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ON B-WEAKLY DEMICOMPACT OPERATORS ON BANACH LATTICES

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Abstract. Aqzzouz and Elbour proved that an operator T on a Banach lattice E is b-weakly compact if and only if $\|Tx_n\| \to 0$ as $n \to \infty$ for each b-order bounded weakly null sequence $\{x_n\}$ in E_+. In this present paper, we introduce and study new concept of operators that we call b-weakly demicompact, use it to generalize known classes of operators which defined by b-weakly compact operators. An operator T on a Banach lattice E is said to be b-weakly demicompact if for every b-order bounded sequence $\{x_n\}$ in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. As consequence, we obtain a characterization of KB-spaces in terms of b-weakly demicompact operators. After that, we investigate the relationships between b-weakly demicompact operators and some other classes of operators on Banach lattices espaciallly their relationships with demi Dunford–Pettis operators and order weakly demicompact operators.

Keywords: Banach lattice, KB-space, b-weakly demicompact operator, order weakly demicompact operator, demi Dunford–Pettis operator.

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1. Introduction and Preliminaries

Throughout this paper X, Y will denote Banach spaces and E, F will denote Banach lattices. The set of all bounded linear operators on X is denoted by $\mathscr{L}(X)$. The positive cone of E will be denoted by $E_+ = \{x \in E;\ 0 \leqslant x\}$.

Recall from [1] that a subset A of E is called b-order bounded if it is order bounded in the topological bidual E''. Note that every order bounded subset of E is b-order bounded, however, the converse is not true in general. But a Banach lattice E is said to have property (b) if each subset A of E is order bounded whenever it is b-order bounded. An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it carries each b-order bounded subset of E into a relatively weakly compact subset of X. The class of b-weakly compact operators was introduced by Alpay, Altin and Tonyali in [1] on vector lattices. After that, a series of papers, which gave different characterizations of this class of operators, were published (see [2, 3]). In [4], the authors proved that an operator $T : E \to E$ is b-weakly compact if and only if $\|Tx_n\| \to 0$ for every b-order bounded sequence $\{x_n\} \subset E_+$ satisfying $x_n \to 0$ for the topology $\sigma(E, E')$ (see [4, Theorem 2.2]). This characterization is proved by B. Altin in [3] for the first time with Banach lattices with order continuous norms.

Recall from [5] that an operator $T : \mathscr{D}(T) \subseteq X \to X$, where $\mathscr{D}(T)$ is a subspace of X, is said to be demicompact if, for every bounded sequence (x_n) in the domain $\mathscr{D}(T)$ such that $(x_n - Tx_n)$ converges to $x \in X$, there is a convergent subsequence of (x_n). Note that each compact operator is demicompact, but the opposite is not always true. In fact, let $Id_X : X \to X$ be the identity operator of a Banach space X of infinite dimension. It is clear that $-Id_X$ is demicompact but it is not compact. Jeribi [6] used the class of demicompact operators to obtain some results on Fredholm and spectral theories.

Let us recall from [7] that an operator $T : \mathscr{D}(T) \subseteq X \to X$ is said to be weakly demicompact if, every bounded sequence (x_n) in $\mathscr{D}(T)$ such that $(x_n - Tx_n)$ weakly converges in X, has a weakly convergent subsequence. This class includes both the weakly compact and demicompact operators.

Recall from [8] that an operator $T : X \to X$ is said to be demi Dunford–Pettis (DDP in short), if for every sequence $\{x_n\}$ in X such that $x_n \to 0$ in $\sigma(X, X')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. In [8, Theorem 2.4], the authors showed that an operator T from X into X is demi Dunford–Pettis if and only if X has the Schur property.

In [9], Benkhaled et al provided a systematic approach to the demicompactness criteria by using the framework of the theory of Banach lattices. More precisely, they introduced the class of order weakly demicompact operators on Banach lattices. An operator T from a Banach lattice E into itself is said to be order weakly demicompact if for every order bounded sequence $\{x_n\}$ in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$.

