A STUDY ON A CLASS OF \(p\)-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS\(^1\)

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In this paper, we study and introduce the majorization properties of a new class of analytic \(p\)-valent functions of complex order defined by the generalized hypergeometric function. Some known consequences of our main result will be given. Moreover, we investigate the coefficient estimates for this class.

Mathematics Subject Classification (2000): 30C45.
Key words: majorization, \(p\)-valent functions, hypergeometric functions

1. Introduction

Let \(\mathcal{A}_p\) be the class of functions \(f(z)\) normalized by

\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}, \quad p \in \mathbb{N}, \quad (1.1) \]

which are analytic and \(p\)-valent in the unit disc \(\mathbb{U}\). Let \(f\) and \(g\) be analytic in the open unit disc \(\mathbb{U}\). We say that \(f\) is majorized by \(g\) in \(\mathbb{U}\) and write

\[ f(z) \prec g(z) \quad (z \in \mathbb{U}), \quad (1.2) \]

if there exists a function \(\varphi\), analytic in \(\mathbb{U}\) such that

\[ |\varphi(z)| \leq 1, \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}). \quad (1.3) \]

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

For \(f(z)\) and \(g(z)\) are analytic in \(\mathbb{U}\), we say that \(f\) is subordinate to \(g\) if there exists the Schwarz function \(\omega\), analytic in \(\mathbb{U}\), with \(\omega(0) = 0\) and \(|\omega(z)| < 1\) such that \(f(z) = g(\omega(z))\), \(z \in \mathbb{U}\). We denote this subordination by \(f(z) \prec g(z)\). If \(g(z)\) is univalent in \(\mathbb{U}\), then the subordination is equivalent to \(f(0) = g(0)\) and \(f(\mathbb{U}) \subset g(\mathbb{U})\).

If \(f(z)\) and \(g(z)\) belong to \(\mathcal{A}_p\), then the Hadamard product \(f \ast g\) is defined by

\[ f(z) \ast g(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}, \quad p \in \mathbb{N}. \]

El-Ashwah [2] studied the following \(p\)-valent function, which defined by generalized hypergeometric functions

\[ {}_pF_q(a_1, b_1; z^p) = z^p + \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_r)_n z^{p+n}}{(b_1)_n \cdots (b_s)_n n!}, \quad p \in \mathbb{N}, \]

\(^1\) The work presented here was partially supported by UKM grant AP-2013-009.
where \(a_i \in \mathbb{C}, b_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, (i = 1, \ldots, r, q = 1, \ldots, s)\), and \(r \leq s + 1; r, s \in \mathbb{N}_0\), and \((x)_n\) is the Pochhammer symbol defined by

\[
(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1, & n = 0, \\ x(x + 1) \cdots (x + n - 1), & n = \{1, 2, 3, \ldots\}. \end{cases}
\]

Let \(\mathcal{L}_{\lambda_1, \lambda_2, p} \in \mathcal{A}_p\) is defined by

\[
\mathcal{L}_{\lambda_1, \lambda_2, p}^{m, b} = z^p + \sum_{n=1}^{\infty} \left[ \frac{p + (\lambda_1 + \lambda_2)n + b}{p + \lambda_2n + b} \right]^{m} z^{p+n}; \quad p \in \mathbb{N},
\]

where \(m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda_2 \geq \lambda_1 \geq 0\).

Corresponding to \(r \mathcal{G}_s(a_1, b_1; z^p), \mathcal{L}_{\lambda_1, \lambda_2, p}^{m, b}\) and using the Hadamard product, we define a new generalized differential operator \(D_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1)\) as follows:

**Definition 1.1.** Let \(f \in \mathcal{A}_p\), then a generalized differential operator \(D_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z) : \mathcal{A}_p \to \mathcal{A}_p\) is given as

\[
D_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z) = (r \mathcal{G}_s(a_1, b_1; z^p) \ast \mathcal{L}_{\lambda_1, \lambda_2, p}^{m, b} \ast f(z))
\]

\[
= z^p + \sum_{n=1}^{\infty} \left[ \frac{p + (\lambda_1 + \lambda_2)n + b}{p + \lambda_2n + b} \right]^{m} \frac{(a_1)_n \cdots (a_r)_n a_{p+n} z^{p+n}}{(b_1)_n \cdots (b_s)_n n!}.
\]

It follows from the above definition that

\[
(p + \lambda_2n + b) D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_1, b_1) f(z)
\]

\[
= (p + \lambda_2 n - p \lambda_1 + b) D_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z) + \lambda_1 z(D_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z))'.
\]

**Remark 1.1.** It should be remarked that the linear operator \(\mathcal{G}_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z)\) is a generalization of many operators considered earlier. Let us see some of the examples:

For \(\lambda_2 = b = 0\), the operator \(\mathcal{G}_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z)\) reduces to the operator was given by Selvaraj and Karthikeyan [1].

