REVERSIBLE AJW-ALGEBRAS

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The main result states that every special AJW-algebra can be decomposed into the direct sum of totally irreversible and reversible subalgebras. In turn, every reversible special AJW-algebra decomposes into a direct sum of two subalgebras, one of which has purely real enveloping real von Neumann algebra, and the second one contains an ideal, whose complexification is a C*-algebra and the annihilator of this complexification in the enveloping C*-algebra of this subalgebra is equal to zero.

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AJW-algebra, reversible AJW-algebra, AW*-algebra, Enveloping C*-algebra.

1. Introduction

This article is devoted to abstract Jordan operator algebras, which are analogues of abstract W*-algebras (AW*-algebras) of Kaplansky. These Jordan operator algebras can be characterized as a JB-algebra satisfying the following conditions

(1) in the partially ordered set of all projections any subset of pairwise orthogonal projections has the least upper bound in this JB-algebra;

(2) every maximal associative subalgebra of this JB-algebra is generated by it's projections (i.e. coincides with the least closed subalgebra containing all projections of the given subalgebra).

In the articles [3, 4] the second author introduced analogues of annihilators for Jordan algebras and gave algebraic conditions equivalent to (1) and (2). Currently, these JB-algebras are called AJW-algebras or Baer JB-algebras in the literature. Further, in [5] a classification of these algebras has been obtained. It should be noted that many of facts of the theory of JBW-algebras and their proofs hold for AJW-algebras. For example, similar to a JBW-algebra an AJW-algebra is the direct sum of special and purely exceptional Jordan algebras [5].

It is known from the theory of JBW-algebras that every special JBW-algebra can be decomposed into the direct sum of totally irreversible and reversible subalgebras. In turn, every reversible special JBW-algebra decomposes into a direct sum of a subalgebra, which is the hermitian part of a von Neumann algebra and a subalgebra, enveloping real von Neumann algebra of which is purely real [6, 7]. In this paper we prove a similar result for AJW-algebras, the proof of which requires a different approach. Namely, we prove that for every special AJW-algebra A there exist central projections e, f, g ∈ A, e + f + g = 1 such that (1) eA is reversible and there exists a norm-closed two sided ideal I of C*(eA) such that eA = 1+(I_{sa})_+; (2) fA is reversible and R^*(fA) ∩ iR^*(fA) = {0}; (3) gA is a totally nonreversible AJW-algebra.

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2. Preliminary Notes

We fix the following terminology and notations.

Let $\mathscr{A}$ be a real Banach $*$-algebra. $\mathscr{A}$ is called a real C*-algebra, if $\mathscr{A}_c = \mathscr{A} + i\mathscr{A} = \{a + ib : a, b \in \mathscr{A}\}$, can be normed to become a (complex) C*-algebra, and keeps the original norm on $\mathscr{A}$ [11].

Let A be a JB-algebra, $P(A)$ be a set of all projections of $A$. Further we will use the following standard notations: 

\begin{align*}
\{aba\} = U_{ab} := 2a(ab) - a^2b, \quad \{abc\} = a(bc) + (ac)b - (ab)c
\end{align*}

and $\{aAb\} = \{acb : c \in A\}$, where $a, b, c \in A$. A JB-algebra $A$ is called an AJW-algebra, if the following conditions hold:

1. In the partially ordered set $P(A)$ of projections any subset of pairwise orthogonal projections has the least upper bound in $A$;

2. every maximal associative subalgebra $A_0$ of the algebra $A$ is generated by its projections (i.e. coincides with the least closed subalgebra containing $A_0 \cap P(A)$).

Let 

\begin{align*}
(S)^\perp &= \{a \in A : (\forall x \in S) U_a x = 0\}, \\
\perp(S) &= \{x \in A : (\forall a \in S) U_a x = 0\}, \\
\perp(S^\perp) &= \perp(S) \cap A^+_0.
\end{align*}

Then for a JB-algebra $A$ the following conditions are equivalent:

1. $A$ is an AJW-algebra;

2. for every subset $S \subseteq A^+_0$ there exists a projection $e \in A$ such that $(S)^\perp = U_e(A)$;

3. for every subset $S \subseteq A$ there exists a projection $e \in A$ such that $\perp(S^\perp) = U_e(A^+_0)$ [3].

