

# On some Improperly Posed Problem for a Degenerate Nonlinear Parabolic Equation

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**Abstract.** We consider the non-characteristic Cauchy problem for the degenerate nonlinear parabolic equation  $|u|^\alpha u_t - \Delta u - \gamma|u|^{-\beta}u = 0$  under some assumptions on  $\alpha, \beta$  and  $\gamma$ . The problem is improperly posed in the sense of Hadamard. We derive for such solutions an estimate in terms of the Cauchy data and a prescribed bound of the solution.

**Keywords:** *Degenerate parabolic equations, non-characteristic Cauchy problem, fast diffusion*

**AMS subject classification:** 35 K 65

## 1. Introduction

The Cauchy problem is not well-posed for the parabolic equation in the sense of Hadamard, if its Cauchy surface is not characteristic. But the estimation on the continuous dependence of solutions holds under their prescribed bound and the bound of their Cauchy data, if the principal part of the equation is linear. Such a estimation for the heat equation was given, e.g., by Cannon [3] with an explicit form, where the heat kernel was considered as that of some integral equation. This problem is related to a model of the oil-well drilling. The solution  $u(x, t)$  means the temperature of the oil at depth  $x$  and time  $t$ , when the space dimension is one. Many authors used the method in [3] to more general situation (c.f., e.g., [9, 17]). The precise bibliography is referred to [9].

The above estimation yields at the same time the unique continuation property of the non-characteristic Cauchy problem for solutions of the parabolic equation. Previously to these results Mizohata [13] established the unique continuation property for linear parabolic equations of second order, where their lower order terms are allowed to be nonlinear in the sense of Lipschitz condition. Continuously to [13], Ohya [14] extended Mizohata's result to some parabolic equations of fourth order. Their method is due to the singular integral operator, from which the theory of pseudo differential operators developed. Afterwards Saut and Scheurer [16] gave another simple proof for the result of [13] without using the theory of pseudo differential operators. Their method is to use a skilful weight function and to yield an  $L^2$ -energy estimate equipped with it.

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From another viewpoint of the unique continuation property in non-characteristic Cauchy problem, there is the work of Lin [11], whose theorem is as follows: if the solution  $u(x, t)$  of the heat equation vanishes at  $(x_0, t_0)$  with infinite order as to the  $x$ -variable, then  $u(x, t_0) = 0$  identically. In [11] it is remarkable that the coefficients of the zeroth order term in the equation are allowed to be unbounded, more precisely,  $L^{(N+2)/2}$ -integrable, where  $N$  is the dimension of the space. The nonlinearity in the sense of Lipschitz condition means that the coefficient is essentially bounded, if it is expressed in the linear form.

Here we explain the terminology "nonlinearity". If the function  $f(u)$  satisfies

$$|f(u) - f(v)| \leq K|u - v|$$

for some constant  $K$ , we say that  $f(u)$  is nonlinear with respect to  $u$  in the sense of Lipschitz condition, or simply it satisfies the Lipschitz condition. If for some differential equation its lower order terms are nonlinear in the sense of Lipschitz condition and  $u, v$  are two solutions, the difference  $u - v$  satisfies another differential inequalities, whose coefficients of lower order terms are essentially bounded. Thus the uniqueness problem is reduced to the stability of null solutions concerning it. When  $f(u)$  does not satisfy the nonlinearity in the sense of Lipschitz condition, the above argument is not correct. For example, Varin [18] treated the semilinear equation

$$Lu + u\varphi(|u|) = 0, \quad (1.1)$$

where  $L$  is a linear parabolic operator of second order, in particular,  $L = \partial_t - \Delta$ . He proved the three cylinder theorem for solutions of equation (1.1). Particularly, if  $\varphi = 0$ , this result is due to Glagoleva [5]. In [18] it is assumed that  $\varphi(\eta) \in C^1(0, \infty)$ ,  $\varphi(\eta) > 0$ ,  $\varphi'(\eta) < 0$  and  $|\varphi'| \leq \frac{1}{\eta}$ , for example,  $\varphi(\eta) = -\log \eta$ . The nonlinear term  $u\varphi(|u|)$  does not satisfy generally the Lipschitz condition.

A nonlinearity of another type appears in parabolic equations, for example, the Navier-Stokes equation. The backward uniqueness also is not well-posed in the sense of Hadamard for solutions of parabolic equations. For the Navier-Stokes equation, Masuda [12] proved the stability of null solutions with respect to the backward unique continuation property, under the homogeneous Dirichlet boundary condition.

In this paper we consider the equation

$$|u|^\alpha u_t - \Delta u - \gamma|u|^{-\beta} u = 0 \quad (1.2)$$

where  $\Delta = \sum_{i=1}^N \partial_{x_i}^2$ . Throughout this paper we assume that

$$\alpha = 0 \quad \text{or} \quad \alpha \geq 1 \quad (1.3)$$

and

$$\beta < 1 \quad \text{and} \quad \beta\gamma \geq 0. \quad (1.4)$$

Though it seems to be strange that the condition  $0 < \alpha < 1$  is removed in (1.3), this is unavoidable in the proof of our Proposition, which will be used for our aim.

If  $0 < \beta < 1$ , the lower order term  $|u|^{-\beta}u$  does not satisfy the Lipschitz condition and is different from the nonlinear part of equation (1.1) in [18]. Let  $\alpha = 0, \gamma \geq 0$  and  $0 < \beta < 1$ . Then equation (1.2) becomes

$$u_t - \Delta u - \gamma|u|^{-\beta}u = 0. \quad (1.5)$$

We examine the existence of  $C^1$ -solutions of equation (1.5) as follows:

We consider equation (1.5) in a cylindrical domain  $D$ , whose lateral boundary is sufficiently regular. Let  $\psi$  be any sufficiently smooth function in  $\bar{D}$ . Then there exists at least one solution  $u$  of equation (1.5) in  $D$  such that  $u$  equals  $\psi$  on the parabolic boundary of  $D$ . That is, the existence theorem of the initial boundary value problem holds for equation (1.5), even if the nonlinear term does not satisfy the Lipschitz condition. But the uniqueness is not assured. This is due to [10: p. 457/Theorem 6.2]. The regularity of  $u$  is assured as follows:  $u \in C^1(D)$  and  $\partial_{x_i}\partial_{x_j}u \in C(D)$  ( $i, j = 1, \dots, N$ ) which is referred to [10].

Secondly, let  $\gamma = 0$  and  $\alpha \geq 1$ . Then equation (1.2) has the form

$$|u|^\alpha u_t = \Delta u \quad (\alpha \geq 1) \quad (1.6)$$

which is the fast diffusion. This concerns the plasma physics and  $u$  means the density of some substance. So, physically it is natural to assume that  $u \geq 0$ . In the fast diffusion the extinction time occurs. In [1] the behavior of  $u$  near the extinction time was studied. If  $\alpha = 1$ , condition (1.4) is known as Okuda-Dawson diffusion.

From the viewpoint of mathematics the definite sign of the solution  $u$  of equation (1.6) needs not to be assumed. We consider the equation

$$mv_t = \Delta(|v|^{m-1}v) \quad (1.7)$$

independently of equation (1.6), where  $m = \frac{1}{\alpha+1}$ . Equations (1.6) and (1.7) are tied up with the relation  $u = |v|^{m-1}v$ . The Cauchy problem for equation (1.7) was studied in [2, 7], where an existence theorem was proved. The situation is different from the case of  $m > 1$ , the porous media equation. The precise references are referred to [8]. If some assumptions are imposed on the initial data, the Cauchy problem (1.7) has a  $C^\infty$  non-negative solution  $u$  for  $t > 0$ . Applying the classical regularity theorem to equation (1.7), we obtain  $u \in C^\infty$ .

By Sabinina [15] it is known that if the non-negative solution  $u$  of equation (1.6) satisfies  $u = 0$  at some point  $(x_0, t_0)$ , then  $u(x, t_0) = 0$  for all  $x$ . The method in [15] is the maximum principle. On the other hand, the author and Yamashiro [6] considered the non-characteristic Cauchy problem for equation (1.6) and proved an estimation of the continuous dependence of non-negative solutions, under their bound and their Cauchy data. Here we assumed  $N = 1$  and the definite sign of solutions. However the  $L^2$ -norm of  $u_t$  can be estimated and the Cauchy surface is allowed to be not convex. The weight function of the estimate used in [6] is primitive and different from [16]. The proof in [6] needs essentially to assume that  $u \geq 0$ .