For \(m = 0\), the operator \(\mathcal{G}_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z)\) reduces to the operator was given by El-Ashwah [2].

For \(m = 0\) and \(p = 1\), the operator \(\mathcal{G}_{\lambda_1, \lambda_2, p}^{m, b}(a_1, b_1) f(z)\) reduces to the well-known operator introduced by Dziok and Srivastava [3].

For \(\lambda_2 = b = 0\) and \(p = 1\), we get the operator studied by Selvaraj and Karthikeyan [4].

For \(m = 0\), \(r = 2\), \(s = 1\) and \(p = 1\), we obtain the operator which was given by Mohlov [5].

For \(r = 1\), \(s = 0\), \(a_1 = 1\), \(\lambda_1 = 1\), \(\lambda_2 = b = 0\) and \(p = 1\), we get the Sălăgean derivative operator [6].

For \(r = 1\), \(s = 0\), \(a_1 = 1\), \(\lambda_2 = b = 0\) and \(p = 1\), we get the generalized Sălăgean derivative operator introduced by Al-Oboudi [7].

For \(r = 1\), \(s = 0\), \(a_1 = \delta + 1\) and \(p = 1\), we obtain the operator introduced by Ruscheweyh [8].

For \(r = 1\), \(s = 0\), \(a_1 = \delta + 1\) and \(p = 1\), we obtain the operator studied by El-Yagubi and Darus [9].

For \(m = 0\), \(r = 2\) and \(s = 1\), \(a_2 = 1\) and \(p = 1\), we obtain the operator studied by Carlson and Shaffer [10].
For \( r = 1, s = 0, a_1 = 1, \lambda_2 = 0 \) and \( p = 1 \), we get the operator introduced by Cátás [11].

Next, by using the generalized differential operator \( D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1) \), we study the class \( S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) as follows:

**Definition 1.2.** Let \( f \in \mathcal{A}_p \), then \( f \in S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) of \( p \)-valent functions of complex order \( \gamma \neq 0 \) in \( \mathbb{U} \), if and only if

\[
\left\{ 1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)f(z))^{(j+1)}}{(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)f(z))^{(j)}} - p + j \right) \right\} < \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{U},
\]

where \( p \in \mathbb{N}, m, b, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C}\setminus\{0\}, \lambda_2 \geq \lambda_1 \geq 0, -1 \leq B < A \leq 1, a_i \in \mathbb{C}, b_q \in \mathbb{C}\setminus\{0,-1,-2,\ldots\} \) \( (i = 1, \ldots, r; q = 1, \ldots, s) \) and \( r \leq s + 1; r, s \in \mathbb{N}_0 \).

Clearly, we have the following relationships:

(i) when \( m = 0, p = 1, j = 0, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1 \) and \( B = -1 \), then the class \( S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) reduces to the class \( S(\gamma) \).

(ii) when \( m = 0, p = 1, j = 1, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1 \) and \( B = -1 \), then the class \( S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) reduces to the class \( C(\gamma) \).

(iii) when \( m = 0, p = 1, j = 0, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1 \) and \( B = -1 \) and \( \gamma = 1 - \alpha \), then the class \( S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) reduces to the class \( S^*(\alpha) \) for \( 0 < \alpha < 1 \).

The classes \( S(\gamma) \) and \( C(\gamma) \) are said to be classes of starlike and convex of complex order \( \gamma \neq 0 \) in \( \mathbb{U} \), were considered by Nasr and Aouf [12] and \( S^*(\alpha) \) denote the class of starlike functions of order \( \alpha \) in \( \mathbb{U} \).

### 2. Majorization Problem

A majorization problem for functions \( f \) belong to the class \( S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) is considered.

