixed, as given above. Moreover, we have the following result

Theorem 3.4 *A minimal hypersurface M^n of $G_k\mathbb{F}^m$ whose unit normal is ξ is of 1-type in E^N with the type-eigenvalue λ if and only if either M^n is (a portion of) a totally geodesic $\mathbb{R}P^{m-2} \subset \mathbb{R}P^{m-1} = G_1\mathbb{R}^m$ in the case $\mathbb{F} = \mathbb{R}$, or else $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, $k = l$, $m = 2k$ is even, and the following conditions hold:*

(i) $\lambda = \frac{4n}{k}$ *and M^n is mass-symmetric in* $S^{N-1}_{I/2}(\sqrt{\frac{m}{8}})$;

(ii) $\sigma(\xi,\xi) = \frac{2}{k}(I - 2\tilde{x})$, *i.e.* $\xi^2 = \frac{1}{k}I$.

Proof. The statement of the part when $\mathbb{F} = \mathbb{R}$ (the totally geodesic case) is clear from the above discussion and the result of [5]. When $\alpha = 0$, the two conditions of Theorem 3.2 reduce to $\bar{A}_{\sigma(\xi,\xi)}X = (2md - \lambda)X$ and $\sigma(AX, \xi) = 0$, for every $X \in \Gamma(TM)$. It follows therefore,

$$\tilde{\nabla}\sigma(\xi,\xi) = -\bar{A}_{\sigma(\xi,\xi)}X + \bar{D}_X\sigma(\xi,\xi) = (\lambda - 2md)X.$$

Since $\tilde{\nabla}_X\tilde{x} = X$, $\tilde{x}$ being the position vector in E^N, we have

$$\tilde{\nabla}_X[\sigma(\xi,\xi) - (\lambda - 2md)\tilde{x}] = 0.$$

Thus, the bracketed expression represents a constant vector field K along M normal to the Grassmannian and $\sigma(\xi,\xi) = (\lambda - 2md)\tilde{x} + K$. Hence,

$$\bar{A}_{\sigma(\xi,\xi)}X = (2md - \lambda)X + \bar{A}_KX$$

for all $X \in \Gamma(TG)$. If M is not totally geodesic, then $\lambda = \frac{2mdn}{n+1}$ so that (3.11) yields $\bar{A}_{\sigma(\xi,\xi)}\xi = (2md - \lambda)\xi$. Therefore, $\bar{A}_KX = (XK - KX)(I - 2P) = 0$, i.e. $XK = KX$ for every $X \in \Gamma(TG)$. This is equivalent to $K = \rho I$ for some constant ρ. Namely, since the Grassmannian is a symmetric space on which the group $U_{\mathbb{F}}(m)$ acts transitively and the embedding Φ is equivariant, without loss of generality we may assume that the point $P_0 = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$ is in M. At that point we have

$$X = \begin{pmatrix} O & B \\ B^* & O \end{pmatrix},\ B \in M_{k\times l}(\mathbb{F}) \ \text{ and } \ K = \begin{pmatrix} C & O \\ O & D \end{pmatrix},\ C \in \mathbf{H}(k),\ D \in \mathbf{H}(l).$$

The equality $XK = KX$ implies $BD = CB$ for every matrix $B \in M_{k\times l}(\mathbb{F})$. If we denote with indexed lowercase letters the elements of the corresponding matrices then $\sum_{r=1}^{l} b_{ir}d_{rj} = \sum_{s=1}^{k} c_{is}b_{sj}$ holds for every $1 \le i \le k$,