In this paper we consider the non-characteristic Cauchy problem for equation (1.2) under assumptions (1.3) and (1.4). And we prove an estimation of the continuous dependence for solutions of equation (1.2) under the prescribed bound, where the definite

sign of  $u$  is not assumed, but  $C^1$ -regularity of  $u$  is assumed. The  $L^2$ -norm of  $u$  and  $\nabla_x u$  will be estimated. However, the Cauchy surface is supposed to be strongly convex. If it is not so, we must take the Holmgren transformation. So the situation is more complicated and we cannot yet solve the required problem. When  $u_1$  and  $u_2$  are two solutions of equation (1.2), the difference  $u_1 - u_2$  does not satisfy any differential equations or any differential inequality. Thus our Theorem in the next section means that the null solution of equation (1.2) is stable for the required uniqueness on the non-characteristic Cauchy problem. Our method is to use the weight function devised in [16], in order to obtain the required estimate.

Lastly, we take notice that there is the work of Dinh [4] concerning an existence theorem for the non-characteristic Cauchy problem of linear parabolic equations, in the category of functions with Gevrey class.

## 2. Theorem

Let  $x = (x_1, \dots, x_N)$ ,  $N \geq 1$ . We write  $x' = (x_1, \dots, x_{N-1})$  and  $y = x_N$ . Let  $a$  be a fixed number with  $0 < a < 1$  and  $Q$  be a cylindrical domain such that

$$Q = \left\{ (x', y, t) \mid |x'|^2 + t^2 < a^2 \text{ and } 0 < y < a \right\}.$$

From now on let  $\delta$  and  $\kappa$  be two positive numbers less than 1 such that

$$\delta < \kappa^5 \quad \text{and} \quad \kappa^4 < \min(a^2, 2 - \sqrt{2}). \quad (2.1)$$

We set

$$\varphi(x', y, t) = (y - \delta)^2 + \delta^2(|x'|^2 + t^2).$$

Figures 1 and 2

We define the elliptic and parabolic surfaces

$$\left. \begin{array}{l} S : \varphi = \frac{1}{2}\delta^2 \\ \Gamma : y = \kappa(|x'|^2 + t^2). \end{array} \right\}$$

The  $y$ -coordinates of the intersection of  $S$  and  $\Gamma$  are given by

$$y = \delta \left( 1 - \frac{\delta}{2\kappa} \pm \frac{1}{2} \sqrt{\left(\frac{\delta}{\kappa}\right)^2 - \frac{4\delta}{\kappa} + 2} \right).$$

The interior of the radical sign is positive, because  $\frac{\delta}{\kappa} < 2 - \sqrt{2}$  from (2.1). Let  $D$  be the domain enclosed by  $\Gamma$  and  $S \cap \{y < \delta\}$  (see Figure 1). The maximum of the  $y$ -coordinate of  $\bar{D}$  is

$$\delta \left( 1 - \frac{\delta}{2\kappa} - \frac{1}{2} \sqrt{\left(\frac{\delta}{\kappa}\right)^2 - \frac{4\delta}{\kappa} + 2} \right).$$

We immediately see that

$$|x'|^2 + t^2 < \frac{\delta}{\kappa}, \quad \frac{\delta^2}{2\kappa} < \delta - y \quad \text{in } D \tag{2.2}$$

and  $D \subset Q$  from (2.1).

More generally we define the elliptic surface

$$S_\eta : \varphi = \left(\frac{\delta}{\eta}\right)^2$$

where  $1 < \eta \leq \sqrt{2}$ . We write by  $D_\eta$  the domain enclosed by  $\Gamma$  and  $S_\eta \cap \{y < \delta\}$  (see Figure 2). Further, set  $\Gamma_\eta = \Gamma \cap \partial D_\eta$ . We see that  $S = S_{\sqrt{2}}$  and  $D = D_{\sqrt{2}}$  and  $S_1$  contains the origin. Let  $\theta$  be any number with  $\frac{1}{\eta} < \theta < 1$ . Then  $D_{\theta\eta} \subset D_\eta$ . For such a  $\theta$  we define  $\theta' = \frac{1+\theta}{2}$ . We see that  $\theta < \theta' < 1$  and  $\theta'\eta > 1$ . When  $u$  belongs to  $C^1(Q)$  and is a solution of equation (1.2) in  $Q$ , in the distribution sense, we say simply that  $u$  satisfies equation (1.2) in  $Q$ . From now on we denote by  $C$  all constants independent of  $\alpha, \beta, \gamma, \delta, \eta$  and  $\kappa$ . We put  $\rho(\alpha) = \min(1, \alpha)$ .

Our aim is to prove the following

**Theorem.** *Suppose that  $u$  is in  $C^1(\bar{Q})$  and satisfies equation (1.2) in  $Q$ , further*

$$|u|, |\nabla_x u|, |u_t|, |u|^{\frac{2-\beta}{2}} \leq M \quad \text{in } D_\eta \tag{2.3}$$

and  $M^\alpha < \frac{1}{6\kappa^2}$ . Suppose also that

$$\left. \begin{aligned} \int_{\Gamma_\eta} (u^2 + |\nabla_x u|^2 + u_t^2 + |u|^{2-\beta}) d\sigma &\leq \varepsilon^2 \\ \max(\delta^{-2}\kappa^{-2}, \rho(\alpha)|\gamma(\beta - \alpha)|) &\leq \tau \\ 5\kappa^2 \max((2 + \alpha)^2 N^2 \delta^{-4}, M^{2\alpha} + \rho(\alpha)M^{2(\alpha-\beta)}) &\leq \tau \end{aligned} \right\} \tag{2.4}$$

where

$$\tau = -\frac{\eta^2}{\delta^2(\eta^2 - 1)} \log \frac{\varepsilon}{M}.$$

Then

$$\begin{aligned} & \int_{D_{\theta\eta}} (u^2 + |\nabla_x u|^2) dxdt \\ & \leq C \frac{\kappa^2}{\delta^8} (1 + \alpha)(1 + |\gamma|)(1 + M^{2\alpha}) \\ & \quad \times \frac{1}{(1 - \theta)^2} \exp \left[ \frac{2(1 - \theta'^2)}{\theta'^2(\eta^2 - 1)} \log \varepsilon \right] \exp \left[ \frac{2(\theta'^2\eta^2 - 1)}{\theta'^2(\eta^2 - 1)} \log M \right] \end{aligned} \quad (2.5)$$

where  $C$  does not depend on  $\theta, \varepsilon$  and  $M$ .

If  $\varepsilon \rightarrow 0$ , then the left-hand side of (2.5) converges to 0, too. Hence  $u$  vanishes identically in  $D_{\theta\eta}$ , if  $\varepsilon = 0$ .

### 3. Main Estimate

In this section we impose the assumptions in the previous section, except those of our Theorem. Let  $u$  be a function in  $C^1(\bar{Q})$  satisfying equation (1.2) in  $Q$  and let  $\varphi$  be the function in the previous section. We suppose (2.3) and set  $v = e^{\tau\varphi}u$ , where  $\tau$  is any real number different from that in our Theorem. Hereafter we assume that all constants do not depend on the numbers  $\alpha, \beta, \gamma, \delta, \eta$  and  $\kappa$ .

Under the above assumptions we have

**Proposition.** *Suppose that*

$$\left. \begin{aligned} & M^\alpha \leq \frac{1}{6\kappa^2} \\ & \max(\delta^{-2}\kappa^{-2}, \rho(\alpha)|\gamma(\beta - \alpha)|) \leq \tau \\ & 5\kappa^2 \max((2 + \alpha)^2 N^2 \delta^{-4}, M^{2\alpha} + \rho(\alpha)M^{2(\alpha - \beta)}) \leq \tau. \end{aligned} \right\}$$

Then

$$\begin{aligned} & \int_{D_\eta} |\nabla_x v|^2 dxdt + \frac{1}{1 + \alpha} \left(\frac{\delta}{\kappa}\right)^2 \tau^2 \int_{D_\eta} v^2 dxdt \\ & \leq C \frac{1 + M^{2\alpha}}{\delta^2} \int_{\partial D_\eta} (v_t^2 + |\nabla_x v|^2 + \tau^2 v^2 + |\gamma| e^{\beta\tau\varphi} |v|^{2 - \beta}) d\sigma \end{aligned}$$

where the constant  $C$  does not depend on  $\tau$  and  $M$ .