The purpose of this work is to pursue the analysis started in [9] in order to define a new class of operators on Banach lattices that we call b-weakly demicompact operators and study some of their properties.

This paper is organized in the following way. In Section 2, the notion of b-weakly demicompact operators is introduced (see Definition 2.1). Note that the class of b-weakly demicompact operators contains that of b-weakly compact operators (see Proposition 2.1). After that, a new characterization of KB-space Banach lattices in terms of b-weak demicompactness of its identity (see Theorem 2.2) is established. In Section 3, our aim is to study the connections between b-weakly demicompact operators and some other operators on Banach lattices spacial their relationships with order weakly demicompact (see Proposition 3.1 and Corollary 3.1) and demi Dunford–Pettis operators (see Propositions 3.2 and 3.3).

To state our results, we need to fix some notations and recall some definitions that will be used in this paper. Let E be a vector lattice, for each $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E : x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence $\{x_\alpha\}$ such that $x_\alpha \downarrow 0$ in E, the sequence $\{x_\alpha\}$ converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence $\{x_\alpha\}$ is decreasing, its infimum exists and $\inf\{x_\alpha\} = 0$. A Banach lattice E is said to be a KB-space whenever each increasing norm bounded sequence of E_+ is norm convergent. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf\{x, y\} = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. A Banach lattice E is said to have weakly sequentially continuous lattice operations whenever $x_n \to 0$ in $\sigma(E, E')$ implies $|x_n| \to 0$ in $\sigma(E, E')$. In an AM-space the lattice operations are weakly sequentially continuous.

We will use the term operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \in E_+$. We write $S \leqslant T$ if $(T - S)x \geqslant 0$ for every $x \in E_+$. We say that S is dominated by T.

For the theory of Banach lattices theory and operators, we refer the reader to the monographs [10, 11].

2. Definition and Some Properties

We commence by the following definition.

Definition 2.1. Let E be a Banach lattice. An operator $T : E \to E$ is said to be b-weakly demicompact if for every b-order bounded sequence $\{x_n\}$ in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. The collection of b-weakly demicompact operators will be denoted by $\mathscr{W}\mathscr{D}_b(E)$.

Example 2.1. For every scalar $\alpha \neq 1$, $\alpha Id_E \in \mathscr{W}\mathscr{D}_b(E)$.

Our following result proves that the class of b-weakly demicompact operators includes that of b-weakly compact operators.

Proposition 2.1. *Let E be a Banach lattice. If $T : E \to E$ is an b-weakly compact operator, then T is b-weakly demicompact.*

$\lhd$ Let $\{x_n\}$ be a b-order bounded sequence of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We have to show that $\|x_n\| \to 0$ as $n \to \infty$. By reason of T is b-weakly compact and in view of [4, Theorem 2.2], we obtain that $\|Tx_n\| \to 0$ as $n \to \infty$. Since we can write

$$\|x_n\| \leqslant \|x_n - Tx_n\| + \|Tx_n\|$$

for each n, then $\|x_n\| \to 0$ as $n \to \infty$. $\rhd$

Remark 2.1. The converse of Proposition 2.1 is false in general. Let $Id_{L^\infty[0,1]}$ be the identity operator from $L^\infty[0, 1]$ into itself. Clearly that $-Id_{L^\infty[0,1]}$ is b-weakly demicompact (see Example 2.1). On the other side, $L^\infty[0, 1]$ is not KB-space. Hence, $Id_{L^\infty[0,1]}$ is not b-weakly compact (see [1, Proposition 2.10]).

Note that the class of b-weakly demicompact operators lacks the vector space structure. To illustrate this, we give the following example.