$1 \le j \le l$, and every choice of b_{ij}. Fix i and j and choose $b_{ij} = 1$, all other b's$= 0$. Then $d_{jj} = c_{ii}$ for every i and j in their ranges. Take next $b_{ir} = 1$, $r \ne j$, for any chosen $1 \le r \le l$, set all other b's zero to get $d_{rj} = 0$, for all $r \ne j$. Similarly, $c_{is} = 0$ for $s \ne i$. It follows, therefore, that both C and D are diagonal matrices with identical diagonal entries at every position. In other words, $K = \rho I_m$ and since K is constant so is ρ, hence this holds at every point of G. Therefore,

$$\sigma(\xi,\xi) = \left(\frac{2mdn}{n+1} - 2md\right)\tilde{x} + \rho I,$$

and taking the inner product with I we get $\rho = \frac{2dk}{n+1}$ and

$$\sigma(\xi,\xi) = \frac{2dk}{n+1}\left(I - \frac{m}{k}\tilde{x}\right).$$

In light of the formula (3.5) this is equivalent to

$$n\sigma(\xi,\xi) = \sum_{i=1}^{n} \sigma(e_i, e_i) = n\tilde{H}.$$

Additionally, from the expression for σ we get $\xi^2 = \frac{1}{l}(I + \frac{l-k}{k}\tilde{x})$, since $dkl = n+1$. At point P_0 we have $\xi = \begin{pmatrix} O & B \\ B^* & O \end{pmatrix}$, for some $B \in M_{k\times l}(\mathbb{F})$ and thus

$$\xi^2 = \begin{pmatrix} BB^* & 0 \\ 0 & B^*B \end{pmatrix} = \frac{1}{l}\left[\begin{pmatrix} I_k & O \\ O & I_l \end{pmatrix} + \frac{l-k}{k}\begin{pmatrix} I_k & O \\ O & O \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{k}I_k & O \\ O & \frac{1}{l}I_l \end{pmatrix},$$

implying $BB^* = \frac{1}{k}I_k$ and $B^*B = \frac{1}{l}I_l$. Computing BB^*B two ways we conclude $k = l$. Therefore, $m = 2k$ is even and $\xi^2 = \frac{1}{k}I_m$. Since $n+1 = dkl$, $m = 2k$, $k = l$, the value $\lambda = \frac{2mdn}{n+1}$ simplifies to

$$\lambda = 2md - \frac{8}{m} = \frac{4n}{k} \quad \text{and therefore}$$

$$\sigma(\xi,\xi) = \frac{2}{k}\,(I - 2\tilde{x}). \tag{3.14}$$

Note that this condition implies the condition $\sigma(AX,\xi) = 0$, since differentiating (3.14) yields

$$\tilde{\nabla}_X \sigma(\xi,\xi) = -\bar{A}_{\sigma(\xi,\xi)}X - 2\sigma(AX,\xi) = -\frac{4}{k}X,$$

which gives $\sigma(AX,\xi)=0$ and $\bar{A}_{\sigma(\xi,\xi)}X=\frac{4}{k}X$, the both conditions of Theorem 3.2 with the prescribed value of λ given above, proving the converse. Finally, when $k=l$ and $m=2k$, from before we have

$$\sum_{i=1}^{n}\sigma(e_i,e_i)=n\sigma(\xi,\xi)=-\frac{4n}{k}(\tilde{x}-I/2).$$

On the other hand,

$$\Delta(\tilde{x}-I/2)=\Delta\tilde{x}=-n\tilde{H}=-\sum_{i=1}^{n}\sigma(e_i,e_i)=\frac{4n}{k}(\tilde{x}-I/2).$$

If we put $\tilde{x}_t=\tilde{x}-I/2$, then $\tilde{x}=\frac{I}{2}+\tilde{x}_t$ and $\Delta x_t=\frac{4n}{k}x_t$, giving the 1-type decomposition. Moreover, the center of mass $\tilde{x}_0=I/2$ is at the same time the center of the sphere $S^{N-1}_{kI/m}(\sqrt{\frac{kl}{2m}})$ containing the Grassmannian, so M is mass-symmetric in that sphere. □

Because of the Hopf's lemma, when $\Delta\alpha$ does not change sign on a compact hypersurface M it follows that $\alpha=$ const. In view of (iii) of Theorem 3.3 that happens when α does not change sign and $\lambda-trA^2$ remains sufficiently large or sufficiently small, for example when trA^2 is constant.