Let $A$ be a real or complex $*$-algebra, and let $S$ be a nonempty subset of $A$. Then the set $R(S) = \{x \in A : sx = 0 \text{ for all } s \in S\}$ is called the right annihilator of $S$ and the set $L(S) = \{x \in A : xs = 0 \text{ for all } s \in S\}$ is called the left annihilator of $S$. A $*$-algebra $A$ is called a Baer $*$-algebra, if the right annihilator of any nonempty set $S \subseteq A$ is generated by a projection, i.e. $R(S) = gA$ for some projection $g \in A$ ($g^2 = g = g^*$). If $S = \{a\}$ then the projection $1 - g$ such that $R(S) = gA$ is called the right projection and denoted by $r(a)$. Similarly one can define the left projection $l(a)$. A (real) C*-algebra $A$, which is a Baer (real) $*$-algebra, is called an (real) AW*-algebra [1, 8]. Real AW*-algebras were introduced and investigated in [1, 2]. In these papers it was shown that for a real AW*-algebra $\mathscr{A}$ the C*-algebra $M = \mathscr{A} + i\mathscr{A}$ is not necessarily a complex AW*-algebra.

Let $A$ be an AJW-algebra. By [5, Theorem 2.3] we have the equality $A = A_I \oplus A_{II} \oplus A_{III}$, where $A_I$ is an AJW-algebra of type I, $A_{II}$ is an AJW-algebra of type II and $A_{III}$ is an AJW-algebra of type III [5]. By [5, Theorem 3.7] $A_I$, in its turn, is a direct sum of the following form

\begin{align*}
A_I = A_\infty \oplus A_1 \oplus A_2 \oplus \ldots,
\end{align*}

where $A_n$ for every $n$ either is $\{0\}$ or an AJW-algebra of type $I_n$, $A_\infty$ is a direct sum of AJW-algebras of type $I_\infty$, with $a$ infinite. If $A = A_1 \oplus A_2 \oplus \ldots$ then $A$ is called an AJW-algebra of type I$_{\infty}$ and denoted by $A_{I_{\infty}}$ and if $A = A_{\infty}$ then $A$ is called an AJW-algebra of type $I_{\infty}$ and denoted by $A_{I_{\infty}}$. We say that $A$ is properly infinite if $A$ has no nonzero central modular projection. The fact that an AJW-algebra $A_{II}$ of type II is a JC-algebra can be proved similar to JJB-algebras [9]. Therefore, it is isomorphic to some AJW-algebra defined in [14] (i.e. to some AJW-algebra of self-adjoint operators), and by virtue of [14] $A_{II} = A_{II} \oplus A_{II_{\infty}}$, where $A_{II}$ is a modular AJW-algebra of type II and $A_{II_{\infty}}$ is an AJW-algebra of type II, which is properly infinite. So, we have the decomposition

\begin{align*}
A = A_{I_{\infty}} \oplus A_{I_{\infty}} \oplus A_{II_{\infty}} \oplus A_{II_{\infty}} \oplus A_{II_{\infty}}.
\end{align*}
Reversible AJW-algebras

It is easy to verify that the part \( A_{f_{fin}} \oplus A_{II_1} \) is modular, and \( A_{f_{\infty}} \oplus A_{II_{\infty}} \oplus A_{III} \) is properly infinite (i.e. properly nonmodular).

