THE EDGE $C_k$ GRAPH OF A GRAPH

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For any integer $k \geq 4$, the edge $C_k$ graph $E_k(G)$ of a graph $G = (V, E)$ has all edges of $G$ as its vertices, and two vertices in $E_k(G)$ are adjacent if their corresponding edges in $G$ are either incident or belongs to a copy of $C_k$. In this paper, we obtained the characterizations for the edge $C_k$ graph of a graph $G$ to be connected, complete, bipartite, etc. It is also proved that the edge $C_4$ graph has no forbidden subgraph characterization. Moreover, the dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(G)$ are studied.

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Key words: edge $C_k$ graph, triangular line graph, line graph, convergent, periodic, mortal, transition number.

1. Introduction

For graph theory terminology and notation in this paper we follow the book [3]. All graphs considered in this paper are finite, unoriented, without loops and multiple edges.

Graph theory [3] is an established area of research in combinatorial mathematics. It is also one of the most active areas of mathematics that has found large number of applications in diverse areas including not only computer science, but also chemistry, physics, biology, anthropology, psychology, geography, history, economics, and many branches of engineering. Graph theory has been especially useful in computer science, since after all, any data structure can be represented by a graph. Furthermore, there are applications in networking, in the design of computer architectures, and in general, in virtually every branch of computer science. However, to date most of the research in graph theory has only considered graphs that remain static, i.e., they do not change with time. A wealth of such literature has been developed for static graph theory. Our purpose is to classify dynamic graphs, i.e., graphs that change with time. Dynamic graphs appear in almost all fields of science. This is especially true of computer science, where almost always the data structures (modeled as graphs) change as the program is executed. Very little is known about the properties of dynamic graphs.

The study of graph dynamics has been receiving wide attention, since Ore’s work on the line graph operator $L(G)$ (see [5, 6]). The edge $C_k$ graph $E_k(G)$ of a graph $G$ is defined in [5] as follows: The edge $C_k$ graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices $e$ and $f$ are adjacent if, and only if, the corresponding edges in $G$ either incident or opposite edges of some cycle $C_k$. So for any two edges in $G$ are adjacent if, and only if, they belong to a common $P_3$ or $C_k$ in $G$. When $k = 3$, the definition coincides with triangular line graph of a graph [2], and when $k = 4$, the definition coincides with $E_4$-graph of a graph [4].

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2. Edge $C_k$ Graph of a Graph

The edge $C_k$ graph $E_k(G)$ of a graph $G$ is defined in [5] as follows: The edge $C_k$ graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices $e$ and $f$ are adjacent if, and only if, the corresponding edges in $G$ either incident or opposite edges of some cycle $C_k$. So for any two edges in $G$ are adjacent if, and only if, they belong to a common $P_3$ or $C_k$ in $G$. When $k = 3$, the definition coincides with triangular line graph of a graph [2], and when $k = 4$, the definition coincides with $E_4$-graph of a graph [4]. Clearly the edge $C_k$ graph coincides with the line graph for any acyclic graph. But they differ in many properties. As a case, for a connected graph $G$, $E_k(G) = G$ if and only if $G = C_n$, $n \neq k$. The following result characterizes graphs whose $E_k$ graph is isomorphic to their line graph.

**Theorem 1.** For a graph $G$, $E_k(G) \cong L(G)$ if, and only if, $G$ is $C_k$-free.

**Theorem 2.** For any graph $G$, $E_k(G)$ is connected if, and only if, exactly one component of $G$ contains edges.

**Theorem 3.** For any graph $G$, the edge $E_k$ graph is complete then $\text{diam}(G) \leq \left\lfloor \frac{k}{2} \right\rfloor$.

$\Leftarrow$ Since $E_k(G)$ is complete then by the definition of $E_k(G)$ any two edges must either incident or belongs to a cycle of length $k$. Suppose that $\text{diam}(G) > \left\lceil \frac{k}{2} \right\rceil$. That is there exists two vertices $u$ and $v$ in $G$ with $d(u, v) > \left\lceil \frac{k}{2} \right\rceil$. Clearly $u$ and $v$ can not be in the same cycle of length $\left\lfloor \frac{k}{2} \right\rfloor$. Let $u'$ and $v'$ be any two vertices adjacent to $u$ and $v$ respectively. Then the edges $uu'$ and $vv'$ are not adjacent and does not belongs to a cycle of length $k$. This proves, a contradiction. $\Rightarrow$

**Remark.** The converse need not be true for example $C_5$ has diameter 2 but $E_4(G)$ is not complete.

In [4], the authors prove that: For a connected graph $G$, $E_4(G)$ is complete if, and only if, $G$ is complete multipartite. But the same can not be generalized for $k \geq 5$. For example for the peterson graph $\mathcal{P}$, $E_5(\mathcal{P})$ is clearly complete graph $K_{15}$. But $\mathcal{P}$ is not complete bipartite. However, we have the following:

**Theorem 4.** For a connected graph $G$, $E_k(G)$ is complete if, and only if, every edge of $G$ belongs to a $C_k$. 
**Corollary 5.** Let $G$ be a complete $r$-multipartite graph for some $r \geq \frac{2}{3}$, $E_k(G)$ is complete.

In [4], the authors proved that the edge $C_4$ graph $E_4(G)$ has no forbidden subgraphs. We now prove that the edge $C_k$ graph $E_k(G)$ also have no forbidden subgraph characterization.

**Theorem 6.** There is no forbidden subgraph characterization for $E_k(G)$ for any $k \geq 3$.

We can assume that $k \geq 5$, since the edge $C_3$ of a graph is nothing but triangular line graph and when $k = 4$, the result follows from the above result. We shall prove that given any graph $G$, we can find a graph $H$ such that $G$ is an induced subgraph of $E_k(H)$. For any graph $G$, let $H = G \times K_2$. Clearly $H$ contains 2 copies of $G$ say $G$ and $G'$. Now let $H'$ be the graph obtained from $H$ by subdivide each edge of one copy of $G$ in $H$ into $k - 4$ edges.