**Proof.** First we assume that  $u \in C^2(Q) \cap C^1(\bar{Q})$ . In the last we remove this assumption. We denote formally the left-hand side of (1.2) by  $f$  and write simply  $\nabla$  instead of  $\nabla_x$ . Then

$$\left. \begin{aligned} & u_t = e^{-\tau\varphi} (v_t - \tau\varphi_t v) \\ & \Delta u = e^{-\tau\varphi} (\Delta v - 2\tau\nabla\varphi \cdot \nabla v + (\tau^2 |\nabla\varphi|^2 - \tau\Delta\varphi)v) \end{aligned} \right\}.$$

So, equation (1.2) becomes

$$\begin{aligned} -e^{\tau\varphi} f &= [\Delta v + \tau^2 |\nabla\varphi|^2 v + \tau e^{-\alpha\tau\varphi} \varphi_t |v|^\alpha v + \gamma e^{\beta\tau\varphi} |v|^{-\beta} v] \\ & \quad - [2\tau\nabla\varphi \cdot \nabla v + e^{-\alpha\tau\varphi} |v|^\alpha v_t] \\ & \quad - \tau\Delta\varphi \cdot v \\ & =: A - B - E. \end{aligned} \quad (3.1)$$

From now on we yield an estimate of  $v$  on  $D_\eta$ . We denote by  $\|\cdot\|$  and  $(\cdot, \cdot)$  the  $L^2(D_\eta)$ -norm and the  $L^2(D_\eta)$ -inner product, respectively, and by  $\langle \cdot, \cdot \rangle$  the  $L^2(\partial D_\eta)$ -inner product. Easily,

$$\frac{4}{3}(A - B - E)^2 \geq A^2 + B^2 - 2AB - 4E^2.$$

Hence

$$\begin{aligned} & \frac{4}{3}\|e^{\tau\varphi}f\|^2 \\ & \geq \left\| \Delta v + \tau^2|\nabla\varphi|^2v + \tau e^{-\alpha\tau\varphi}\varphi_t|v|^\alpha v + \gamma e^{\beta\tau\varphi}|v|^{-\beta}v \right\|^2 \\ & \quad + \left\| 2\tau\nabla\varphi \cdot \nabla v + e^{-\alpha\tau\varphi}|v|^\alpha v_t \right\|^2 \\ & \quad - 2\left( \Delta v + \tau^2|\nabla\varphi|^2v + \tau e^{-\alpha\tau\varphi}\varphi_t|v|^\alpha v + \gamma e^{\beta\tau\varphi}|v|^{-\beta}v, 2\tau\nabla\varphi \cdot \nabla v + e^{-\alpha\tau\varphi}|v|^\alpha v_t \right) \\ & \quad - 4\tau^2((\Delta\varphi)^2, v^2) \\ & =: I_1 + I_2 - 2I_3 - 4\tau^2((\Delta\varphi)^2, v^2). \end{aligned} \tag{3.2}$$

Now we calculate  $I_3$ , which is of the form

$$\begin{aligned} -I_3 &= -2\tau(\Delta v, \nabla\varphi \cdot \nabla v) - 2\tau^3(|\nabla\varphi|^2v, \nabla\varphi \cdot \nabla v) \\ & \quad - 2\tau^2(\varphi_t e^{-\alpha\tau\varphi}|v|^\alpha v, \nabla\varphi \cdot \nabla v) - 2\gamma\tau(e^{\beta\tau\varphi}|v|^{-\beta}v, \nabla\varphi \cdot \nabla v) \\ & \quad - (\Delta v, e^{-\alpha\tau\varphi}|v|^\alpha v_t) - \tau^2(|\nabla\varphi|^2 e^{-\alpha\tau\varphi}, |v|^\alpha v v_t) \\ & \quad - \tau(\varphi_t e^{-2\alpha\tau\varphi}, |v|^{2\alpha} v v_t) - \gamma(e^{(\beta-\alpha)\tau\varphi}, |v|^{\alpha-\beta} v v_t). \end{aligned} \tag{3.3}$$

Hereafter we use often integration by parts without saying. We estimate each term on the right-hand side of (3.3). First

$$\begin{aligned} -(\Delta v, \nabla\varphi \cdot \nabla v) &= \sum_{i,j} (\partial_{x_i}\partial_{x_j}\varphi, \partial_{x_i}v \cdot \partial_{x_j}v) \\ & \quad + \sum_{i,j} (\partial_{x_j}\varphi, \partial_{x_i}v \cdot \partial_{x_i}\partial_{x_j}v) \\ & \quad - \int_{\partial D_\eta} \partial_\nu v \cdot (\nabla\varphi \cdot \nabla v) d\sigma \end{aligned}$$

where  $\nu$  is the exterior normal on  $\partial D_\eta \cap \{t\}$ ,  $\partial_\nu = \nu \cdot \nabla$ , and

$$\begin{aligned} \sum_{i,j} (\partial_{x_j}\varphi, \partial_{x_i}v \cdot \partial_{x_i}\partial_{x_j}v) &= \frac{1}{2}(\nabla\varphi, \nabla(|\nabla v|^2)) \\ &= -\frac{1}{2}(\Delta\varphi, |\nabla v|^2) + \frac{1}{2} \int_{\partial D_\eta} \partial_\nu \varphi \cdot |\nabla v|^2 d\sigma. \end{aligned}$$

Since  $\nabla(\Delta\varphi) = 0$ ,

$$-\frac{1}{2}(\Delta\varphi, |\nabla v|^2) = \frac{1}{2}(\Delta\varphi, v\Delta v) - \frac{1}{2} \int_{\partial D_\eta} \Delta\varphi \cdot v\partial_\nu v d\sigma.$$

Hence we have

$$\begin{aligned}
-(\Delta v, \nabla \varphi \cdot \nabla v) &= \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) \\
&\quad + \frac{1}{2} (v, \Delta \varphi \cdot \Delta v) \\
&\quad + J_1
\end{aligned} \tag{3.4}$$

where

$$|J_1| \leq C (\langle |\nabla \varphi|, |\nabla v|^2 \rangle + \langle \Delta \varphi, |v| |\nabla v| \rangle).$$

From now on we assume that any constant  $C$  does not depend on  $\tau$  and  $M$ , not only on  $\alpha, \beta, \gamma, \delta, \eta$  and  $\kappa$ .

We continue the following calculations :

$$\begin{aligned}
&-(|\nabla \varphi|^2 v, \nabla \varphi \cdot \nabla v) \\
&= -\frac{1}{2} (|\nabla \varphi|^2, \nabla \varphi \cdot \nabla v^2) \\
&= \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi \cdot \partial_{x_i} \varphi \cdot \partial_{x_j} \varphi, v^2) + \frac{1}{2} (|\nabla \varphi|^2 \Delta \varphi, v^2) \\
&\quad - \frac{1}{2} \int_{\partial D_\eta} |\nabla \varphi|^2 \partial_\nu \varphi \cdot v^2 d\sigma
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&-(e^{-\alpha\tau\varphi} \varphi_t |v|^\alpha v, \nabla \varphi \cdot \nabla v) \\
&= -\frac{1}{\alpha+2} (e^{-\alpha\tau\varphi} \varphi_t, \nabla \varphi \cdot \nabla (|v|^{\alpha+2})) \\
&= \frac{1}{\alpha+2} (e^{-\alpha\tau\varphi} \varphi_t \Delta \varphi, |v|^{\alpha+2}) - \frac{\alpha}{\alpha+2} \tau (e^{-\alpha\tau\varphi} \varphi_t |\nabla \varphi|^2, |v|^{\alpha+2}) \\
&\quad - \frac{1}{\alpha+2} \int_{\partial D_\eta} e^{-\alpha\tau\varphi} \varphi_t \partial_\nu \varphi \cdot |v|^{\alpha+2} d\sigma.
\end{aligned} \tag{3.6}$$

Here we have used the fact that  $\nabla \varphi_t = O$ . Further,

$$\begin{aligned}
&-(e^{\beta\tau\varphi} |v|^{-\beta} v, \nabla \varphi \cdot \nabla v) \\
&= -\frac{1}{2-\beta} (e^{\beta\tau\varphi} \nabla \varphi, \nabla (|v|^{2-\beta})) \\
&= \frac{1}{2-\beta} (e^{\beta\tau\varphi} \Delta \varphi, |v|^{2-\beta}) + \frac{\beta}{2-\beta} \tau (e^{\beta\tau\varphi} |\nabla \varphi|^2, |v|^{2-\beta}) \\
&\quad - \frac{1}{2-\beta} \int_{\partial D_\eta} e^{\beta\tau\varphi} \partial_\nu \varphi \cdot |v|^{2-\beta} d\sigma.
\end{aligned} \tag{3.7}$$