Example 2.2. Let $E = c_0$ and $Id_E : E \to E$ be the identity operator. Clearly, $T = 2Id_E$ and $S = -Id_E$ are b-weakly demicompact operators (see Example 2.1). But the operators $T + S = Id_E$ and $-S = Id_E$ are not. In fact, put $x_n = e_n$ for every n, where e_n is the sequence with the nth entry equals to 1 and others are zero. So that $\{x_n\}$ is an b-order bounded and weakly nul sequence of E_+. Moreover, $\|x_n - (T + S)x_n\|_E = \|x_n - (-S)x_n\|_E = 0$. Note that $\|x_n\|_E = \|e_n\|_\infty = 1$. This proves that $\|x_n\|_E$ does not converges to zero.

Remark 2.2. Note that the set of all b-weakly demicompact is not norm closed in $\mathscr{L}(E)$. In fact, consider $T_n = \frac{n+2}{n} Id_{c_0}$ for each $n \in \mathbb{N}^*$ where Id_{c_0} be the identity operator on c_0. Clearly that $\{T_n\} \subseteq \mathscr{W}\mathscr{D}_b(E)$ (see Example 2.1). On the other hand, since we can write

$$\left\|T_n - Id_{c_0}\right\| = \left\|\frac{n+2}{n} Id_{c_0} - Id_{c_0}\right\| = \left\|\frac{2}{n} Id_{c_0}\right\| = \frac{2}{n},$$

it follows that $\lim_{n\to\infty} \|T_n - Id_{c_0}\| = 0$. Therefore, $\{T_n\}$ is norm convergent to Id_{c_0}. But, in view of Example 2.2, Id_{c_0} is not b-weakly demicompact.

The following result asserts that an b-weakly compact perturbation of an b-weakly demicompact operator is b-weakly demicompact.

Theorem 2.1. *Let T and S operators on a Banach lattices E. If T is b-weakly demicompact and S is b-weakly compact, then $T+S$ is b-weakly demicompact operator.*

$\lhd$ Let $\{x_n\}$ be a b-order bounded sequence of E_+ such that $x_n \to 0$ in $\sigma(E,E')$ and $\|x_n-(T+S)x_n\| \to 0$ as $n\to\infty$. We have to show that $\|x_n\|\to 0$ as $n\to\infty$. Using the b-weak compactness of S and from [4, Theorem 2.2], we infer that $\|Sx_n\|\to 0$ as $n\to\infty$. From the following inequality

$$\|x_n - Tx_n\| = \|x_n - Tx_n - Sx_n + Sx_n\| \leqslant \|x_n-(S+T)x_n\| + \|Sx_n\|,$$

then $\|x_n - Tx_n\| \to 0$ as $n\to\infty$. Thus, the b-weak demicompactness of T implies that $\|x_n\|\to 0$ as $n\to\infty$. Therefore, $T+S$ is b-weakly demicompact. $\rhd$

The following result discus the b-weak demicompactnees of 2×2 matrix operators.

Proposition 2.2. *Let E, F be two Banach lattices and $T_1 : E \to F$, $T_2 : F\to F$ be two operators. If T_2 is b-weakly demicompact, then the matrix operator $\widetilde{T} : \widetilde{E}\to\widetilde{E}$ defined by*

$$\widetilde{T} = \begin{pmatrix} 0 & 0 \\ T_1 & T_2 \end{pmatrix}$$

is b-weakly demicompact with $\widetilde{E} = E\oplus F$.

$\lhd$ Let $\{\tilde{x}_n = (x_n, y_n)\}$, $x_n\in E$ and $y_n \in F$, be a b-order bounded sequence of $\big(\widetilde{E}\big)_+$ such that $\tilde{x}_n \to 0$ in $\sigma(\widetilde{E},\widetilde{E}')$ and $\|\tilde{x}_n - \widetilde{T}\tilde{x}_n\|_{\widetilde{E}} \to 0$ as $n\to\infty$. Our aim is to prove that $\|\tilde{x}_n\|_{\widetilde{E}}\to 0$ as $n\to\infty$. Since we can write