We claim that $G$ is induced subgraph of $E_k(H')$. For any $v \in V(G)$, $E_k(G)$ contains vertices of the form $vu'$, where $u'$ is the corresponding vertex in $G'$. Now for any two adjacent vertices $u$ and $v$, the corresponding vertices $uu'$ and $vv'$ are also adjacent in $E_k(G)$, since the vertices $u, u', u_1, u_2, \ldots, u_{k-4}, v', v$ forms a cycle of length $k$ in $H'$. Now if $u$ and $v$ are non adjacent adjacent vertices of $G$ then $uu'$ and $vv'$ are also non adjacent vertices in $E_k(H')$.

Thus the subgraph induced by the set \{uu' : u \in G\} of vertices in $H'_k(G)$ contains $G$ as a induced subgraph. This completes the proof. $	riangleright$

**Theorem 7.** For a connected graph $G$, $E_k(G)$ is bipartite if, and only if, $G$ is either a path or an even cycle of length $r \neq k$.

Suppose that $E_k(G)$ is bipartite. Suppose that $G$ has a vertex of degree at least 3, then $G$ contains a cycle of length 3. Hence, the degree of every vertex is at least 2. Since $G$ connected, $G$ must be a path or a cycle. Now, if $G$ is an odd cycle of length $r$, then $r$ cannot be odd or equal to $k$. since, if $r$ is odd then $L(G)$ is also a cycle which is a subgraph of $E_k(G)$ and if $r = k$, then $E_k(G) = K_1$ and hence $E_k(G)$ cannot be bipartite in both cases.

Finally if $r$ is even and $r \neq k$ then $E_k(G) = G$, which is bipartite.

Conversely, suppose that $G$ is either a path or an even cycle of length $r \neq k$, then $E_k(G)$ is either a path or an even cycle respectively. Hence $E_k(G)$ is bipartite. $	riangleright$

**Corollary 8.** For a connected graph $G$, $E_k(G)$ is a tree if, and only if, $G$ is a path.

3. Dynamical Properties

First we recall some graph dynamical terminologies from [6]. Let $G$ be any graph. The $n^{th}$-iterated graph is iteratively defined as follows: $E_0^0(G) = G$, $E_k^0(G) = E_k(G)$, $E_k^n(G) = E_k(E_k^{n-1}(G))$, $n \geq 2$. We say that $G$ is convergent under $E_k$ if \{E_k^n(G) : n \in N\} is finite. If $G$ is not convergent under $E_k$, then $G$ is divergent under $E_k$. A graph $G$ is periodic if there is some natural number $n$ with $G = E_k^n(G)$. The smallest such number is called the period of $G$. The transition number $t(x)$ of a convergent graph $G$ is defined as zero if $G$ is periodic and as the smallest number $n$ such that $E_k^n(G)$ is periodic. A graph $G$ is mortal if for some $n \geq N$, $E_k^n(G) = \phi$ the empty graph.

**Theorem 9.** The graphs $P_n$, $K_{1,3}$, $C_n$ ($n \neq k$) are the only $E_k$ convergent graphs.

If $G$ contains a vertex of degree $> 3$, then $E_k(G)$ contains $K_4$. In the subsequent iterations the clique size goes on increasing and hence $G$ diverges. So, for convergent graphs $\delta(G) \leq 3$.

If $G$ is a tree which is neither $P_n$ nor $K_{1,3}$, then $K_4$ is contained at least in the third iterated graph and hence $G$ cannot converge. $	riangleright$

**Corollary 10.** For $E_k(G)$, the only periodic graphs are the cycles $C_n$, $n \neq k$ and they have period one.
The paths $P_n$ converge to and $K_{1,3}$ converges to the triangle. Consider the graphs which are not trees. If $G$ is not a cycle, then $G$ contains a cycle with a pendant edge as a subgraph (need not be induced). Then $K_4$ is a subgraph at least in the second iteration and hence in the subsequent iterations the clique size will go on increasing and hence cannot converge. All cycles except $C_k$ are fixed under $E_k$ and $C_k$ is not convergent. Thus, the proof follows from the fact that a graph $G$ is convergent if, and only if, $G$ is either periodic or there is some positive integer $n$ with $E_k^n(G)$ periodic.

**Corollary 11.** The transition number $t(K_{1,3}) = 1$ and for $n \neq k$, $t(K_n) = 0$.

**Corollary 12.** For $E_k(G)$, the paths are the only mortal graphs.

< Among the convergent graphs, cycles other than $C_k$ are fixed and $K_{1,3}$ converges to $K_3$. The paths are the only graphs converging to $\phi$.>

**References**


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**РЕБЕРНЫЙ $C_k$-ГРАФ ГРАФА**

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Для любого целого $k \geq 4$ реберный $C_k$-граф $E_k(G)$ графа $G$ содержит все ребра графа $G$ в качестве вершин, при этом две вершины смежны в $E_k(G)$, если соответствующие им ребра в графе $G$ либо инцидентны, либо принадлежат копии $C_k$. В статье утверждено, что реберный $C_k$-граф графа $G$ является связным, полным, дуальным и т. д. Доказано также, что реберный $C_k$-граф не содержит характеризованный запрещенного подграф. Кроме того, исследованы такие характеристики динамических графов как сходимость, периодичность, моральность и число переходов графа $E_k(G)$.

Ключевые слова: реберный $C_k$-граф, треугольный динейный граф, сходимость, периодичность, моральность, число переходов.