3. Reversibility of AJW-algebras

Let \( A \) be a special AJW-algebra on a complex Hilbert space \( H \). By \( R^*(A) \) we denote the uniformly closed real \( * \)-algebra in \( B(H) \), generated by \( A \), and by \( C^*(A) \) the \( C^* \)-algebra, generated by \( A \). Thus the set of elements of kind

\[
\sum_{i=1}^{n} \prod_{j=1}^{m_i} a_{ij} \quad (a_{ij} \in A)
\]

is uniformly dense in \( R^*(A) \). Let \( iR^*(A) \) be the set of elements of kind \( ia, a \in R^*(A) \). Then

\[
C^*(A) = R^*(A) + iR^*(A) \quad [7, 13].
\]

**Lemma 3.1.** The set \( R^*(A) \cap iR^*(A) \) is a uniformly closed two sided ideal in \( C^*(A) \).

\(< \) If \( a, b \in R^*(A) \) and \( c = id \in R^*(A) \cap iR^*(A) \), then \( (a+ib)c = ac + ibid = ac - bd \in R^*(A) \). Similarly \( (a+ib)c \in iR^*(A) \), i.e. \( (a+ib)c \in R^*(A) \cap iR^*(A) \). Since \( R^*(A) \cap iR^*(A) \) is uniformly closed and the set of elements of kind \( a \in R^*(A) \), \( b \in A \) is uniformly dense in \( C^*(A) \), we have \( R^*(A) \cap iR^*(A) \) is a left ideal in \( C^*(A) \). By the symmetry \( R^*(A) \cap iR^*(A) \) is a right ideal. \( \triangleright \)

Let \( R \) be a \( * \)-algebra, \( R_{sa} \) be the set of all self-adjoint elements of \( R \), i.e. \( R_{sa} = \{ a \in R : a^* = a \} \).

**Definition 3.2.** A JC-algebra \( A \) is said to be reversible if \( a_1a_2 \ldots a_n + a_na_{n-1} \ldots a_1 \in A \) for all \( a_1, a_2, \ldots, a_n \in A \).

Similar to JW-algebras we have the following criterion.

**Lemma 3.3.** An AJW-algebra \( A \) is reversible if and only if \( A = R^*(A)_{sa} \).

\(< \) It is clear that, if \( A = R^*(A)_{sa} \), then \( A \) is reversible since

\[
\left( \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} a_i \right)^* = \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} a_i \in R^*(A)_{sa} = A,
\]

for all \( a_1, a_2, \ldots, a_n \in A \). Conversely, let \( A \) be a reversible AJW-algebra. The inclusion \( A \subset R^*(A)_{sa} \) is evident. If \( a = \sum_{i=1}^{n} \prod_{j=1}^{m_i} a_{ij} \in R^*(A)_{sa} \), then

\[
a = \frac{1}{2}(a + a^*) = \frac{1}{2} \sum_{i=1}^{n} \left( \prod_{j=1}^{m_i} a_{ij} + \prod_{j=1}^{m_i} a_{ij} \right) \in A.
\]

Hence the converse inclusion holds, i.e. \( R^*(A)_{sa} = A \). \( \triangleright \)

**Lemma 3.4.** Let \( A \) be an AJW-algebra and let \( I \) be a norm-closed ideal of \( A \). Then there exists a central projection \( g \) such that \( \dagger(\dagger(I_{sa})_+) = gA_+ \).

\(< \) Since \( A \) is an AJW-algebra there exists a projection \( g \) in \( A \) such that

\[
\dagger(I_{sa})_+ = U_{(1-g)}(A_+), \quad \dagger(\dagger(I_{sa})_+) = U_g(A_+),
\]

where \( \dagger(S)_+ = \{ x \in A_+ : (\forall a \in S) U_a x = 0 \} \) for \( S \subset A \).

Let \( (a_\lambda) \) be an approximate identity of the JB-subalgebra \( I \) and \( a \) be an arbitrary positive element in \( I \). Then there exists a maximal associative subalgebra \( A_o \) of \( A \) containing \( a \). Let
$v_\mu$ be an approximate identity of $A_\nu$. Then $(v_\mu) \subseteq (u_\lambda)$ and $\|av_\mu - a\| \to 0$. Let $b \in A_+$ and $U_\nu b = 0$ for every $\mu$. Then $U_{\lambda}U_\nu b = U_{\lambda}U_\nu b = 0$ and $U_\nu U_{\lambda}U_\nu b = 0$, where $c$ is an element in $A$ such that $b = c^2$. Hence $U_\lambda U_{\lambda}U_\nu c^2 = 0$, $(U_\lambda U_{\lambda}U_\nu c)^2 = 0$, $U_\lambda U_{\lambda}U_\nu c = 0$ and $U_\lambda U_{\lambda}U_\nu c = U_\lambda U_\nu c = 0$ for every $\mu$. We have

