SOLUTIONS OF THE DIFFERENTIAL INEQUALITY
WITH A NULL LAGRANGIAN: HIGHER INTEGRABILITY
AND REMOVABILITY OF SINGULARITIES. II

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The aim of this paper is to establish a result on removability of singularities for solutions of the differential inequality with a null Lagrangian. Also, we obtain integral estimates for wedge products of closed differential forms and for minors of a Jacobian matrix.


Key words: null Lagrangian, removability of singularities, integral estimates, closed differential forms, minors of a Jacobian matrix.

Introduction

In this paper we continue to study the properties of solutions \( v: V \to \mathbb{R}^m, V \subset \mathbb{R}^n \), of the following inequality

\[
F(v'(x)) \leq KG(v'(x)) + H(x) \quad \text{a.e. } V
\]

constructed by means of a continuous function \( F: \mathbb{R}^{m \times n} \to \mathbb{R} \), a null Lagrangian \( G: \mathbb{R}^{m \times n} \to \mathbb{R} \), a measurable function \( H: V \to \mathbb{R} \), and a constant \( K \geq 1 \). Here \( v'(x) \) denotes the differential of \( v \) at \( x \in V \). Using the higher integrability theorem of the previous paper [8], we establish a result on removability of singularities for solutions to (1).

Many investigations have dealt with the problem of removable singularities for quasiconformal mappings and mappings with bounded distortion (for example, see [1, 2, 4], [9]–[29] and the bibliography therein). Painlevé’s theorem, a classical result in complex function theory, states that sets of zero length are removable for bounded holomorphic functions. More precisely, if \( E \) is a closed subset of linear measure zero in a planar domain \( V \) and \( v \) is a bounded function holomorphic in \( V \setminus E \), then \( v \) extends to a bounded holomorphic function of \( V \). Observe that the class of planar mappings with 1-bounded distortion coincides with the class of holomorphic functions. The strongest removability conjecture, stated in [14] as the counterpart of Painlevé’s theorem for mappings with bounded distortion, suggests that sets of Hausdorff \( \alpha \)-measure zero, \( \alpha \leq n/(K + 1) \leq n/2 \), are removable for bounded mappings with \( K \)-bounded distortion in \( \mathbb{R}^n \). In the case \( n = 2 \) this conjecture was verified

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by K. Astala [1] for \( \alpha < 2/(K+1) \) and K. Astala, A. Clop, J. Mateu, J. Orobitg, I. Uriarte-Tuero [2] for \( \alpha = 2/(K+1) \) (also see [3]). The higher integrability results for mappings with bounded distortion are closely related to the removability problems. Caccioppoli-type estimates (one of the key ingredients in proofs of higher integrability results) can be used as the basic tool for proving removability theorems. More precisely, if there exists \( p(n, K) < n \) such that Caccioppoli-type estimates hold for \( p > p(n, K) \), then a close set \( E \) with the Hausdorff dimension \( \dim_H(E) < n - p(n, K) \) is removable for bounded mappings with \( K \)-bounded distortion in \( \mathbb{R}^n \) (for example, see [13, 14, 15, 16]). In such way removability results have been established for the classes of mappings that are close to solutions of linear elliptic partial differential equations and for the classes of quasiregular mappings of several \( n \)-dimensional variables (for example, see [5, 6]). Mappings of these classes, as mappings with bounded distortion, can be considered as solutions to (1) with specific functions \( F, G, \) and \( H \). Our removability result (Theorem 1.1) contain partially the known results on removability of singularities for mappings of these classes.

In this paper, using the Hodge decomposition theory developed by T. Iwaniec and G. Martin [13, 14, 15], we also obtain integral estimates for wedge products of closed differential forms (Theorem 2.3) and for minors of a Jacobian matrix (Theorem 2.1). These estimates are extensions of integral estimates derived in [15]. They have been used in the proof of the higher integrability theorem in [8].

Some results of this paper have been announced in [7].

This paper is organized as follows. In §1 we establish a result on removability of singularities for solutions to (1). We derive integral estimates for wedge products of closed differential forms and for minors of a Jacobian matrix in §2.

We use the notation and terms from [8].

### 1. Removability of Singularities

Using the higher integrability theorem from [8], we establish the following result on removability of singularities for solutions to (1).