Now we calculate

$$\begin{aligned}
-(\Delta v, e^{-\alpha\tau\varphi} |v|^\alpha v_t) &= (e^{-\alpha\tau\varphi} |v|^\alpha, \nabla v \cdot \nabla v_t) \\
&\quad + \alpha (e^{-\alpha\tau\varphi} |v|^{\alpha-2} v, |\nabla v|^2 v_t) \\
&\quad - \alpha \tau (e^{-\alpha\tau\varphi} \nabla \varphi \cdot \nabla v, |v|^\alpha v_t) \\
&\quad - \int_{\partial D_\eta} e^{-\alpha\tau\varphi} \partial_\nu v \cdot |v|^\alpha v_t d\sigma
\end{aligned}$$



with

$$\begin{aligned}
 & (e^{-\alpha\tau\varphi}|v|^\alpha, \nabla v \cdot \nabla v_t) \\
 &= \frac{1}{2}(e^{-\alpha\tau\varphi}|v|^\alpha, (|\nabla v|^2)_t) \\
 &= -\frac{1}{2}\alpha(e^{-\alpha\tau\varphi}|v|^{\alpha-2}vv_t, |\nabla v|^2) + \frac{1}{2}\alpha\tau(e^{-\alpha\tau\varphi}\varphi_t, |v|^\alpha|\nabla v|^2) \\
 &\quad + \frac{1}{2}\int_{\partial D_\eta} e^{-\alpha\tau\varphi}|v|^\alpha|\nabla v|^2 \cos(\tilde{\nu}, t) d\sigma
 \end{aligned}$$

where  $(\tilde{\nu}, t)$  is the angle between the  $t$ -axis and the exterior normal  $\tilde{\nu}$  on  $\partial D_\eta$ . If  $\alpha = 0$ , the integral  $(e^{-\alpha\tau\varphi}|v|^{\alpha-2}vv_t, |\nabla v|^2)$  does not converge in general, but it is regarded as 0, because its coefficients are  $\alpha$ . From the above we have

$$\begin{aligned}
 -(\Delta v, e^{-\alpha\tau\varphi}|v|^\alpha v_t) &= \frac{1}{2}\alpha(e^{-\alpha\tau\varphi}|v|^{\alpha-2}vv_t, |\nabla v|^2) \\
 &\quad - \alpha\tau(e^{-\alpha\tau\varphi}\nabla\varphi \cdot \nabla v, |v|^\alpha v_t) \\
 &\quad + \frac{1}{2}\alpha\tau(e^{-\alpha\tau\varphi}\varphi_t, |v|^\alpha|\nabla v|^2) \\
 &\quad + J_2
 \end{aligned} \tag{3.8}$$

where

$$|J_2| \leq C(\langle |u|^\alpha, |v_t||\nabla v| \rangle + \langle |u|^\alpha, |\nabla v|^2 \rangle).$$

We calculate each term on the right-hand side of (3.8). First,

$$(e^{-\alpha\tau\varphi}|v|^{\alpha-2}vv_t, |\nabla v|^2) = (|u|^{\alpha-2}uu_t, |\nabla v|^2) + \tau(\varphi_t|u|^\alpha, |\nabla v|^2). \tag{3.9}$$

From (3.1) we have

$$\begin{aligned}
 & - (e^{-\alpha\tau\varphi}\nabla\varphi \cdot \nabla v, |v|^\alpha v_t) \\
 &= -\left(\nabla\varphi \cdot \nabla v, \Delta v + \tau^2|\nabla\varphi|^2v + \tau e^{-\alpha\tau\varphi}\varphi_t|v|^\alpha v\right. \\
 &\quad \left.+ \gamma e^{\beta\tau\varphi}|v|^{-\beta}v - 2\tau\nabla\varphi \cdot \nabla v - \tau\Delta\varphi \cdot v + e^{\tau\varphi}f\right).
 \end{aligned}$$

This is written as

$$\begin{aligned}
 & - (e^{-\alpha\tau\varphi}\nabla\varphi \cdot \nabla v, |v|^\alpha v_t) \\
 &= -(\nabla\varphi \cdot \nabla v, \Delta v) - \tau^2(\nabla\varphi \cdot \nabla v, |\nabla\varphi|^2v) \\
 &\quad - \tau(\nabla\varphi \cdot \nabla v, e^{-\alpha\tau\varphi}\varphi_t|v|^\alpha v) - \gamma(\nabla\varphi \cdot \nabla v, e^{\beta\tau\varphi}|v|^{-\beta}v) \\
 &\quad + 2\tau(1, (\nabla\varphi \cdot \nabla v)^2) + \tau(\nabla\varphi \cdot \nabla v, \Delta\varphi \cdot v) - (\nabla\varphi \cdot \nabla v, e^{\tau\varphi}f).
 \end{aligned} \tag{3.10}$$

Now we calculate each term on the right-hand side of (3.10). Recall (3.4) and (3.5). We see that

$$\begin{aligned}
 & - (\nabla\varphi \cdot \nabla v, e^{-\alpha\tau\varphi}\varphi_t|v|^\alpha v) \\
 &= -\frac{1}{\alpha+2}(e^{-\alpha\tau\varphi}\varphi_t, \nabla\varphi \cdot \nabla|v|^{\alpha+2}) \\
 &= \frac{1}{\alpha+2}(e^{-\alpha\tau\varphi}\varphi_t\Delta\varphi, |v|^{\alpha+2}) - \frac{\alpha}{\alpha+2}\tau(e^{-\alpha\tau\varphi}\varphi_t|\nabla\varphi|^2, |v|^{\alpha+2}) \\
 &\quad - \frac{1}{\alpha+2}\int_{\partial D_\eta} e^{-\alpha\tau\varphi}\varphi_t\partial_\nu\varphi \cdot |v|^{\alpha+2}d\sigma,
 \end{aligned}$$

$$\begin{aligned}
& -(\nabla\varphi \cdot \nabla v, e^{\beta\tau\varphi}|v|^{-\beta}v) \\
& = -\frac{1}{2-\beta}(e^{\beta\tau\varphi}, \nabla\varphi \cdot \nabla|v|^{2-\beta}) \\
& = \frac{1}{2-\beta}(e^{\beta\tau\varphi}\Delta\varphi, |v|^{2-\beta}) + \frac{\beta}{2-\beta}\tau(e^{\beta\tau\varphi}|\nabla\varphi|^2, |v|^{2-\beta}) \\
& \quad - \frac{1}{2-\beta} \int_{\partial D_\eta} e^{\beta\tau\varphi} \partial_\nu \varphi \cdot |v|^{2-\beta} d\sigma
\end{aligned}$$

and

$$(\nabla\varphi \cdot \nabla v, \Delta\varphi \cdot v) = \frac{1}{2}(\Delta\varphi, \nabla\varphi \cdot \nabla v^2) = -\frac{1}{2}((\Delta\varphi)^2, v^2) + \frac{1}{2} \int_{\partial D_\eta} \Delta\varphi \cdot \partial_\nu \varphi \cdot v^2 d\sigma.$$

Hence from (3.10) it follows that

$$\begin{aligned}
& - (e^{-\alpha\tau\varphi} \nabla\varphi \cdot \nabla v, |v|^\alpha v_t) \\
& = \frac{1}{2}(v, \Delta\varphi \cdot \Delta v) + \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) + \frac{1}{2}\tau^2(|\nabla\varphi|^2 \Delta\varphi, v) \\
& \quad + \frac{1}{2}\tau^2(\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) + \frac{1}{\alpha+2}\tau(e^{-\alpha\tau\varphi} \varphi_t \Delta\varphi, |v|^{\alpha+2}) \\
& \quad - \frac{\alpha}{\alpha+2}\tau^2(e^{-\alpha\tau\varphi} \varphi_t |\nabla\varphi|^2, |v|^{\alpha+2}) + \frac{\gamma}{2-\beta}(e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) \\
& \quad + \frac{\beta\gamma}{2-\beta}\tau(e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\
& \quad + 2\tau(1, (\nabla\varphi \cdot \nabla v)^2) - \frac{1}{2}\tau((\Delta\varphi)^2, v^2) - (\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f) \\
& \quad + J_3
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
|J_3| \leq C & \left( \langle |\nabla\varphi|, |\nabla v|^2 \rangle + \langle \Delta\varphi, |v| |\nabla v| \rangle + \tau^2 \langle |\nabla\varphi|^3, v^2 \rangle \right. \\
& \left. + \tau \langle |\varphi_t| |\nabla\varphi|, |u|^\alpha v^2 \rangle + |\gamma| \langle e^{\beta\tau\varphi} |\nabla\varphi|, |v|^{2-\beta} \rangle + \tau \langle |\nabla\varphi| \Delta\varphi, v^2 \rangle \right).
\end{aligned}$$