**Theorem.** Let \( f \in \mathcal{A}_p \) and suppose that \( g \in S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \). If \( (D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)f(z))^{(j)} \) is majorized by \( (D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)g(z))^{(j)} \) in \( \mathbb{U} \), then

\[
(D_{\lambda_1,\lambda_2,p}^{m+1,b}(a_1,b_1)f(z))^{(j)} \leq (D_{\lambda_1,\lambda_2,p}^{m+1,b}(a_1,b_1)g(z))^{(j)} \quad \text{for} \quad |z| \leq r_0,
\]

where \( r_0 = r_0(p,\gamma,\lambda_1,\lambda_2,b,A,B) \) is the smallest positive root of the equation

\[
r^3 \left[ \gamma(A - B) + \left( \frac{p + \lambda n + b}{\lambda_1} \right) B - \left[ \frac{p + \lambda n + b}{\lambda_1} + 2|B| \right] r^2 \right] - \left[ \gamma(A - B) - \left( \frac{p + \lambda n + b}{\lambda_1} \right) B + 2 \right] r + \left( \frac{p + \lambda n + b}{\lambda_1} \right) = 0, \quad \text{for} \quad -1 \leq B < A \leq 1; \quad \lambda_2 \geq \lambda_1 \geq 0; \quad b \in \mathbb{N}_0; \quad P \in \mathbb{N}; \quad \gamma \in \mathbb{C}\setminus\{0\}.
\]

Since \( g \in S_{\lambda_1,\lambda_2,p}^{m,b,j}[a_1,b_1,A,B,\gamma] \) we can get from (1.6), that

\[
1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)g(z))^{(j+1)}}{(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)g(z))^{(j)}} - p + j \right) = \frac{1 + Aw(z)}{1 + Bw(z)},
\]

where

\[
w(z) = \frac{A}{B} \left( \frac{z(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)f(z))^{(j+1)}}{(D_{\lambda_1,\lambda_2,p}^{m,b}(a_1,b_1)f(z))^{(j)}} - p + j \right).
\]
where \( \gamma \in \mathbb{C}\setminus\{0\}, j, p \in \mathbb{N}, p > j \) and \( w \) is analytic in \( \mathbb{U} \) with
\[
w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}).
\]
From (2.3), we get
\[
z\left( \frac{D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)g(z))^{(j+1)}}{D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)g(z))^{(j)}} = \frac{(p - j) + \left[ \gamma(A - B) + (p - j)B \right]w(z)}{1 + Bw(z)} \tag{2.4}
\]
By noting that
\[
z\left( D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z) = \left( \frac{p + \lambda_2n + b}{\lambda_1} \right) D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z)) (D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)f(z))
\]
and by virtue of (2.4) and (2.5) we get
\[
\left| \left( D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)g(z))^{(j+1)} \right| \leq \frac{p + \lambda_2n + b}{\lambda_1} \frac{1 + |B||z|}{\gamma(A - B) + \left( \frac{p + \lambda_2n + b}{\lambda_1} \right) |B||z|} \left| \left( D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)g(z))^{(j+1)} \right| \tag{2.6}
\]
Next, since \( D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j)} \) is majorized by \( (D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)g(z))^{(j)} \) in the unit disc \( \mathbb{U} \), thus from (1.3) we have
\[
(D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)g(z))^{(j)} = \varphi(z)(D_{\lambda_1, 2, p}^{(j)}(a_1, b_1)g(z))^{(j)}.
\]
Differentiating it with respect to \( z \) and multiplying by \( z \) we get
\[
z(D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j+1)} = z\varphi'(z)(D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j)} + z\varphi(z)(D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j+1)}.
\]
Now by using (2.5) in the above equation, it yields
\[
\frac{D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j)}}{p + \lambda_2n + b} + \varphi(z)(D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j)} \tag{2.7}
\]
Thus, by noting that \( \varphi \in \Omega \) satisfies the inequality
\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U}), \tag{2.8}
\]
by using (2.6) and (2.8) in (2.7), we get
\[
\left| \left( D_{\lambda_1, 2, p}^{(j+1)}(a_1, b_1)f(z))^{(j)} \right| \leq \left| \varphi(z) \right| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \frac{|z|(1 + |B||z|)}{\frac{p + \lambda_2n + b}{\lambda_1} - \gamma(A - B) + \left( \frac{p + \lambda_2n + b}{\lambda_1} \right) |B||z|} \tag{2.9}
\]
\[\]
which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$\left| \left( D_{\lambda_1,\lambda_2,p}^{m+1,b}(a_1,b_1)f(z) \right)^{(j)} \right| \leq \frac{\phi(\rho)}{(1 - r^2) \left[ \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right) - \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right) \right] B} \left| \left( D_{\lambda_1,\lambda_2,p}^{m+1,b}(z) \right)^{(j)} \right|$$

(2.10)

where

$$\phi(\rho) = -r(1 + |B|r)\rho^2 + (1 - r^2)$$

(2.11)

takes its maximum value at \( \rho = 1 \) with \( r_0 = r_0(p, \gamma, \lambda_1, \lambda_2, b, A, B) \). Here \( r_1(p, \gamma, \lambda_1, \lambda_2, b, A, B) \) is the smallest positive root of the equation (2.2).