Let us assume now that $\alpha=\text{const}\neq 0$. Then, from (3.12) by replacing Y with $AY+\frac{n\alpha}{2}Y$ and referring to the property (ii) of Theorem 3.2, we obtain

$$aA^3X+bA^2X+cAX=0, \tag{3.15}$$

where

$$a=n\alpha,\qquad b=2mdn-(n+1)\lambda+\frac{3}{2}\,(n\alpha)^2,\qquad \text{and}$$
$$c=\frac{n\alpha}{2}\,[2mdn-(n+1)\lambda+(n\alpha)^2]$$

are all constant. Therefore, a 1-type real hypersurface of G with constant nonzero mean curvature can have at most three distinct principal curvatures, all of them constant. As a matter of fact, if $\varkappa$ is a principal curvature it satisfies the equation $\varkappa(a\varkappa^2+b\varkappa+c)=0$ whose solutions are

$$\varkappa_1=0,\quad \varkappa_2=-\frac{n\alpha}{2},\quad \text{and}\quad \varkappa_3=\frac{1}{n\alpha}\,[(n+1)\lambda-2mdn-(n\alpha)^2]. \tag{3.16}$$

A Grassmannian can have a totally umbilical submanifold only if $\operatorname{codim} M\geq \operatorname{rank} G$ [6, p. 71], thus such hypersurfaces exist only in the projective spaces and moreover only as a totally geodesic $\mathbb{R}P^{m-2}$ in

$\mathbb{R}P^{m-1} = G_1\mathbb{R}^m$ ([9], [16], [18]). Therefore considering real hypersurfaces of Grassmannians with constant nonzero mean curvature we have

Theorem 3.5 *A 1-type hypersurface of $G_k\mathbb{F}^m$ with constant nonzero mean curvature α can have only two or three distinct principal curvatures, all of them constant, chosen from the three values given above. If it has three distinct principal curvatures they are the three given in (3.16) and if it has only two constant principal curvatures one of them is equal to $\varkappa_3$ above.*

Note that the values $\varkappa_i$ above, $i = 2, 3$, agree with the values of principal curvatures of 1-type hypersurfaces in the complex and quaternionic projective spaces found in [11] and [12] when $k = 1$. Unlike 1-type geodesic hyperspheres in the complex and quaternionic projective spaces (which do have constant mean curvature), the candidates for 1-type hypersurfaces with constant mean curvature are likely not to be found among geodesic spheres in higher-rank Grassmannians, since they do not have constant mean curvature in general. This follows from the classical result saying that a symmetric space is harmonic when and only when it has rank one. One characterization of harmonic spaces is that small geodesic spheres have constant mean curvature [a]. On the other hand, some geodesic spheres still may be candidates for a 1-type hypersurfaces among non-constant mean curvature hypersurfaces. These investigations will be carried out in another paper.

Let us now consider submanifolds for which $\tilde{x}$ is mass-symmetric and of 1-type. In our case, mass-symmetric means that $\tilde{x}_0 = \frac{k}{m}I$, which is the center of the sphere containing the Grassmannian. Recall that in the case of compact M, $\tilde{x}_0$ is the center of mass. From (3.2) we get

$$-nH - \sum_i \sigma(e_i, e_i) = \lambda\left(\tilde{x} - \frac{k}{m}I\right). \tag{3.17}$$

Since $\tilde{x}$ and I are normal to the Grassmannian we get immediately $H = 0$, i.e. such a submanifold must be minimal. Further, taking the inner product of (3.17) with $\tilde{x}$ and using (2.1) we get $\lambda = \frac{2mn}{kl}$, thus the conditions which characterize submanifolds that are mass-symmetric and of 1-type are $H = 0$ and

$$\sum_{i=1}^{n} \sigma(e_i, e_i) = \frac{2mn}{kl}\left(\frac{k}{m}I - \tilde{x}\right). \tag{3.18}$$

[a]For these remarks I am indebted to Jürgen Berndt.