$$\begin{aligned}\big\|\tilde{x}_n - \widetilde{T}\tilde{x}_n\big\|_{\widetilde{E}} &= \big\|(x_n,y_n) - \widetilde{T}(x_n,y_n)\big\|_{\widetilde{E}} = \big\|(x_n,y_n) - (0, T_1x_n + T_2y_n)\big\|_{\widetilde{E}} \\ &= \big\|(x_n, y_n - T_1x_n - T_2y_n)\big\|_{\widetilde{E}} = \big\|x_n\big\|_E + \big\|y_n - T_1x_n - T_2y_n\big\|_F,\end{aligned}$$

for each $n\in\mathbb{N}$. Then

$$\big\|x_n\big\|_E \to 0 \tag{1}$$

and

$$\big\|y_n - T_1x_n - T_2y_n\big\|_F \to 0 \tag{2}$$

as $n\to\infty$. It is clear from (1) that $\|T_1x_n\|_F\to 0$ as $n\to\infty$. Furthermore, in view of (2) and from the following inequalities:

$$\big\|y_n - T_2y_n\big\|_F \leqslant \big\|y_n - T_1x_n - T_2y_n\big\|_F + \big\|T_1x_n\big\|_F,$$

we infer that $\|y_n - T_2y_n\|_F \to 0$ as $n\to\infty$. The b-weak demicompactness of T_2 asserts that $\|y_n\|_F\to 0$ as $n\to\infty$. Therefore, we conclude that

$$\big\|\tilde{x}_n\big\|_{\widetilde{E}} = \big\|x_n\big\|_E + \big\|y_n\big\|_F \to 0,$$

as $n\to\infty$. This completes the proof. $\rhd$

The next result gives a characterization of KB-spaces in terms of b-weak demicompactness of its identity operator. Also, it is a generalization of Proposition 2.10 of Alpay–Altin–Tonyali [1].

Theorem 2.2. *Let E be a Banach lattice, then the following assertions are equivalent:*
(i) E is a KB-space.
(ii) Every operator $T : E\to E$ is b-weakly compact.
(iii) Every operator $T : E \to E$ is b-weakly demicompact.

(iv) The identity operator of E is b-weakly demicompact.

(v) $\|x_n\| \to 0$ as $n \to \infty$ for every b-order bounded sequence $\{x_n\}$ of E_+ satisfying $x_n \to 0$ for the topology $\sigma(E, E')$.

$\lhd$ $(i) \Longrightarrow (ii)$. The proof follows from [12, Proposition 2.1].

$(ii) \Longrightarrow (iii)$. See Proposition 2.1.

$(iii) \Longrightarrow (iv)$ is obvious.

$(iv) \Longrightarrow (v)$. Let $Id_E : E \to E$ be a b-weakly demicompact and we have to show that for every b-order bounded weakly null sequence $\{x_n\}$ of E_+ we have $\|x_n\| \to 0$ as $n \to \infty$. Let $\{x_n\}$ be such a sequence. It is clear that $\lim_n \|x_n - Id_E x_n\| = 0$. Since Id_E is b-weakly demicompact, we obtain $\|x_n\| \to 0$ as $n \to \infty$.

$(v) \Longrightarrow (i)$. It follows from [4, Corollary 2.3]. $\rhd$

Recall from that the class of b-weakly compact operators satisfies the domination problem, i. e., if E and F are two Banach lattices, and S and T are two operators from E into F such that $0 \leqslant S \leqslant T$ and T is b-weakly compact, then S is b-weakly compact (see [1, Corollary 2.9]). One of the short comings of the class of b-weakly demicompact operators is that it does not satisfy the domination problem. To illustrate this, we give the following example.