**Theorem 1.1** (Removability of singularities). Let \( n, m, k \in \mathbb{N} \) and \( t > k \) such that \( 2 \leq k \leq \min\{n, m\} \), and let \( V \) be a domain in \( \mathbb{R}^n \). Suppose that a continuous function \( F: \mathbb{R}^{m \times n} \to \mathbb{R} \) satisfies

\[
F(\zeta) \geq c_F |\zeta|^k, \quad \zeta \in \mathbb{R}^{m \times n},
\]

with some constant \( c_F > 0 \), a null Lagrangian \( G: \mathbb{R}^{m \times n} \to \mathbb{R} \) is homogeneous of degree \( k \), and a measurable function \( H: V \to \mathbb{R} \) has \( H_+ \in L^1_{\text{loc}}(V) \). Fix \( K \geq 1 \). Let \( p = p(F, G, K) \) be the exponent from [8, Theorem 2.1]. For a closed subset \( E \) of \( V \) with the Hausdorff dimension \( \dim_H(E) < n - p \) every bounded solution \( v \in W^{1,k}_{\text{loc}}(V \setminus E; \mathbb{R}^m) \) to (1) extends to a mapping of the class \( W^{1,k}_{\text{loc}}(V; \mathbb{R}^m) \) which is defined over the whole domain \( V \) and also satisfies (1).

We follow the approach, developed in [14] (also see [15]), to prove Theorem 1.1. There are two key components in the proof. Firstly, the assumption on the size of the set \( E \) implies that \( E \) has zero \( s \)-capacity for an appropriate value of \( s \). Secondly, the Caccioppoli-type estimate holds for this particular value of \( s \). The definition of \( s \)-capacity can be find in [9, 10, 15, 22, 23].

\[ \triangleright \text{Proof of Theorem 1.1.} \] We have \( p < n - \dim_H(E) \). Let \( s \in (p, n - \dim_H(E)) \). From [9, Ch. 3, Theorem 5.11] (also see [10, 15]), we obtain that the set \( E \) has zero \( s \)-capacity.
It is clear that $|E| = 0$. Further, the higher integrability theorem [8, Theorem 2.1] gives the Caccioppoli-type estimate

$$
\|\varphi'\|_{L^r(V;\mathbb{R}^{m\times n})} \leq C \|v \otimes \varphi' + |\varphi|(\varepsilon + \varepsilon^{-k}H_+)\|_{L^s(V)}
$$

(3)

for $\varepsilon > 0$ and $\varphi \in C^\infty_0(V \setminus E)$, where the constant $C = C(F,G,K,s)$ does not depend on the test function $\varphi$ or the mapping $v$. Let $\chi \in C^\infty_0(V)$ and $E' := E \cap \text{supp} \chi$. Then $E'$ has zero $s$-capacity. Therefore there exists a sequence of functions $(\eta_j \in C^\infty_0(V))_{j \in \mathbb{N}}$ such that $0 \leq \eta_j \leq 1$, $\eta_j = 1$ on some neighbourhood of $E'$, $\lim_{j \to \infty} \eta_j = 0$ almost everywhere in $V$, and $\lim_{j \to \infty} \int_V |\eta_j'|^s = 0$. Put $\varphi_j := (1 - \eta_j)\chi \in C^\infty_0(V \setminus E)$ and $v_j := \varphi_j v \in W^{1,s}_0(V;\mathbb{R}^m)$. Then the mappings $v_j$ are bounded in $L^\infty(V;\mathbb{R}^m)$ and converge to $\chi v$ almost everywhere. We have $v_j' = \varphi_j v' + v \otimes \varphi_j'$ and $\varphi_j' = -\chi \eta_j' + (1 - \eta_j)\chi'$. Using (3), we obtain

$$
\|v_j'\|_{L^r(V;\mathbb{R}^{m\times n})} \leq \|\varphi_j v'\|_{L^r(V;\mathbb{R}^{m\times n})} + \|v \otimes \varphi_j'\|_{L^r(V;\mathbb{R}^{m\times n})}
$$

$$
\leq (1 + C) \left(\|v \otimes \varphi_j'\|_{L^r(V;\mathbb{R}^{m\times n})} + \|\varphi_j(\varepsilon + \varepsilon^{-k}H_+)|_{L^s(V)}\right)
$$

$$
\leq (1 + C) \left(\|v\|_{L^\infty(V \setminus E;\mathbb{R}^m)} \|\varphi_j'\|_{L^s(V;\mathbb{R}^n)} + \|\varphi_j(\varepsilon + \varepsilon^{-k}H_+)|_{L^s(V)}\right)
$$

$$
+ \|v\|_{L^\infty(V \setminus E;\mathbb{R}^m)} \|(1 - \eta_j)\chi'\|_{L^r(V;\mathbb{R}^n)} + \|(1 - \eta_j)\chi(\varepsilon + \varepsilon^{-k}H_+)|_{L^s(V)}\right).
$$