Furthermore, if \( 0 \leq \rho \leq r_1(p, \gamma, \lambda_1, \lambda_2, b, A, B) \), then the function \( \psi(\rho) \) defined by

$$\psi(\rho) = -\sigma(1 + |B|\sigma)\rho^2 + (1 - \sigma^2)$$

(2.12)

is seen to be an increasing function on the interval \( 0 \leq \rho \leq 1 \), so that

$$\psi(\rho) \leq \psi(0) = (1 - \sigma^2) \left[ \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right) - \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right) \right]$$

(2.13)

Hence upon setting \( \rho = 1 \) in (2.10) we conclude that (2.1) of Theorem 2.1 holds true for

$$|z| \leq r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$$

where \( r_1(p, \gamma, \lambda_1, \lambda_2, A, B) \) is the smallest positive root of equation (2.2).

Putting \( A = 1 \) and \( B = -1 \) in Theorem 2.1, we have the following result:

**Corollary 2.1.** Let \( f \in \mathcal{A}_p \) and suppose that \( g \in S^{m,b,j}_{\lambda_1,\lambda_2,p}(a_1,b_1,\gamma) \). If \( (D_{\lambda_1,p}^m(a_1,b_1)f(z))^{(j)} \) is majorized by \( (D_{\lambda_1,p}^m(a_1,b_1)g(z))^{(j)} \) in \( \mathbb{U} \), then

$$\left| \left( D_{\lambda_1,p}^{m+1,b}(a_1,b_1)f(z) \right)^{(j)} \right| \leq \left| \left( D_{\lambda_1,p}^{m+1,b}(a_1,b_1)g(z) \right)^{(j)} \right| \quad \text{for} \quad |z| \leq r_0,$$

(2.14)

where

$$r_0 = r_0(p, \gamma, \lambda_1, \lambda_2, b) = \frac{k - \sqrt{k^2 - 4 \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right)}}{2|2\gamma - \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right)}$$

$$k = 2 + \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right) + 2\gamma - \left( \frac{p + \lambda_2 n + b}{\lambda_1} \right),$$

\( p \in \mathbb{N}; \quad \gamma, \lambda_1 \in \mathbb{C}\backslash\{0\}, \quad b \in \mathbb{N}_0, \quad \lambda_2 \geq 0 \)

and \( S^{m,b,j}_{\lambda_1,\lambda_2,p}(a_1,b_1,\gamma) \) be a special case of \( S^{m,b,j}_{\lambda_1,\lambda_2,p}(a_1,b_1,A,B,\gamma) \) when \( A = 1 \) and \( B = -1 \).
Setting \( p = 1, \ m = \lambda_2 = b = 0, \ \lambda_1 = 1, \ j = 0, \ r = 2, \ s = 1, \ a_1 = b_1 \) and \( a_2 = 1 \) in Corollary 2.1, we get the following corollary:

**Corollary 2.2.** Let \( f \in \mathcal{A}_p \) and suppose that \( g \in S(\gamma) \). If \( f(z) \) is majorised by \( g(z) \) in \( U \), then
\[
|f'(z)| \leq |g'(z)| \quad (|z| < r_3),
\]
where
\[
r_0 = r_0(\gamma) = \frac{3 + 2\gamma - 1 - \sqrt{9 + 24\gamma - 1 + 12\gamma - 1^2}}{2\gamma - 1},
\]
which is a known result obtained by Altintas et al. [13].

For \( \gamma = 1 \), the Corollary 2.2 reduces to the following result:

**Corollary 2.3.** Let \( f(z) \in \mathcal{A}_p \) and suppose that \( g \in S^* = S^*(0) \). If \( f(z) \) is majorised by \( g(z) \) in \( U \), then
\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq 2 - \sqrt{3}),
\]
which is a known result obtained by MacGregor [14].