Combining (3.8) and (3.9) with (3.11), we have

$$\begin{aligned}
& -(\Delta v, e^{-\alpha\tau\varphi}|v|^\alpha v_t) \\
& = \frac{1}{2}\alpha(|u|^{\alpha-2} u u_t, |\nabla v|^2) + \frac{1}{2}\alpha\tau(\varphi_t |u|^\alpha, |\nabla v|^2) + \frac{1}{2}\alpha\tau(v, \Delta\varphi \cdot \Delta v) \\
& \quad + \alpha\tau \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) + \frac{1}{2}\alpha\tau^3(|\nabla\varphi|^2 \Delta\varphi, v^2) \\
& \quad + \frac{1}{2}\alpha\tau^3(\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) + \frac{\alpha}{\alpha+2}\tau^2(e^{-\alpha\tau\varphi} \varphi_t \Delta\varphi, |v|^{\alpha+2}) \\
& \quad - \frac{1}{\alpha+2}\alpha^2\tau^3(\varphi_t |\nabla\varphi|^2, |u|^\alpha v^2) + \frac{\alpha\gamma}{2-\beta}\tau(e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) \\
& \quad + \frac{\alpha\beta\gamma}{2-\beta}\tau^2(e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\
& \quad + 2\alpha\tau^2(1, (\nabla\varphi \cdot \nabla v)^2) - \frac{1}{2}\alpha\tau^2((\Delta\varphi)^2, v^2) + J_2 + \alpha\tau J_3 \\
& \quad - \alpha\tau(\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f) + \frac{1}{2}\alpha\tau(\varphi_t |u|^\alpha, |\nabla v|^2).
\end{aligned} \tag{3.12}$$

Lastly, we calculate the remained terms on the right-hand side of (3.3). First,

$$\begin{aligned}
& - (|\nabla\varphi|^2 e^{-\alpha\tau\varphi}, |v|^\alpha v v_t) \\
& = -\frac{1}{\alpha+2} (|\nabla\varphi|^2 e^{-\alpha\tau\varphi}, (|v|^{\alpha+2})_t) \\
& = -\frac{\alpha}{\alpha+2} \tau (|\nabla\varphi|^2 \varphi_t e^{-\alpha\tau\varphi}, |v|^{\alpha+2}) - \frac{1}{\alpha+2} \int_{\partial D_\eta} |\nabla\varphi|^2 e^{-\alpha\tau\varphi} |v|^{\alpha+2} \cos(\tilde{\nu}, t) d\sigma
\end{aligned} \tag{3.13}$$

because  $(|\nabla\varphi|^2)_t = 0$ . Next,

$$\begin{aligned}
& - (e^{-2\alpha\tau\varphi} \varphi_t, |v|^{2\alpha} v v_t) \\
& = -\frac{1}{2(\alpha+1)} (e^{-2\alpha\tau\varphi} \varphi_t, (|v|^{2\alpha+2})_t) \\
& = \frac{1}{2(\alpha+1)} (e^{-2\alpha\tau\varphi} \varphi_{tt}, |v|^{2\alpha+2}) - \frac{\alpha}{\alpha+1} \tau (e^{-2\alpha\tau\varphi} \varphi_t^2, |v|^{2\alpha+2}) \\
& \quad - \frac{1}{2(\alpha+1)} \int_{\partial D_\eta} e^{-2\alpha\tau\varphi} \varphi_t |v|^{2\alpha+2} \cos(\tilde{\nu}, t) d\sigma
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
& - (e^{(\beta-\alpha)\tau\varphi}, |v|^{\alpha-\beta} v v_t) \\
& = -\frac{1}{\alpha-\beta+2} (e^{(\beta-\alpha)\tau\varphi}, (|v|^{\alpha-\beta+2})_t) \\
& = \frac{\beta-\alpha}{\alpha-\beta+2} \tau (e^{(\beta-\alpha)\tau\varphi} \varphi_t, |v|^{\alpha-\beta+2}) \\
& \quad - \frac{1}{\alpha-\beta+2} \int_{\partial D_\eta} e^{(\beta-\alpha)\tau\varphi} |v|^{\alpha-\beta+2} \cos(\tilde{\nu}, t) d\sigma.
\end{aligned} \tag{3.15}$$

Combining (3.3) with (3.4) - (3.7) and (3.12) - (3.15), we finally conclude that

$$\begin{aligned}
-I_3 & = 2\tau \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) + \tau (v, \Delta\varphi \cdot \Delta v) \\
& \quad + 2\tau^3 \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi \cdot \partial_{x_i} \varphi \cdot \partial_{x_j} \varphi, v^2) + \tau^3 (|\nabla\varphi|^2 \Delta\varphi, v^2) \\
& \quad + \frac{2}{\alpha+2} \tau^2 (\varphi_t \Delta\varphi, |u|^\alpha v^2) - \frac{2\alpha}{\alpha+2} \tau^3 (\varphi_t |\nabla\varphi|^2, |u|^\alpha v^2) \\
& \quad + \frac{2\gamma}{2-\beta} \tau (e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) + \frac{2\beta\gamma}{2-\beta} \tau^2 (e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\
& \quad + \frac{1}{2} \alpha (|u|^{\alpha-2} u u_t, |\nabla v|^2) + \frac{1}{2} \alpha \tau (\varphi_t |u|^\alpha, |\nabla v|^2) \\
& \quad + \frac{1}{2} \alpha \tau (v, \Delta\varphi \cdot \Delta v) + \alpha \tau \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) \\
& \quad + \frac{1}{2} \alpha \tau^3 (|\nabla\varphi|^2 \Delta\varphi, v^2) + \frac{1}{2} \alpha \tau^3 (\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) \\
& \quad + \frac{\alpha}{\alpha+2} \tau^2 (\varphi_t \Delta\varphi, |u|^\alpha v^2) - \frac{1}{\alpha+2} \alpha^2 \tau^3 (\varphi_t |\nabla\varphi|^2, |u|^\alpha v^2) \\
& \quad + \frac{\alpha\gamma}{2-\beta} \tau (e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) + \frac{\alpha\beta\gamma}{2-\beta} \tau^2 (e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\
& \quad + 2\alpha\tau^2 (1, (\nabla\varphi \cdot \nabla v)^2) - \frac{1}{2} \alpha \tau^2 ((\Delta\varphi)^2, v^2) \\
& \quad + \frac{1}{2} \alpha \tau (\varphi_t |u|^\alpha, |\nabla v|^2) - \frac{\alpha}{\alpha+2} \tau^3 (|\nabla\varphi|^2 \varphi_t, |u|^\alpha v^2) \\
& \quad + \frac{\tau}{2(\alpha+1)} (\varphi_{tt}, |u|^{2\alpha} v^2) - \frac{\alpha}{\alpha+1} \tau^2 (\varphi_t^2, |u|^{2\alpha} v^2) \\
& \quad + \frac{\gamma(\beta-\alpha)}{\alpha-\beta+2} \tau (\varphi_t, |u|^{\alpha-\beta} v^2) - \alpha \tau (\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f) \\
& \quad + J_4
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
J_4 &= 2\tau J_1 - \tau^3 \int_{\partial D_\eta} |\nabla\varphi|^2 \partial_\nu\varphi \cdot v^2 d\sigma - \frac{2}{\alpha+2}\tau^2 \int_{\partial D_\eta} \varphi_t \partial_\nu\varphi \cdot |u|^\alpha v^2 d\sigma \\
&\quad - \frac{2\gamma}{2-\beta}\tau \int_{\partial D_\eta} e^{\beta\tau\varphi} \partial_\nu\varphi \cdot |v|^{2-\beta} d\sigma + J_2 + \alpha\tau J_3 \\
&\quad - \frac{1}{\alpha+2}\tau^2 \int_{\partial D_\eta} |\nabla\varphi|^2 |u|^\alpha v^2 \cos(\tilde{\nu}, t) d\sigma - \frac{1}{2(\alpha+1)}\tau \int_{\partial D_\eta} \varphi_t |u|^{2\alpha} v^2 \cos(\tilde{\nu}, t) d\sigma \\
&\quad - \frac{\gamma}{\alpha-\beta+2} \int_{\partial D_\eta} |u|^{\alpha-\beta} v^2 \cos(\tilde{\nu}, t) d\sigma.
\end{aligned} \tag{3.17}$$