Passing to the limit over $j$, we get

$$
\limsup_{j \to \infty} \|v_j'\|_{L^r(V;\mathbb{R}^{m\times n})} \leq (1 + C) \left(\|v\|_{L^\infty(V \setminus E;\mathbb{R}^m)} \|\chi'\|_{L^r(V;\mathbb{R}^n)} + \|\chi(\varepsilon + \varepsilon^{-k}H_+)|_{L^s(V)}\right). 
$$

(4)

Therefore the sequence $(v_j)_{j \in \mathbb{N}}$ is bounded in $W^{1,s}(V;\mathbb{R}^m)$. Hence there exists its subsequence converged weakly in $W^{1,s}(V;\mathbb{R}^m)$ to a mapping in this Sobolev space. Clearly, this limit coincides with $\chi v$ almost everywhere in $V$.

Therefore $\chi v \in W^{1,s}_0(V;\mathbb{R}^m)$ for all test functions $\chi \in C^\infty_0(V)$. This yields $v \in W^{1,s}_0(V;\mathbb{R}^m)$. Since $v$ is a solution to (1) almost everywhere in $V$, the higher integrability theorem [8, Theorem 2.1] implies $v \in W^{1,k}_{\text{loc}}(V;\mathbb{R}^m)$. □

**Remark 1.2.** The assumption that $v$ is bounded is of course rather more than we really need in the proof of Theorem 1.1. All that is required is that the sequence $(\|v \otimes \varphi_j'\|_{L^r(V;\mathbb{R}^{m\times n}))_{j \in \mathbb{N}}$ remains bounded as $j \to \infty$. Thus, for instance, Theorem 1.1 can be extended to the case $v \in L^p(V \setminus E;\mathbb{R}^m)$, $p \geq ns/(n-s)$, if in addition we require the stronger restriction that the set $E$ has zero $r$-capacity for $r = sp/(p-s)$. That this requirement is sufficient follows from H"older’s inequality applied to $\|v \otimes \varphi_j'\|_{L^r(V;\mathbb{R}^{m\times n})}$ instead of using the trivial bound $\|v\|_{L^\infty(V \setminus E;\mathbb{R}^m)} \|\varphi_j'\|_{L^r(V;\mathbb{R}^n)}$ used above.

### 2. Integral Estimates

In the proof of the higher integrability of solutions to (1) ([8, Theorem 2.1]) we have used the following theorem on integral estimates for minors of a Jacobian matrix. This theorem is an extension of T. Iwaniec and G. Martin’s result on integral estimates for Jacobians (cf. [15, Theorems 7.8.1 and 13.7.1]).
Theorem 2.1. Let $n, m, k \in \mathbb{N}$ with $2 \leq k \leq \min(m, n)$. Then for every distribution $v = (v_1, \ldots, v_n) \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^m)$ with $v' \in L^p(\mathbb{R}^n; \mathbb{R}^{m \times n})$, $1 \leq p < \infty$, and for every $I = (i_1, \ldots, i_k) \in \Gamma_n^k$, $J = (j_1, \ldots, j_k) \in \Gamma_n^k$ we have the inequality

$$
\left| \int \frac{v'|^p - k \partial v_j}{\partial x_I} \right| \leq C \left| 1 - \frac{p}{k} \right| \int |v'|^p
$$

with some constant $C = C(k)$ depended only on $k$.

Remark 2.2. In the case $k = n = m$ Theorem 2.1 coincides with T. Iwaniec and G. Martin's result on integral estimates for Jacobians (see [15, Theorems 7.8.1 and 13.7.1]).

In the proof of Theorem 2.1 we need the following modification of T. Iwaniec and G. Martin's result on integral estimates for wedge products of closed differential forms (cf. [15, Theorem 13.6.1]).