$$\|U_\lambda U_\nu c - U_\lambda a\| = \|U_\lambda U_\nu c - a\| \to 0$$

because $\|av_\mu - a\| \to 0$ and the operator $U_\lambda$ is norm-continuous. Hence $U_\lambda a = 0$. We may assume that $a = d^2$ for some element $d \in A$. Then

$$U_d U_\lambda a = U_d U_\lambda d^2 = (U_d b)^2 = 0, \quad U_d b = 0.$$

Thus $U_d U_\lambda b = U_d b = 0$. Therefore, if $b \in (u_\lambda)_+$ then $b \in (I_{sa})_+$. Hence $\left(\left(\left(u_\lambda\right)_+\right)_+\right) \subseteq \left(\left(I_{sa}\right)_+\right)_+$. It is clear that $\left(\left(I_{sa}\right)_+\right) \subseteq \left(\left(u_\lambda\right)_+\right)_+$ and

$$\left(\left(I_{sa}\right)_+\right) = \left(\left(u_\lambda\right)_+\right).$$

This implies that $\left(\left((u_\lambda)\right)_+\right) = U_{1-g}(A_+)$ and

$$\sup_{\lambda} u_\lambda = g.$$

Let us prove that $U_g(A)$ is an ideal of $A$. Indeed, let $x$ be an arbitrary element in $A$. Then $U_x u_\lambda \in I_{sa}$, i.e. $U_x u_\lambda \in U_g(A)$. By [9, Proposition 3.3.6] and the proof of [9, Lemma 4.1.5] we have $U_x$ is a normal operator in $A$. Hence

$$\sup_{\lambda} u_\lambda \in U_g(A).$$

At the same time

$$\sup_{\lambda} u_\lambda \in U_g(A).$$

Hence $U_x g \in U_g(A)$. By [9, 2.3.10] we have

$$A(\lambda x)^2 = 2gU_x g + U_x g^2 + U_x g^2 = 2gU_x g + U_x g + U_g x^2.$$
Lemma 3.6. Let $A$ be an AJW-algebra and let $J$ be the set of elements $a \in A$ such that $bac + c^*ab^* \in A$ for all $b, c \in C^*(A)$. Then $J$ is a norm-closed ideal in $A$. Moreover $J$ is a reversible AJW-algebra.

\(\triangleleft\) Let $a, b \in J$, $s, t \in C^*(A)$. Then
\[
s(a + b)t + t^*(a + b)s^* = (sat + t^*as^*) + (sbt + t^*bs^*) \in A,
\]
i.e. $J$ is a linear subspace of $A$. Now, if $a \in J$, $b \in A$, $s, t \in C^*(A)$, then
\[
s(ab + ba)t + t^*(ab + ba)s^* = (sa(bt) + (bt)^*as^*) + ((sb)at + t^*a(sb)^*) \in A,
\]
i.e. $J$ is a norm-closed ideal of $A$.

Let $a_1 \in J$, $a_2, \ldots, a_n \in A$ and $a = \prod_{i=2}^{n} a_i$. Then $a_1a + a^*a_1 \in A$ by the definition of $J$. Let us show that $a_1a + a^*a_1 \in J$; then, in particular, in the case of $a_2, \ldots, a_n \in J$ this will imply that $J$ is reversible. For all $b, c \in C^*(A)$ we have
\[
b(a_1a + a^*a_1)c + c^*(a_1a + a^*a_1)b^* = (b(a_1ac) + (ac)^*a_1b^*) + ((ba^*)a_1c + c^*a_1(ba^*)^*) \in A,
\]
i.e. $a_1a + a^*a_1 \in J$. \(\triangleright\)

Definition 3.7. An AJW-algebra $A$ is said to be totally nonreversible, if the ideal $J$ in Lemma 3.6 is equal to $\{0\}$, i.e. $J = \{0\}$.