Example 2.3. Consider the positive operators $S, T : C^1([0,1]) \to c_0$ defined by

$$S(f) = \left(\int_0^1 r_1^+ f(x)\,dx,\ \int_0^1 r_2^+ f(x)\,dx,\ \ldots \right)$$

and

$$T(f) = \left(\int_0^1 f(x)\,dx,\ \int_0^1 f(x)\,dx,\ \ldots \right),$$

where r_n denotes the sequence of the Rademacher functions on $[0,1]$. Clearly, $0 \leqslant S \leqslant T$ holds. Put $\widetilde{E} = C^1([0,1]) \oplus c_0$. Consider now the matrix operators $0 \leqslant \widetilde{S} \leqslant \widetilde{T} : \widetilde{E} \to \widetilde{E}$ defined by

$$\widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 2Id_{c_0} \end{pmatrix} \quad \text{and} \quad \widetilde{S} = \begin{pmatrix} 0 & 0 \\ S & Id_{c_0} \end{pmatrix}.$$

Since $2Id_{c_0}$ is b-weakly demicompact (see Example 2.1), then $\widetilde{T}$ is b-weakly demicompct (see Proposition 2.2). On the other side, $\widetilde{S}$ is not b-weakly demicompact. In fact, consider the sequence $\widetilde{x}_n = (0, e_n)$ for every n, where e_n is the sequence with the nth entry equals to 1 and others are zero. This sequence is b-order bounded and $\widetilde{x}_n \to 0$ in $\sigma(\widetilde{E}, (\widetilde{E})')$. Moreover, $\|\widetilde{x}_n - \widetilde{S}\widetilde{x}_n\|_{\widetilde{E}} = 0$. If $\widetilde{S}$ is b-weakly demicompact then $\|\widetilde{x}_n\|_{\widetilde{X}} = \|e_n\|_\infty = 1 \to 0$ which is impossible.

Let us recall that an operator $T : E \to E$ is called central if it is dominated by a multiple of the identity operator that is T is a central operator if and only if there exists some scalar $\lambda > 0$ such that $|Tx| \leqslant \lambda |x|$ holds for all $x \in E$ (see [13]). In the following result, we prove that the dominance problem for central b-weakly demicompact operators is valid.

Theorem 2.3. *Let $S, T : E \to E$ be two positive operators on a Banach lattice such that $0 \leqslant S \leqslant T \leqslant I$ holds. If T is a b-weakly demicompact operator, then S is b-weakly demicompact.*

$\lhd$ Let $\{x_n\}$ be a b-order bounded sequence of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. From [13, Theorem 3.30], we have

$$|x_n - Sx_n| = |(I - S)(|x_n|)| = |I - S|\,|(|x_n|)| = (I - S)(x_n),$$

for all n. From the following inequalities

$$|(I-T)(x_n)| = (I-T)(x_n) \leqslant (I-S)(x_n),$$

we get $\|(I-T)(x_n)\| \leqslant \|(I-S)(x_n)\|$ for all n. This implies that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The b-weak demicompactness of T implies that $\|x_n\| \to 0$ as $n \to \infty$ and so S is b-weakly demicompact. $\rhd$

We close this section by the following remark.

Remark 2.1. As the class of b-weakly compact operators [14], the set of b-weakly demicompact operators does not satisfy the duality property.

More precisely, there is a b-weakly demicompact operator T from E into E whose dual T' from E' into E' is not b-weakly demicompact operator and conversely, there is an operator T from E into E which is not b-weakly demicompact operator while its dual T' from E' into E' is one. In fact, the identity operator Id_{l^1} is b-weakly demicompact but its dual which is the identity operator Id_{l^∞} is not b-weakly demicompact.

Conversely, the identity operator Id_{c_0} is not b-weakly demicompact, but its dual which is the identity operator Id_{l^1} is b-weakly demicompact.

3. Relationships with Order Weakly Demicompact (Respective Demi Dunford–Pettis) Operators

As mentioned in the introduction, this section is dedicated to investigate the relationships between the class of b-weakly demicompact and some other operators on Banach lattices.

The following result proves that the class of order weakly demicompact operators contains the class of b-weakly demicompact operators.