When $\mathbb{F} = \mathbb{R}$, Brada and Niglio ([4], [5]) classified 1-type minimal submanifolds of the real Grassmannian $G_k\mathbb{R}^m$, showing that they must be isometric to (open portions) of the products of lower dimensional Grassmannians satisfying some dimension restrictions (see Example 1). Those products that are mass-symmetric in S_C^{N-1} satisfy $m_1 = m_2 = ... = m_s$ and $k_1 = k_2 = ... = k_s$ as in Example 2. Thus we have

Corollary 3.1 *The only mass-symmetric 1-type submanifolds of $G_k\mathbb{R}^m$ are (open portions of) the canonically embedded products*

$$G_{k_1}\mathbb{R}^{m_1} \times \cdots \times G_{k_s}\mathbb{R}^{m_s}$$

in which all m_i have the same value r, all k_i have the same value q, and $m = rs,\ k = qs$.

The Examples 2-5 provide minimal mass-symmetric submanifolds of type 1 in $G_k\mathbb{C}^m$ and $G_k\mathbb{H}^m$.

4. Some eigenvalue estimates

We now give some estimates of the first nonzero eigenvalue λ_1 of the Laplace operator on a compact submanifold M of volume V in $G_k\mathbb{F}^m$. Note first that for a compact minimal Lagrangian and quaternion-Lagrangian submanifold M of $G_k\mathbb{C}^m$ and $G_k\mathbb{H}^m$ respectively, one has from Examples 3 and 4 that $2m \in$ Spec (M), thus for those submanifolds $\lambda_1 \le 2m$. Also Example 5 yields $\lambda_1 \le 6m$ for a compact minimal anti-Lagrangian submanifold of $G_k\mathbb{H}^m$. In each case the equality is achieved when the order of the immersion is 1.

Since

$$\langle \Delta\tilde{x}, \tilde{x}\rangle = -\sum_i \langle \sigma(e_i, e_i), \tilde{x}\rangle = n \qquad \text{and}$$
$$\langle \Delta\tilde{x}, \Delta\tilde{x}\rangle = n^2\langle H, H\rangle + \sum_{i,j} \langle \sigma(e_i, e_i), \sigma(e_j, e_j)\rangle,$$

by using the L^2-product $(\cdot,\cdot)$ on M, we have

$$(\Delta\tilde{x}, \Delta\tilde{x}) \ge \lambda_1(\Delta\tilde{x}, \tilde{x}) = \lambda_1 \int_M \langle \Delta\tilde{x}, \tilde{x}\rangle = n\lambda_1 V$$

and thus

$$\lambda_1 \le \frac{n}{V}\int_M \alpha^2\, dV + \frac{1}{nV}\int_M \|\sum_i \sigma(e_i, e_i)\|^2\, dV.$$

If M^n is a real hypersurface of dimension $n = g - 1 = dkl - 1$, by means of (2.1) and (3.5) we further compute

$$\begin{aligned}\|\sum_i \sigma(e_i, e_i)\|^2 &= \|2dkI - 2dm\tilde{x} - \sigma(\xi,\xi)\|^2\\ &= 2d^2mkl - 4dm + \|\sigma(\xi,\xi)\|^2\\ &= 2dm(n-1) + 2Re\,\mathrm{tr}\,\xi^4\\ &\leq 4 + 2dm(n-1),\end{aligned}$$

by the Cauchy-Schwarz inequality as in he proof of Prop. 5 (iii) in [10]. Hence

$$\begin{aligned}\lambda_1 &\leq \frac{2dm(n-1)}{n} + \frac{n}{V}\int_M \alpha^2\, dV + \frac{2}{nV}\int_M \mathrm{tr}\,\xi^4\, dV\\ &\leq \frac{4 + 2dm(n-1)}{n} + \frac{n}{V}\int_M \alpha^2\, dV.\end{aligned}$$

This upper bound is consistent with those found in the projective spaces (see [9], [12]).