Let $\Lambda^I = \Lambda^I(\mathbb{R}^n)$, $l \in \mathbb{N} \cup \{0\}$, be the space of all $l$-exterior forms on $\mathbb{R}^n$. For $I = (i_1, \ldots, i_l) \in \Gamma_n^l$, we denote the $l$-exterior form $dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ by $dx_I$. We use the convention that $dx_I = 1$ if $l = 0$. For $\omega \in \Lambda^I$ we have $\omega = \sum_{I \in \Gamma_n} \gamma_I dx_I$ with some coefficients $\gamma_I \in \mathbb{C}$. We put $|\omega| = \left( \sum_{I \in \Gamma_n} |\gamma_I|^2 \right)^{1/2}$. For $p \geq 1$ we denote by $L^p(\mathbb{R}^n; \Lambda^I)$ the space of differential $l$-forms on $\mathbb{R}^n$ with coefficients in $L^p(\mathbb{R}^n)$.

Theorem 2.3. Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n$. Consider $p_1, \ldots, p_k, \varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R}$ and $l_1, \ldots, l_k \in \mathbb{N}$ such that $1 < p_{\varepsilon} < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$, $-1 < \varepsilon_< \leq -\frac{p_{\varepsilon}-1}{p_{\varepsilon}}$, and $\hat{l} := n - l_1 - \cdots - l_k \geq 0$. Let $\hat{I} = (\hat{i}_1, \ldots, \hat{i}_l) \in \Gamma_n^l$. Suppose that $(\varphi_1, \ldots, \varphi_k)$ be a $k$-tuple of closed differential forms with $\varphi_\kappa \in L^{(1-\varepsilon_{\kappa})p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_\kappa})$. Then

$$
\int \frac{\varphi_1 \wedge \cdots \wedge \varphi_k \wedge dx_{\hat{I}}}{|\varphi_1|^{\varepsilon_1} \cdots |\varphi_k|^{\varepsilon_k}} \leq C|\varphi_1|^{1-\varepsilon_1}L^{(1-\varepsilon_{\kappa})p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_\kappa}) \cdots |\varphi_k|^{1-\varepsilon_k}L^{(1-\varepsilon_{\kappa})p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_\kappa}),
$$

where $\varepsilon := \max(|\varepsilon_1|, \ldots, |\varepsilon_k|)$ and the constant $C = C(p_1, \ldots, p_k)$ depends only on $p_1, \ldots, p_k$.

Remark 2.4. In the case $\hat{l} = 0$, i.e. $dx_I = 1$, Theorem 2.3 coincides with [15, Theorem 13.6.1]. For proving Theorem 2.3 we use the Hodge decomposition technique developed in [13, 14, 15] and follow the proof of Theorem 6.1 in [15].

< Proof of Theorem 2.3. Observe that $(1 - \varepsilon_{\kappa})p_{\varepsilon} > \frac{p_{\varepsilon}+1}{2} > 1$ and $|\varepsilon_{\kappa}| < 1/2$, $\kappa = 1, \ldots, k$. We have $\varphi_\kappa L^{p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_\kappa}) = L^{p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_\kappa})$. Denote by $W^{1,p}(\mathbb{R}^n; \Lambda^I)$, $0 \leq l \leq n$, $p \geq 1$, the space of differential $l$-forms on $\mathbb{R}^n$ with coefficients in $W^{1,p}(\mathbb{R}^n)$. We can consider the following Hodge decomposition in $L^{p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_{\kappa+1}})$ ([14, Theorem 6.1], also see [15, §10.6]):

$$
\frac{\varphi_\kappa}{|\varphi_\kappa|^{\varepsilon_{\kappa}}} = d\alpha_\kappa + d^*\beta_\kappa
$$

with some $\alpha_\kappa \in W^{1,p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_{\kappa+1}})$ and $\beta_\kappa \in W^{1,p_{\varepsilon}^k}(\mathbb{R}^n; \Lambda^{l_{\kappa+1}})$. Here $d$ is the exterior derivative, and $d^*$ is its formal adjoint, the coexterior derivative. The forms $d\alpha_\kappa$ and $d^*\beta_\kappa$, $\kappa = 1, \ldots, k$, are uniquely determined and can be expressed by means of the Hodge projection operators