Proposition 3.1. *Let E be a Banach lattice. Every b-weakly demicompact operator $T : E \to E$ is order weakly demicompact.*

$\lhd$ Let $T : E \to E$ be a b-weakly demicompact operator. We have to show that $\|x_n\| \to 0$ as $n \to \infty$ for every order bounded sequence $\{x_n\}$ of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Let $\{x_n\}$ be such a sequence, then $\{x_n\}$ is a b-order bounded sequence of E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Thus, the b-weak demicompactness of T implies that $\|x_n\| \to 0$ as $n \to \infty$. $\rhd$

Remark 3.1. Note that an order weakly compact operator need not be b-weakly demicompact operator. In fact, the identity operator of the Banach lattice c_0 is order weakly compact (because the norm of c_0 is order continuous norm, see [9, Theorem 2.1]) but it is not b-weakly demicompact (because c_0 is not KB-space, see Theorem 2.2).

If Banach lattice E has the property (b), then the class of b-weakly demicompact operators on E coincides with that of order weakly demicompact operators on E.

Corollary 3.1. *Let E be a Banach lattice has property (b), then each order weakly demicompact operator from E into E is b-weakly demicompact.*

The space of b-weakly demicompact operators is bigger than the space of demi Dunford–Pettis operators.

Proposition 3.2. *Every demi Dunford–Pettis operator from a Banach lattice E into itself is b-weakly demicompact.*

$\lhd$ Consider $T : E \to E$ be a demi Dunford–Pettis operator and we have to show that it is b-weakly demicompact. Let $\{x_n\}$ be a b-order bounded of E_+ such that $x_n \to 0$ in $\sigma(E, E')$

and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The fact that T is demi Dunford–Pettis, we obtain $\|x_n\| \to 0$ as $n \to \infty$ and the proof is finished. $\rhd$

However, as the next example shows, the converse of Proposition 3.2 is not true in general.

EXAMPLE 3.1. The identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is b-weakly demicompact. On the other hand, $L^1[0,1]$ does not have the Schur property, it follows from [8, Theorem 2.4] that $Id_{L^1[0,1]}$ is not demi Dunford–Pettis.

To give a sufficient condition under which the classes of demi Dunford–Pettis operators coincide with the class of b-weakly demicompact operators, we need to present the following lemma.

Lemma 3.1 [15, Lemma 2.8]. *Let E be a Banach lattice. Then every positive norm bounded increasing net $\{x_\alpha\}$ of E is b-order bounded, i. e., $\{x_\alpha\}$ is order bounded in the topological bidual E''.*

Proposition 3.3. *Let E be an AM-space. Let $T : E \to E$ be an operator such that $Id_E - T \geqslant 0$. Then the following assertions are equivalent:*

(1) *T is demi Dunford–Pettis.*

(2) *T is b-weakly demicompact.*

$\lhd$ (1) $\Longrightarrow$ (2). It follows from Proposition 3.2.

(2) $\Longrightarrow$ (1). Let $T : E \to E$ be a b-weakly demicompact operator and we have to show that T is demi Dunford–Pettis. Let $\rho(x) = \|x\|$ for every $x \in E$, then ρ is a continuous lattice seminorm on E. Suppose that T is not demi Dunford–Pettis. Then, there exists a sequence $\{x_n\}$ of E such that $x_n \to 0$ in $\sigma(E, E')$, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ and $\|x_n\| \geqslant 1$. By [10, Theorem 4.31], E has weakly sequentially continuous lattice operations. So, we may assume that $\{x_n\} \subseteq E_+$. Corollary 2.3.5 of [11] shows that for every $0 < c < 1$, there exists a subsequence $\{k_n\} \subseteq \mathbb{N}$ and a disjoint sequence $\{y_n\} \subseteq E_+$ such that $y_n \leqslant x_{k_n}$, $\|y_n\| \geqslant c$ for all $n \in \mathbb{N}$. Since $y_n \leqslant x_{k_n}$ and $x_n \to 0$ in $\sigma(E, E')$ as $n \to \infty$, the sequence $\{y_n\}$ is norm bounded. So, the sequence $u_n = \sum_{i=1}^{n} y_i$ is an increasing norm bounded sequence. Hence, from Lemma 3.1, there exists $x'' \in E''_+$ such that $0 \leqslant u_n \leqslant x''$. So, $\{u_n\}$ is a b-order bounded sequence in E_+ such that $u_n \to 0$ in $\sigma(E, E')$. Also, the fact that $Id_E - T \geqslant 0$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then $\|u_n - Tu_n\| \to 0$ as $n \to \infty$. Thus, the b-weak demicompactness of T implies that $\|u_n\| \to 0$ as $n \to \infty$. Since we can write $y_n = u_n - u_{n-1}$, then $\|y_n\| \to 0$ as $n \to \infty$. This gives a contradiction. Therefore T is demi Dunford–Pettis. $\rhd$