We arrange the right-hand side of (3.16) as follows:

$$\begin{aligned}
-I_3 &= (\alpha+2)\tau \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi, \partial_{x_i} v \cdot \partial_{x_j} v) + \tau(v, \Delta\varphi \cdot \Delta v) \\
&\quad + 2\tau^3 \sum_{i,j} (\partial_{x_i} \partial_{x_j} \varphi \cdot \partial_{x_i} \varphi \cdot \partial_{x_j} \varphi, v^2) + \frac{\alpha+2}{2}\tau^3 (|\nabla\varphi|^2 \Delta\varphi, v^2) \\
&\quad + \tau^2 (\varphi_t \Delta\varphi, |u|^\alpha v^2) - \frac{\alpha(\alpha+3)}{\alpha+2}\tau^3 (\varphi_t |\nabla\varphi|^2, |u|^\alpha v^2) \\
&\quad + \frac{\gamma(\alpha+2)}{2-\beta}\tau (e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) + \frac{\beta\gamma(\alpha+2)}{2-\beta}\tau^2 (e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\
&\quad + \frac{1}{2}\alpha (|u|^{\alpha-2} uu_t, |\nabla v|^2) + \alpha\tau (\varphi_t |u|^\alpha, |\nabla v|^2) + \frac{1}{2}\alpha\tau (v, \Delta\varphi \cdot \Delta v) \\
&\quad + \frac{1}{2}\alpha\tau^3 (\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) + 2\alpha\tau^2 (1, (\nabla\varphi \cdot \nabla v)^2) \\
&\quad - \frac{1}{2}\alpha\tau^2 ((\Delta\varphi)^2, v^2) + \frac{\tau}{2(\alpha+1)} (\varphi_{tt}, |u|^{2\alpha} v^2) - \frac{\alpha}{\alpha+1}\tau^2 (\varphi_t^2, |u|^{2\alpha} v^2) \\
&\quad + \frac{\gamma(\beta-\alpha)}{\alpha-\beta+2}\tau (\varphi_t, |u|^{\alpha-\beta} v^2) - \alpha\tau (\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f) \\
&\quad + J_4 \\
&=: \sum_{i=1}^{18} K_i + J_4.
\end{aligned} \tag{3.18}$$

We set

$$I_4 = K_2 + K_4 + K_{11}.$$

Then

$$\begin{aligned}
2I_4 &= (\alpha+2)\tau (v, \Delta\varphi \cdot \Delta v) + (\alpha+2)\tau^3 (|\nabla\varphi|^2 \Delta\varphi, v^2) \\
&= (\alpha+2)\tau (\Delta\varphi \cdot v, \Delta v + \tau^2 |\nabla\varphi|^2 v + \tau e^{-\alpha\tau\varphi} \varphi_t |v|^\alpha v + \gamma e^{\beta\tau\varphi} |v|^{-\beta} v) \\
&\quad - (\alpha+2)\tau (\Delta\varphi \cdot v, \tau e^{-\alpha\tau\varphi} \varphi_t |v|^\alpha v + \gamma e^{\beta\tau\varphi} |v|^{-\beta} v).
\end{aligned}$$

Thus

$$\begin{aligned}
2I_4 &\geq -I_1 \\
&\quad - \frac{1}{4}(\alpha+2)^2 \tau^2 ((\Delta\varphi)^2, v^2) \\
&\quad - (\alpha+2)\tau^2 (\Delta\varphi \cdot \varphi_t, |u|^\alpha v^2) - (\alpha+2)\gamma\tau (e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta})
\end{aligned} \tag{3.19}$$

where we have used the trivial inequality  $AB \geq -A^2 - \frac{1}{4}B^2$ . Further, we set

$$I_5 = K_7 + K_8 + K_{17} - \frac{1}{2}\gamma(\alpha + 2)\tau(e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}).$$

Then

$$\begin{aligned} I_5 &= \frac{\beta\gamma(\alpha+2)}{2(2-\beta)}\tau(e^{\beta\tau\varphi} \Delta\varphi, |v|^{2-\beta}) \\ &\quad + \frac{\beta\gamma(\alpha+2)}{2-\beta}\tau^2(e^{\beta\tau\varphi} |\nabla\varphi|^2, |v|^{2-\beta}) \\ &\quad + \frac{\gamma(\beta-\alpha)}{\alpha-\beta+2}\tau(\varphi_t |u|^{\alpha-\beta}, v^2). \end{aligned}$$

For the time being let  $\alpha \geq 1$ . We use the inequality  $\Delta\varphi \geq 0$  and assumption (1.4). Then

$$I_5 \geq \frac{\gamma(\beta-\alpha)}{\alpha-\beta+2}\tau(\varphi_t |u|^{\alpha-\beta}, v^2). \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.18), we obtain from (3.2)

$$\begin{aligned} \frac{4}{3}\|e^{\tau\varphi} f\|^2 &\geq -\frac{1}{4}(\alpha + 2)^2\tau^2((\Delta\varphi)^2, v^2) - (\alpha + 2)\tau^2(\Delta\varphi \cdot \varphi_t, |u|^\alpha v^2) \\ &\quad + 2(\alpha + 2)\tau\sum_{i,j}(\partial_{x_i}\partial_{x_j}\varphi, \partial_{x_i}v \cdot \partial_{x_j}v) \\ &\quad + 4\tau^3\sum_{i,j}(\partial_{x_i}\partial_{x_j}\varphi \cdot \partial_{x_i}\varphi \cdot \partial_{x_j}\varphi, v^2) + 2\tau^2(\varphi_t\Delta\varphi, |u|^\alpha v^2) \\ &\quad - \frac{2\alpha}{\alpha+2}(\alpha + 3)\tau^3(\varphi_t|\nabla\varphi|^2, |u|^\alpha v^2) + \alpha(|u|^{\alpha-2}uu_t, |\nabla v|^2) \\ &\quad + 2\alpha\tau(\varphi_t|u|^\alpha, |\nabla v|^2) + \alpha\tau^3(\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) \\ &\quad + 4\alpha\tau^2(1, (\nabla\varphi \cdot \nabla v)^2) - (\alpha + 4)\tau^2((\Delta\varphi)^2, v^2) \\ &\quad + \frac{\tau}{\alpha+1}(\varphi_{tt}, |u|^{2\alpha}v^2) - \frac{2\alpha}{\alpha+1}\tau^2(\varphi_t^2, |u|^{2\alpha}v^2) \\ &\quad + \frac{2\gamma(\beta-\alpha)}{\alpha-\beta+2}\tau(\varphi_t|u|^{\alpha-\beta}, v^2) - 2\alpha\tau(\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f) \\ &\quad + 2J_4. \end{aligned} \quad (3.21)$$

Using the inequalities

$$\frac{1}{4}(\alpha + 2)^2 + (\alpha + 4) \leq 2(\alpha + 2)^2$$

and

$$2\alpha\tau|(\nabla\varphi \cdot \nabla v, e^{\tau\varphi} f)| \leq 4\alpha\tau^2(1, (\nabla\varphi \cdot \nabla v)^2) + \frac{1}{4}\alpha\|e^{\tau\varphi} f\|^2$$

we rearrange (3.21) as

$$\begin{aligned} &C(|J_4| + (1 + \alpha)\|e^{\tau\varphi} f\|^2) \\ &\geq 2(\alpha + 2)\tau\sum_{i,j}(\partial_{x_i}\partial_{x_j}\varphi, \partial_{x_i}v \cdot \partial_{x_j}v) - \alpha(|u|^{\alpha-1}|u_t||\nabla v|^2) \\ &\quad - 2\alpha\tau(|\varphi_t||u|^\alpha, |\nabla v|^2) - 2(\alpha + 2)^2\tau^2((\Delta\varphi)^2, v^2) \\ &\quad + 4\tau^3\sum_{i,j}(\partial_{x_i}\partial_{x_j}\varphi \cdot \partial_{x_i}\varphi \cdot \partial_{x_j}\varphi, v^2) + \alpha\tau^3(\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi, v^2) \\ &\quad - \alpha\tau^2(\Delta\varphi \cdot \varphi_t, |u|^\alpha v^2) - \frac{2\alpha}{\alpha+2}(\alpha + 3)\tau^3(\varphi_t|\nabla\varphi|^2, |u|^\alpha v^2) \\ &\quad + \frac{\tau}{\alpha+1}(\varphi_{tt}, |u|^{2\alpha}v^2) - \frac{2\alpha}{\alpha+1}\tau^2(\varphi_t^2, |u|^{2\alpha}v^2) \\ &\quad + \frac{2\gamma(\beta-\alpha)}{\alpha-\beta+2}\tau(\varphi_t|u|^{\alpha-\beta}, v^2). \end{aligned} \quad (3.22)$$