Theorem 3.8. Let $A$ be a special AJW-algebra. Then there exist central projections $e, f, g \in A$, $e + f + g = 1$ such that

1. $J = (e + f)A$, $J$ is the ideal from Lemma 3.6;
2. $eA$ is reversible and there exists a norm-closed two sided ideal $I$ of $C^*(eA)$ such that $eA = \frac{1}{2}((I_{ea})_+) +$;
3. $fA$ is reversible and $R^*(fA) \cap iR^*(fA) = \{0\}$;
4. $gA$ is a totally nonreversible AJW-algebra and
\[
gA = \sum_{\omega \in \Omega} C(Q_{\omega}, R \oplus H_{\omega}),
\]
where $\Omega$ is a set of indices, $\{Q_{\omega}\}_{\omega \in \Omega}$ is an appropriate family of extremal compacts and $\{H_{\omega}\}_{\omega \in \Omega}$ is a family of Hilbert spaces.

\(\triangleleft\) We have
\[
A = A_1 \oplus A_2 \oplus \cdots \oplus A_{I_{\infty}} \oplus A_{II_1} \oplus A_{II_{\infty}} \oplus A_{III}
\]
and the subalgebra (without the part $A_2$)
\[
A_1 \oplus A_3 \oplus A_4 \oplus \cdots \oplus A_{I_{\infty}} \oplus A_{II_1} \oplus A_{II_{\infty}} \oplus A_{III}
\]
is reversible. The last statement can be proven similar to [9, Theorem 5.3.10]. By [10] the subalgebra $A_2$ can be represented as follows
\[
A_2 = \sum_{i \in \Xi} C(X_i, R \oplus H_i),
\]
where $\Xi$ is a set of indices, $\{X_i\}_{i \in \Xi}$ is a family of extremal compacts and $\{H_i\}_{i \in \Xi}$ is a family of Hilbert spaces. Hence by [9, Theorem 6.2.5] there exist central projections $h_i$ $g$ such that $A = hA \oplus gA$, $hA$ is reversible and $gA$ is totally nonreversible. For all $a, b_1, \ldots, b_n, c_1, \ldots, c_m$ in $hA$ we have
\[
b_1 \ldots b_nac_1 \ldots c_m + c_m^*c_{m-1} \ldots c_1ab_n b_{n-1} \ldots b_1 \in hA
\]
since $hA$ is reversible. Similarly for all $b, c \in R^*(hA), a \in hA$ we have

$$bac + c^*ab^* \in hA.$$ 

Hence $hA = J$.

By Proposition 3.5 there exist two central projections $e, f$ in $hA$ and a norm-closed two sided ideal $I$ of $C^*(hA)$ such that $e + f = h$, $eA = \frac{1}{2}(I_{sa} + I_{sa})$, $fA$ is a reversible AJW-algebra and $R^*(fA) \cap iR^*(fA) = \{0\}$. This completes the proof. 

Let $A$ be a special AJW-algebra. Despite the fact that for the real AW*-algebra $R^*(A)$ the C*-algebra $\mathcal{M} = R^*(A) + iR^*(A)$ is not necessarily a complex AW*-algebra we consider, that

CONJECTURE. Under the conditions of Theorem 3.8 the following equality is valid

$$ea = I_{sa}.$$ 

References


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ОБРАТИМЫЕ AJW-АЛГЕБРЫ

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Основной результат статьи гласит, что каждая специальная AJW-алгебра раскладывается в прямую сумму тотально неображимой и обратимой подалгебр. В свою очередь, каждая обратимая AJW-алгебра раскладывается в прямую сумму подалгебры, которая содержит идеал такой, что аннулятор комплексификации этого идеала в обертывающей C*-алгебре этой подалгебры равен нулю и подалгебры, обертывающая вещественная алгебра фон Неймана которой является чисто вещественной.

Ключевые слова: AJW-алгебра, обратимая AJW-алгебра, AW*-алгебра, обертывающая *-алгебра.