We can also consider the following inequality for a submanifold of the low order p, defined as the order of the smallest nonzero eigenvalue of the Laplacian in the spectral decomposition of $\tilde{x}$ into (nonzero) vector eigenfunctions.

$$\begin{aligned}nV &= (d\tilde{x}, d\tilde{x}) = (\tilde{x}, \Delta\tilde{x}) = \sum_{t\geq p} \lambda_t \|\tilde{x}_t\|^2\\ &\geq \lambda_p \sum_{t\geq p} \|\tilde{x}_t\|^2 = \lambda_p(\|\tilde{x}\|^2 - \|\tilde{x}_0\|^2)\\ &= \lambda_p\left(\int_M \frac{1}{2}\mathrm{tr}\,\tilde{x}^2 dV - \int_M \langle\tilde{x}_0, \tilde{x}_0\rangle dV\right) = \lambda_p(k/2 - |\tilde{x}_0|^2)V.\end{aligned}$$

Thus $\lambda_p \leq \frac{2n}{k - 2|\tilde{x}_0|^2}$, the equality taking place when $\tilde{x}$ is of type 1 (See [8] for related inequalities involving the center of mass). Note that the center of mass satisfies $\mathrm{tr}\,\tilde{x}_0 = k$ and $\frac{k^2}{2m} \leq |\tilde{x}_0|^2 < \frac{k}{2}$, since the Grassmannian lies in the plane $\{trA = k\}$ whose distance to the origin is $k/\sqrt{2m}$. In particular, if M is mass-symmetric submanifold of $G_k\mathbb{F}^m$ then $\tilde{x}_0 = \frac{k}{m}I$ and we have the following estimate of λ_1.

Corollary 4.1 *Let M^n be a compact connected submanifold of $G_k\mathbb{F}^m$ of real dimension n which is mass-symmetric and of the low order p via $\tilde{x}$.*

Then

$$\lambda_1 \leq \lambda_p \leq \frac{2mn}{kl},$$

and the second inequality becomes an equality if and only if M is a mass-symmetric submanifold of type 1.

This generalizes a result of B. Y. Chen [7, pp. 320-321]. Examples 2-5 give examples of submanifolds for which the second inequality becomes an equality. On the other hand, characterizing submanifolds (1-type and mass-symmetric) for which the second inequality turns to equality is difficult (save for the real case which is resolved in the Corollary 1), especially in the quaternionic case where things are not clear even when $k = 1$, i.e. in the quaternionic projective space. We note that a characterization of mass-symmetric 1-type submanifolds in $\mathbb{H}P^m = G_1\mathbb{H}^{m+1}$ satisfying $\lambda_p = \frac{2n(m+1)}{m}$ which is given in [7, p. 321] is incomplete since it leaves out the minimal anti-Lagrangian submanifolds and the totally complex ones [12]. We can show, however, that the dimension of such submanifold necessarily satisfies $m \leq n \leq 3m$. Namely, from (3.18) with $m \to m+1,\ k = 1,$ and $\ l = m,$ by taking the inner product with itself and using [12, formula (7)] we get

$$n(n+1) + \sum_{q=1}^{3}\sum_{i} |(J_q e_i)_T|^2 = \frac{n^2(m+1)}{m}.$$

If $S_q X := (J_q X)_T$, the tangent part of $J_q X$, then it follows that $\sum_{q=1}^{3} \|S_q\|^2 = \frac{n(n-m)}{m} \geq 0$ and hence $n \geq m$. In the same vein, from (2.2) and (3.18) one gets

$$\sum_{i=1}^{n} \bar{A}_{\sigma(e_i,e_i)} = \frac{2(m+1)n}{m}\, I.$$

Applying this to a unit normal basis vector e_r and using formula (8) of [12] one gets $\sum_{q=1}^{3} J_q(J_q e_r)_T = -\frac{n}{m} e_r$. Therefore,

$$\sum_{q=1}^{3}\sum_{r}\langle J_q(J_q e_r)_N, e_r\rangle = \frac{n-3m}{m}\sum_{r}\langle e_r, e_r\rangle,$$

and finally

$$\sum_{q=1}^{3}\sum_{r} |(J_q e_r)_N|^2 = \frac{3m-n}{m}(4m-n) \geq 0,$$

proving the claim. Equalities $n = m$, $n = 2m$, $n = 3m$ take place, for example, for minimal Lagrangian, totally complex, and minimal anti-Lagrangian submanifolds of $\mathbb{H}P^m$ respectively.

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