$$
E : L^p(\mathbb{R}^n; \Lambda^I) \to dW^{1,p}(\mathbb{R}^n; \Lambda^{l_{\kappa-1}}) \quad \text{and} \quad E^* : L^p(\mathbb{R}^n; \Lambda^I) \to d^*W^{1,p}(\mathbb{R}^n; \Lambda^{l_{\kappa+1}})
$$

defined by [15, §10.6, formulas (10.71) and (10.72)] for $1 < p < \infty$ and $1 \leq l \leq n - 1$. Namely we have

$$
d\alpha_\kappa = E\left( \frac{\varphi_\kappa}{|\varphi_\kappa|^{\varepsilon_{\kappa}}} \right) \quad \text{and} \quad d^*\beta_\kappa = E^*\left( \frac{\varphi_\kappa}{|\varphi_\kappa|^{\varepsilon_{\kappa}}} \right).
$$
Applying [14, Theorem 6.1], we get the following bound for exact term:

$$\|d\alpha_x\|_{L^p(\mathbb{R}^n; \Lambda^k)} \leq C_1(p_x)\|\varphi_x\|_{L^{(1-\varepsilon_x)p_x}(\mathbb{R}^n; \Lambda^k)}^{1-\varepsilon_x}. \quad (9)$$

By [15, § 10.6, formulas (10.73) and (10.74)] we have

$$\text{Ker } E = \{ \varphi \in L^p(\mathbb{R}^n; \Lambda^1) : \text{ } d^* \varphi = 0 \}$$

and

$$\text{Ker } E^* = \{ \varphi \in L^p(\mathbb{R}^n; \Lambda^1) : \text{ } d\varphi = 0 \}$$

for $1 < p < \infty$ and $1 \leq l \leq n - 1$. Then $E^*(\varphi_{\varepsilon_x}) = 0$. Therefore we can write $d^* \beta_x$ as a commutator

$$d^* \beta_x = E^*\left(\frac{\varphi_x}{|\varphi_x|^x}\right) - \frac{E^*(\varphi_x)}{|E^*(\varphi_x)|^{x}}.$$

Applying [15, Theorem 13.2.1] (also see [13, Theorems 8.1 and 8.2]), we obtain

$$\|d^* \beta_x\|_{L^{p_x}(\mathbb{R}^n; \Lambda^k)} \leq C_2(p_x)\|\varepsilon_x\|\|\varphi_x\|_{L^{(1-\varepsilon_x)p_x}(\mathbb{R}^n; \Lambda^k)}^{1-\varepsilon_x}. \quad (10)$$

Using (7), we have

$$\int \frac{\varphi_1 \wedge \cdots \wedge \varphi_k \wedge dx_j}{|\varphi_1|^x \cdots |\varphi_k|^x} = \int (d\alpha_1 + d^* \beta_1) \wedge \cdots \wedge (d\alpha_k + d^* \beta_k) \wedge dx_j$$

$$= \int d\alpha_1 \wedge \cdots \wedge d\alpha_k \wedge dx_j + \int \mathcal{B}. \quad (11)$$

Since $p_1, \ldots, p_k$ represents a Hölder conjugate tuple, by Stokes’ formula via an approximation argument we obtain

$$\int d\alpha_1 \wedge \cdots \wedge d\alpha_k \wedge dx_j = 0. \quad (12)$$

The integrand $\mathcal{B}$ is a sum of wedge products of the type $\psi_1 \wedge \cdots \wedge \psi_k \wedge dx_j$, where $\psi_x$ is either $d\alpha_x$ or $d^* \beta_x$ and at least one $d^* \beta_x$ is always present, with at most $2^k - 1$ terms. Combining Hölder’s inequality with (9) and (10), we get

$$\int \psi_1 \wedge \cdots \wedge \psi_k \wedge dx_j \leq C_3(k)\|\psi_1\|_{L^{p_1}(\mathbb{R}^n; \Lambda^1)} \cdots \|\psi_k\|_{L^{p_k}(\mathbb{R}^n; \Lambda^k)}$$

$$\leq C_4(p_1, \ldots, p_k)\|\varphi_1\|_{L^{(1-\varepsilon_1)p_1}(\mathbb{R}^n; \Lambda^1)} \cdots \|\varphi_k\|_{L^{(1-\varepsilon_k)p_k}(\mathbb{R}^n; \Lambda^k)}^{1-\varepsilon_k}.$$  