We enumerate some properties of  $\varphi$ :

$$\left. \begin{aligned} |\nabla\varphi|^2 &= 4((y - \delta)^2 + \delta^4|x'|^2) \leq 4\varphi \\ \Delta\varphi &= 2(1 + \delta^2(N - 1)) \\ \varphi_t &= 2\delta^2t \\ \nabla(|\nabla\varphi|^2) \cdot \nabla\varphi &= 16((y - \delta)^2 + \delta^6|x'|^2) \\ \sum_{i,j} \partial_{x_i} \partial_{x_j} \varphi \cdot \xi_i \xi_j &= 2(\delta^2|\xi'|^2 + \xi_N^2) \geq 2\delta^2|\xi|^2 \\ \sum_{i,j} \partial_{x_i} \partial_{x_j} \varphi \cdot \partial_{x_i} \varphi \cdot \partial_{x_j} \varphi &= 8((y - \delta)^2 + \delta^6|x'|^2). \end{aligned} \right\}$$

where  $\xi = (\xi_1, \dots, \xi_N) = (\xi', \xi_N) \in \mathbb{R}^N$ . It is immediately seen that inequality (3.22) can be continued by

$$\begin{aligned} & C(|J_4| + (1 + \alpha)\|e^{\tau\varphi} f\|^2) \\ & \geq 4(\alpha + 2)\delta^2\tau(1, |\nabla v|^2) - \alpha(|u|^{\alpha-1}|u_t|, |\nabla v|^2) - 4\alpha\delta^2\tau(|t||u|^\alpha, |\nabla v|^2) \\ & \quad - 8(\alpha + 2)^2N^2\tau^2(1, v^2) + 32\tau^3((y - \delta)^2 + \delta^6|x'|^2, v^2) \\ & \quad + 16\alpha\tau^3((y - \delta)^2 + \delta^6|x'|^2, v^2) - 4\alpha N\delta^2\tau^2(|t||u|^\alpha, v^2) \\ & \quad - \frac{16\alpha}{\alpha+2}(\alpha + 3)\delta^2\tau^3(|t|\varphi, |u|^\alpha v^2) + \frac{2}{\alpha+1}\delta^2\tau(|u|^{2\alpha}, v^2) \\ & \quad - \frac{8\alpha}{\alpha+1}\delta^4\tau^2(t^2|u|^{2\alpha}, v^2) - \frac{4|\gamma(\beta-\alpha)|}{\alpha-\beta+2}\delta^2\tau(|t||u|^{\alpha-\beta}, v^2). \end{aligned} \tag{3.23}$$

Here we set

$$I_6 = 4(\alpha + 2)\delta^2\tau(1, |\nabla v|^2) - \alpha(|u|^{\alpha-1}|u_t|, |\nabla v|^2) - 4\alpha\delta^2\tau(|t||u|^\alpha, |\nabla v|^2).$$

Then from (2.2) and (2.3)

$$I_6 \geq 4(\alpha + 2)\delta^2\tau(1, |\nabla v|^2) - \alpha M^\alpha(1, |\nabla v|^2) - 4\alpha\delta^2\sqrt{\frac{\delta}{\kappa}} M^\alpha\tau(1, |\nabla v|^2).$$

Hence if

$$2\sqrt{\frac{\delta}{\kappa}} M^\alpha \leq 1 \quad \text{and} \quad M^\alpha \leq \delta^2\tau, \tag{3.24}$$

then

$$I_6 \geq (\alpha + 1)\delta^2\tau(1, |\nabla v|^2). \tag{3.25}$$

Next we denote by  $I_7$  the sum of the fifth, sixth and eighth terms on the right-hand side of (3.23), whose coefficients contain  $\tau^3$ . Then from (2.2)

$$I_7 \geq 16(\alpha + 2)\left(\frac{\delta^2}{2\kappa}\right)^2\tau^3(1, v^2) - 24\alpha\delta^4\tau^3(|u|^\alpha, v^2)$$

which implies that

$$I_7 \geq 8\left(\frac{\delta^2}{\kappa}\right)^2\tau^3(1, v^2) \tag{3.26}$$

if

$$M^\alpha \leq \frac{1}{6\kappa^2}. \quad (3.27)$$

Lastly, we denote by  $I_8$  the sum of the remaining terms on the right-hand side of (3.23). Namely,

$$\begin{aligned} I_8 = & -8(\alpha + 2)^2 N^2 \tau^2(1, v^2) - 4\alpha N \delta^2 \tau^2(|t||u|^\alpha, v^2) + \frac{2}{\alpha+1} \delta^2 \tau(|u|^{2\alpha}, v^2) \\ & - \frac{8\alpha}{\alpha+1} \delta^4 \tau^2(t^2|u|^{2\alpha}, v^2) - \frac{4|\gamma(\beta-\alpha)|}{\alpha-\beta+2} \delta^2 \tau(|t||u|^{\alpha-\beta}, v^2). \end{aligned}$$

Easily,

$$I_8 \geq -\left[8(\alpha + 2)^2 N^2 \tau^2 + 4\alpha N \delta^2 \tau^2 M^\alpha + 8\delta^4 \tau^2 M^{2\alpha} + 4|\gamma(\beta - \alpha)|\delta^2 \tau M^{\alpha-\beta}\right](1, v^2).$$

We use the inequalities

$$\left. \begin{aligned} 4\alpha N \delta^2 M^\alpha &\leq 2\alpha^2 N^2 + 2\delta^4 M^{2\alpha} \\ 4|\gamma(\beta - \alpha)|\delta^2 \tau M^{\alpha-\beta} &\leq 2\delta^4 \tau^2 M^{2(\alpha-\beta)} + 2\gamma^2(\beta - \alpha)^2 \end{aligned} \right\}$$

and assume

$$|\gamma(\beta - \alpha)| \leq \tau. \quad (3.28)$$

Then

$$I_8 \geq -10\tau^2 \left[ (\alpha + 2)^2 N^2 + \delta^4 (M^{2\alpha} + M^{2(\alpha-\beta)}) \right] (1, v^2). \quad (3.29)$$

Combining (3.23), (3.25) and (3.26) with (3.29), we obtain that inequality (3.22) can be continued as

$$\begin{aligned} & C(|J_4| + (1 + \alpha)\|e^{\tau\varphi} f\|^2) \\ & \geq (\alpha + 1)\delta^2 \tau(1, |\nabla v|^2) + 8\left(\frac{\delta^2}{\kappa}\right)^2 \tau^3(1, v^2) \\ & \quad - 10\tau^2 \left[ (\alpha + 2)^2 N^2 + \delta^4 (M^{2\alpha} + M^{2(\alpha-\beta)}) \right] (1, v^2) \end{aligned} \quad (3.30)$$

under assumptions (3.24), (3.27) and (3.28). If we assume

$$5(\alpha + 2)^2 N^2 \leq \left(\frac{\delta^2}{\kappa}\right)^2 \tau \quad \text{and} \quad 5(M^{2\alpha} + M^{2(\alpha-\beta)}) \leq \frac{\tau}{\kappa^2}, \quad (3.31)$$

then

$$\frac{5}{2} \left[ (\alpha + 2)^2 N^2 + \delta^4 (M^{2\alpha} + M^{2(\alpha-\beta)}) \right] \leq \left(\frac{\delta^2}{\kappa}\right)^2 \tau.$$

Hence from (3.23) and (3.30) we conclude that

$$(\alpha + 1)\delta^2 \tau(1, |\nabla v|^2) + \left(\frac{\delta^2}{\kappa}\right)^2 \tau^3(1, v^2) \leq C(|J_4| + (1 + \alpha)\|e^{\tau\varphi} f\|^2). \quad (3.32)$$

Next we consider the case of  $\alpha = 0$ . In place of (3.20) we have

$$I_5 \geq \frac{\beta\gamma}{2-\beta} \tau(e^{\beta\tau\varphi}(\Delta\varphi + \varphi_t), |v|^{2-\beta}).$$

From (2.1) and (2.2)

$$\Delta\varphi + \varphi_t \geq 2(1 - \delta^2\kappa^2) \geq 0$$

follows. Accordingly the last term on the right-hand side of (3.22) can be dropped. Thus

$$I_8 \geq -\left[8(\alpha + 2)^2 N^2 \tau^2 + 4\alpha N \delta^2 \tau^2 M^\alpha + 8\delta^4 \tau^2 M^{2\alpha}\right](1, v^2).$$

Without assuming (3.28) we have

$$I_8 \geq -10\tau^2 [(\alpha + 2)^2 N^2 + \delta^4 M^{2\alpha}](1, v^2)$$

in place of (3.29). Hence if we assume

$$5(\alpha + 2)^2 N^2 \leq \left(\frac{\delta^2}{\kappa}\right)^2 \tau \quad \text{and} \quad 5M^{2\alpha} \leq \frac{\tau}{\kappa^2},$$

then (3.32) can be obtained.