There exists a b-weakly demicompact operator from a Banach lattice E into E that is not weakly demicompact. In fact, consider the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$. As the Banach lattice $L^1[0,1]$ is KB-space, it follows from Theorem 2.2 that $Id_{L^1[0,1]}$ is b-weakly demicompact but not weakly demicompact. The next proposition characterize Banach lattices E for which each b-weakly demicompact operator $T : E \to E$ is weakly demicompact. First, we need to recall the following result:

Corollary 3.2 [8, Corollary 3.5]. *Let E be a Banach lattices. Then the following assertions are equivalent:*

(1) *Every demi Dunford–Pettis operator $T : E \to E$ is weakly demicompact.*

(2) *The norm of E' is order continuous.*

Proposition 3.4. *Let E be an AM-space. Every b-weakly demicompact operator $T : E \to E$ such that $Id_E - T \geqslant 0$ is weakly demicompact.*

$\lhd$ Assume that E is an AM-space and let $T : E \to E$ be a b-weakly demicompact operator such that $Id_E - T \geqslant 0$. We have to show that T is weakly demicompact. In view of Proposition 3.3, we obtain that T is demi Dunford–Pettis. Note that if E is an AM-space, the norm of E' is order continuous. Hence, the result follows from Corollary 3.2. $\rhd$

Question 3.1. *Give an operator T from a Banach lattice E into E which is a weakly demicompact; but is not b-weakly demicompact.*

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О b-СЛАБО ДЕМИКОМПАКТНЫХ ОПЕРАТОРАХ НА БАНАХОВЫХ РЕШЕТКАХ

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Аннотация. Акзуз и Эльбур доказали, что оператор T на банаховой решетке E b-слабо компактен тогда и только тогда, когда $\|Tx_n\| \to 0$ при $n \to \infty$ для каждой b-порядково ограниченной последовательность $\{x_n\}$ в E_+, слабо сходящейся к нулю. В настоящей статье мы вводится и изучается новое понятие b-слабо демикомпактного оператора, которое используется для обобщения известных классов операторов, определяемых b-слабо компактными операторами. Оператор T на банаховой решетке E называется b-слабо демикомпактным, если для любой ограниченной последовательности $\{x_n\}$ b-порядка в E_+ такой, что $x_n \to 0$ в $\sigma(E, E')$ и $\|x_n - Tx_n\| \to 0$ при $n \to \infty$, имеем $\|x_n\| \to 0$ при $n \to \infty$. Как следствие, мы получаем характеризацию KB-пространств в терминах b-слабо демикомпактных операторов. Далее, исследованы взаимосвязи между b-слабо демикомпактными операторами и некоторыми другими классами операторов на банаховых решетках, особенно, связи с операторами деми-Данфорда-Петтиса и порядковыми слабо демикомпактными операторами.

Ключевые слова: Банахова решетка, KB-пространство, b-слабо демикомпактный оператор, порядковый слабо демикомпактный оператор, оператор деми Данфорда — Петтиса.

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