3. Coefficient Estimates

The coefficient estimate for the class \( S_{\lambda_1, \lambda_2, \rho}^{m,b}[a_1, b_1, A, B, \gamma] \) is obtained, when \( j = 0 \).

**Definition 3.1.** Let \( S_{\lambda_1, \lambda_2, \rho}^{m,b}[a_1, b_1, A, B, \gamma] \) denote the subclass of \( p \)-valent functions which satisfy the condition
\[
1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z))'}{D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z)} - p \right) < \frac{1 + Az}{1 + Bz},
\]  
\[  \text{for } p \in \mathbb{N}, \ \gamma \in \mathbb{C}\setminus\{0\}, \ \lambda_2 \geq \lambda_1 > 0, \ m, b \in \mathbb{N}_0, \ -1 \leq B < A \leq 1, \ a_i \in \mathbb{C}, \ b_q \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}, \ (i = 1, \ldots, r, \ q = 1, \ldots, s), \ \text{and } r \leq s + 1; \ r, s \in \mathbb{N}_0. \]

**Theorem 3.1.** Let \( f \in \mathcal{A}_p \). If it satisfies the condition:
\[
\sum_{n=1}^{\infty} \left[ \frac{n + |\gamma(A - B) - nB|}{|\gamma|(A - B)} \right] \left[ \frac{z(D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z))'}{D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z)} - p \right] \leq 1,
\]
then \( f \in S_{\lambda_1, \lambda_2, \rho}^{m,b}[a_1, b_1, A, B, \gamma] \).

Let \( f \in S_{\lambda_1, \lambda_2, \rho}^{m,b}[a_1, b_1, A, B, \gamma] \), then we can write (3.1) as follows:
\[
1 + \frac{1}{\gamma} \left( \frac{z(D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z))'}{D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z)} - p \right) = \frac{1 + Aw(z)}{1 + Bw(z)}
\]
which gives
\[
\frac{z(D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z))'}{D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z)} - p = \left[ \gamma(A - B) - B \left( \frac{z(D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z))'}{D_{\lambda_1, \lambda_2, \rho}^{m,b}(a_1, b_1) f(z)} - p \right) \right] w(z).
\]
From (3.3), we obtain

\[
pz^n + \sum_{n=1}^{\infty} \left[ \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} (p+n) a_{p+n} z^{p+n} - p \\
= \left\{ \gamma(A - B) - B \left[ \frac{p^n + \sum_{n=1}^{\infty} \left[ \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} (p+n) a_{p+n} z^{p+n} - p \right] \right\} w(z)
\]

which yields

\[
\sum_{n=1}^{\infty} \left[ \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} a_{p+n} z^n \\
= \left\{ \gamma(A - B) - B \left[ \sum_{n=1}^{\infty} \left[ \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} a_{p+n} z^n \right] \right\} w(z).
\]

Since \(|w(z)| \leq 1\),

\[
\left| \gamma(A - B) - \sum_{n=1}^{\infty} (Bn - \gamma(A - B)) \left[ \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right]^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} a_{p+n} z^n \right| \leq \left| \gamma(A - B) \right|
\]

Letting \(|z| \to 1^+\) through real values, we have

\[
\sum_{n=1}^{\infty} |n + \gamma(A - B) - Bn| \left| \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right|^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} a_{p+n} |z^n| \leq |\gamma(A - B)|,
\]

therefore,

\[
\sum_{n=1}^{\infty} |n + \gamma(A - B) - Bn| \left| \frac{p+(\lambda_1+\lambda_2)n+b}{p+\lambda_2n+b} \right|^m \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n n!} a_{p+n} |z^n| \leq 1.
\]

\textbf{Remark 3.1.} Other works related to different classes of \(p\)-valent functions can be found in [15, 16].

\textbf{References}


Received September 16, 2014.

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ИССЛЕДОВАНИЕ ОДНОГО КЛАССА p-ВАЛЕНТНЫХ ФУНКЦИЙ, ПОРЖДЕННОГО ОБОБЩЕННОЙ ГИПЕРГЕОМЕТРИЧЕСКОЙ ФУНКЦИЕЙ

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Исследованы свойства мажораны для нового класса аналитических p-валентных функций комплексного порядка, порожденной гипергеометрической функцией. При этом некоторые известные следствия полученных результатов. Дана также оценка коэффициентов для этого класса.

Ключевые слова: мажораны, p-валентная функция, гипергеометрическая функция.