Finally, we estimate the surface integral  $J_4$ . If we write

$$|J_4| \leq C \int_{\partial D_\eta} F d\sigma,$$

then from (3.17)

$$\begin{aligned} F &= |u|^\alpha |\nabla v| (|v_t| + |\nabla v|) + |\gamma| e^{\beta\tau\varphi} |u|^\alpha v^{2-\beta} \\ &\quad + (\alpha + 1)\tau \left( |\nabla\varphi| \cdot |\nabla v|^2 + \Delta\varphi \cdot |v| |\nabla v| + |\varphi_t| |u|^{2\alpha} v^2 + |\gamma| e^{\beta\tau\varphi} |\nabla\varphi| |v|^{2-\beta} \right) \\ &\quad + (\alpha + 1)\tau^2 \left( |\varphi_t| |\nabla\varphi| |u|^\alpha v^2 + |\nabla\varphi| \Delta\varphi \cdot v^2 + |\nabla\varphi|^2 |u|^\alpha v^2 \right) + (\alpha + 1)\tau^3 |\nabla\varphi|^3 v^2 \\ &\leq C \left[ M^\alpha (v_t^2 + |\nabla v|^2) + (\alpha + 1)\tau (v^2 + M^{2\alpha} v^2 + |\nabla v|^2 + |\gamma| (1 + M^\alpha) e^{\beta\tau\varphi} |v|^{2-\beta}) \right] \\ &\quad + C \left[ (\alpha + 1)(1 + M^\alpha) \tau^2 v^2 + (\alpha + 1)\tau^3 v^2 \right]. \end{aligned}$$

follows. Using the inequality  $2M^\alpha \leq 1 + M^{2\alpha}$ , we see that

$$F \leq C(\alpha + 1)\tau(1 + M^{2\alpha})(v_t^2 + |\nabla v|^2 + \tau^2 v^2 + |\gamma| e^{\beta\tau\varphi} |v|^{2-\beta}). \quad (3.33)$$

It is enough to complete the proof only for the case of  $\alpha \geq 1$ , from which the case of  $\alpha = 1$  will be obtained, too. We rearrange assumptions (3.24), (3.27) and (3.28), which are satisfied, if

$$M^\alpha \leq \min\left(\frac{1}{2}\sqrt{\frac{\kappa}{\delta}}, \frac{1}{6\kappa^2}\right) \quad \text{and} \quad \max\left(\frac{M^\alpha}{\delta^2}, |\gamma(\beta - \alpha)|\right) \leq \tau.$$

From (2.1),  $\min\left(\frac{1}{2}\sqrt{\frac{\kappa}{\delta}}, \frac{1}{6\kappa^2}\right) = \frac{1}{6\kappa^2}$ . Therefore combining (3.32) and (3.33) with (3.31), we complete the proof for  $u \in C^2(Q) \cap C^1(\bar{Q})$ . If  $u \in C^1(\bar{Q})$ , from our assumptions we see that  $\Delta u \in C^0(\bar{Q})$ . We denote by  $\{u^\varepsilon\}$  the regularized approximation of  $u$ . Obviously,

$$\left. \begin{aligned} \Delta u^\varepsilon &\rightarrow \Delta u \\ |u^\varepsilon|^\alpha (u^\varepsilon)_t &\rightarrow |u|^\alpha u_t \\ |u^\varepsilon|^{-\beta} u^\varepsilon &\rightarrow |u|^{-\beta} u \end{aligned} \right\} \quad (\varepsilon \rightarrow 0)$$

in  $L^2(Q)$ .

Hence the proof is reduced to the previous case that  $u \in C^2(Q) \cap C^1(\bar{Q})$ . This completes the proof ■



### 4. Proof of Theorem

In this section we prove our Theorem. From our assumptions we can apply our Proposition in the previous section. Setting  $v = e^{\tau\varphi}u$ , we have

$$\begin{aligned} & \int_{D_\eta} (\tau^2 v^2 + |\nabla_x v|^2) dxdt \\ & \leq C \frac{\kappa^2}{\delta^4} (1 + \alpha)(1 + M^{2\alpha}) \int_{\partial D_\eta} (\tau^2 v^2 + |\nabla_x v|^2 + v_t^2 + |\gamma| e^{\beta\tau\varphi} |v|^{2-\beta}) d\sigma \end{aligned}$$

where  $\tau$  is the constant defined in the assumption. Contracting the integral domain on the right-hand side, we get

$$\begin{aligned} & \int_{D_{\theta\eta}} e^{2\tau\varphi} (u^2 + |\nabla u|^2) dxdt \tag{4.1} \\ & \leq C \frac{\kappa^2}{\delta^4} \tau^2 (1 + \alpha)(1 + |\gamma|)(1 + M^{2\alpha}) \int_{\partial D_\eta} e^{2\tau\varphi} (u^2 + |\nabla u|^2 + u_t^2 + |u|^{2-\beta}) d\sigma. \end{aligned}$$

Since  $\varphi \geq (\frac{\delta}{\theta\eta})^2$  in  $D_{\theta\eta}$ ,

$$\int_{D_{\theta\eta}} e^{2\tau\varphi} (u^2 + |\nabla u|^2) dxdt \geq \exp \left[ 2\tau \left( \frac{\delta}{\theta\eta} \right)^2 \right] \int_{D_{\theta\eta}} (u^2 + |\nabla u|^2) dxdt.$$

Obviously,  $\varphi \leq \delta^2$  on  $\Gamma_\eta$  and  $\varphi = (\frac{\delta}{\eta})^2$  on  $\partial D_\eta - \Gamma$ . Hence, from our assumptions and the above we obtain

$$\begin{aligned} & \int_{D_{\theta\eta}} (u^2 + |\nabla u|^2) dxdt \\ & \leq C \frac{\kappa^2}{\delta^4} \tau^2 (1 + \alpha)(1 + |\gamma|)(1 + M^{2\alpha}) \tag{4.2} \\ & \quad \times \exp \left[ -2\tau \left( \frac{\delta}{\theta\tau} \right)^2 \right] \left( \varepsilon^2 \exp[2\tau\delta^2] + M^2 \exp \left[ 2\tau \left( \frac{\delta}{\eta} \right)^2 \right] \right). \end{aligned}$$

We use the trivial inequality

$$\left( A \left( \frac{1}{\theta^2} - \frac{1}{\theta'^2} \right) \right)^2 \leq C \exp \left[ 2A \left( \frac{1}{\theta^2} - \frac{1}{\theta'^2} \right) \right] \quad (A \geq 0).$$

Since  $\theta' - \theta = \frac{1-\theta}{2}$ , it follows that

$$A^2 \leq \frac{C}{(1-\theta)^2} \exp \left[ 2A \left( \frac{1}{\theta^2} - \frac{1}{\theta'^2} \right) \right].$$

Putting  $A = \tau \left( \frac{\delta}{\eta} \right)^2$ , we have

$$\tau^2 \delta^4 \exp \left[ -2\tau \left( \frac{\delta}{\theta\eta} \right)^2 \right] \leq \frac{C}{(1-\theta)^2} \exp \left[ -2\tau \left( \frac{\delta}{\theta'\eta} \right)^2 \right].$$

Substituting this into (4.2), that inequality can be continued as

$$\begin{aligned} & \int_{D_{\theta\eta}} (u^2 + |\nabla u|^2) dxdt \\ & \leq C\kappa^2\delta^{-8}(1+\alpha)(1+|\gamma|)(1+M^{2\alpha})\frac{1}{(1-\theta)^2} \\ & \quad \times \exp\left[-2\tau\left(\frac{\delta}{\theta'\eta}\right)^2\right] \left(\varepsilon^2 \exp[2\tau\delta^2] + M^2 \exp\left[2\tau\left(\frac{\delta}{\eta}\right)^2\right]\right). \end{aligned}$$

Obviously,

$$\begin{aligned} \varepsilon^2 \exp\left[2\tau\delta^2\left(1 - \left(\frac{1}{\theta'\eta}\right)^2\right)\right] &= M^2 \exp\left[2\tau\left(\frac{\delta}{\eta}\right)^2\left(1 - \frac{1}{\theta'^2}\right)\right] \\ &= \exp\left[\frac{2(1-\theta'^2)}{\theta'^2(\eta^2-1)} \log \varepsilon\right] \cdot \exp\left[\frac{2(\theta'^2\eta^2-1)}{\theta'^2(\eta^2-1)} \log M\right]. \end{aligned}$$

Therefore we complete the proof of our Theorem.

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Received 25.05.1999