This with (11) and (12) yields (6). ◁

**Proof of Theorem 2.1.** Let $p_{\varepsilon_x} := k, \varepsilon_{\varepsilon_x} := \varepsilon := 1 - \frac{K}{k}$, and $l_{\varepsilon_x} := 1$ for $\varepsilon_x = 1, \ldots, k$. Then $1 < p_{\varepsilon_x} < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$, $l := n - k = n - l_1 - \cdots - l_k \geq 0$, $(1 - \varepsilon_x)p_{\varepsilon_x} = p$, and $\max(|\varepsilon_1|, \ldots, |\varepsilon_x|) = |\varepsilon| = |1 - \frac{K}{k}|$. Let $\varphi_{\varepsilon_x} := dv_{j_{\varepsilon_x}} \in L^{(1-\varepsilon_x)p_{\varepsilon_x}(\mathbb{R}^n; \Lambda^k)}$. Let $\hat{l} = (\hat{i}_1, \ldots, \hat{i}_l) \in \Gamma_n$ be the ordered $l$-tuple such that $\{\hat{i}_1, \ldots, \hat{i}_l\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. We chose the sign $\text{sgn } \hat{l}$ such that $\text{sgn } \hat{l} dx_{\hat{l}} = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

When $p$ lies outside the interval $\left(\frac{k+1}{k}, \frac{3k}{2k}\right)$ the estimate is clear as (5) always holds with $1$ in place $C(k)|1 - \frac{K}{k}|$. In this case $|1 - \frac{K}{k}| \geq \frac{k+1}{2k}$ and inequality (5) holds with $C(k) = \frac{2k}{k-1}$. 

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Suppose that \( k + 1 \leq 2p \leq 3k \). Then \(-1 \leq 2\varepsilon \leq \frac{p-1}{p+1}\) and \(|\varepsilon| \leq 1/2\). Applying Theorem 2.3, we obtain

\[
\int \frac{\partial v_j}{\partial x_l} \, |dv_j|^2 \cdots |dv_k|^2 \leq \int \text{sgn} \left( Jv_{j_1} \wedge \cdots \wedge Jv_{j_k} \wedge dx_l \right) \leq C_1(k) |\varepsilon||dv_j|_{L^p(R^n;\Lambda)}^{1-\varepsilon} \cdots ||dv_k|_{L^p(R^n;\Lambda)}^{1-\varepsilon} \leq C_1(k) \int |v'|^p. \tag{13}
\]

Using the elementary inequalities \( |\frac{\partial v_j}{\partial x_l}| \leq |dv_j| \cdots |dv_k| \) and \(|a - a^{1-\varepsilon}| \leq |\varepsilon|\) for \( 0 \leq a \leq 1 \) and \(-1 < \varepsilon < 1\), we have

\[
\left| \frac{\partial v_j}{\partial x_l} \right| - \left| \frac{\partial v_j}{\partial x_l} \right| \leq \left| \frac{\partial v_j}{\partial x_l} \right| + \left| \frac{\partial v_j}{\partial x_l} \right| \leq \left| \frac{dv_j}{dv_k} \right|_{L^p}^{1-\varepsilon} \leq |\varepsilon||v'|^p.
\]

Combining this with (13), we obtain

\[
\int |v'|^{p-k} \frac{\partial v_j}{\partial x_l} \leq \int \frac{\partial v_j}{\partial x_l} \left| \frac{dv_j}{dv_k} \right|^k + \left| \frac{\partial v_j}{\partial x_l} \right| \leq (C_1(k) + 1)|\varepsilon| \int |v'|^p. \tag{14}
\]


References


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РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНОГО НЕРАВЕНСТВА
С НУЛЬ-ЛАГРАНЖИАНОМ: ПОВЫШАЮЩАЯСЯ ИНТЕГРИРУЕМОСТЬ
И УСТРАНИМОСТЬ ОСОБЕННОСТЕЙ. II

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Целью статьи является установление результата о затягивании особенностей у решений дифференциального неравенства с нуль-лагранжианом. Также получены интегральные оценки для внешних произведений замкнутых дифференциальных форм и для миноров матрицы Якоби.

Ключевые слова: нуль-лагранжян, устраняемость особенностей, интегральные оценки, внешнее произведение замкнутых дифференциальных форм, миноры матрицы Якоби.