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ON POSITIVE INVERTIBILITY AND SPLITTINGS
OF OPERATORS IN ORDERED BANACH SPACES

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*The authors dedicate the paper to Prof. A. G. Kusraev
on the occasion of his sixtieth birthday*

The positive invertibility of operators between Banach spaces, ordered by special closed cones, is characterized by the existence of splittings for the operators into the difference of two operators with appropriate spectral properties. Some results, up to now known only for matrices, are generalized to operators and to order intervals of operators.

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1. Introduction

Let $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ be ordered normed spaces, where the order is being introduced by means of the cones X_+ and Y_+ , respectively. Briefly, we will write X and Y instead of $(X, X_+, \|\cdot\|)$ and Y for $(Y, Y_+, \|\cdot\|)$, respectively. Denote by $L(X, Y)$ the vector space of all linear continuous operators $A: X \rightarrow Y$. We shall write $L(X)$ for $L(X, X)$. An operator $A \in L(X, Y)$ is said to be *positive* if $A(X_+) \subset Y_+$, and is said to be *positively invertible* (or, *inverse-positive*) if it is invertible and $A^{-1}(Y_+) \subset X_+$. We write for these properties simply $A \geq 0$ and $A^{-1} \geq 0$, respectively.

Many iterative methods for solving matrix equations $Ax = b$ in numerical analysis are based on splittings of the corresponding matrix A , see [22]. As usual, a splitting of the matrix A into the difference of two matrices, say $A = U - V$ with an invertible matrix U , is used to associate to A the iteration method

$$Ux^{k+1} = Vx^k + b, \quad k \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This procedure can be rewritten as

$$x^{k+1} = U^{-1}Vx^k + U^{-1}b, \quad k \in \mathbb{N}_0.$$

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It is well known that several questions concerning the matrix A and the iteration method, e. g. the convergence rate of iteration methods, can be answered by means of properties of the operators involved in the splitting (see [21, 22]).

Positively invertible operators are also of interest in connection with the existence of a positive solution for equations

$$Ax = y, \quad y \in Y_+,$$

where $A: X \rightarrow Y$ is an operator between the ordered normed spaces X and Y (see for e. g. [9, 11, 13, 14, 20, 27]).

The positive invertibility of matrices was characterized in [18] by what later has been called B -splittings. This result has been generalized to operators in ordered normed spaces in [25–27]. In the recent work [10] the positivity of generalized inverse matrices, such as Moore-Penrose and group inverses, is studied by means of adapted B -splittings and, conditions are given when such B -splittings exist.

The objective of the present article is to generalize certain results, up to now known only for matrices, to the cases of operators and operator intervals in ordered normed spaces.

2. Preliminaries

Let $(X, X_+, \|\cdot\|)$ be an ordered normed space, where throughout the cone X_+ is understood to be closed. We need some additional properties of cones² in ordered normed spaces, which characterize further relations or the compatibility between the order and the norm in such spaces. The cone X_+ of an ordered vector space (X, X_+) is *reproducing* (or *generating*), if any $x \in X$ has a representation $x = u - v$, where $u, v \in X_+$. The cone X_+ of the ordered normed space $(X, X_+, \|\cdot\|)$ is said to be *normal*, if there exists a constant N (the constant of normality) such that $x, y \in X$ and $0 \leq x \leq y$ imply $\|x\| \leq N \|y\|$. In this case the norm is called *semi-monotone*. A cone X_+ is said to be *regular*, if each monotone increasing sequence of elements of X_+ that is order bounded (from above), is norm-Cauchy. In a Hilbert space any closed normal cone is regular (see [17]). If a cone in an ordered normed space is regular and possesses interior points then it is normal and, the norm is order continuous (see [24, Theorem I.5.2]). Each closed regular cone in a Banach space is normal (see [15, Theorem 5.1]).

In the vector space $L(X, Y)$ of all linear continuous operators $A: X \rightarrow Y$, where X is an ordered normed space and Y stands for the ordered normed space $(Y, Y_+, \|\cdot\|)$, the subset $L_+(X, Y) = \{A \in L(X, Y): A(X_+) \subset Y_+\}$ is, in general, a closed wedge and, under the assumption that the linear set $X_+ - X_+$ is dense in X , it is a cone in $L(X, Y)$ (see [23]).

If $Y = \mathbb{R}^1$ and $Y_+ = \mathbb{R}_+^1$, then $L_+(X, Y)$ is denoted by X'_+ and is called the dual wedge or cone, respectively.

If $X = Y$ (this means also $Y_+ = X_+$ and the identity of the norms) then $L(X, X)$ is denoted by $L(X)$ and $L_+(X, Y)$ by $L_+(X)$.

There are known several conditions for the cone $L_+(X, Y)$ to be reproducing, normal and regular (see [2, 3, 5, 24]). We formulate them in the form we will use them later.

Theorem 2.1. 1. *If the cones X_+ and Y_+ are both normal, $\text{int}(Y_+) \neq \emptyset$ and (Y, Y_+) is a Dedekind complete vector lattice then the wedge $L_+(X, Y)$ is reproducing, i. e., any $A \in L(X, Y)$ can be represented as the difference of two positive operators: $A = U - V$, where $U, V \in L_+(X, Y)$.*

² In several cases the set X_+ may be supposed to be a wedge.

2. Let $\overline{X}_+ \neq X$. For the cone $L_+(X, Y)$ to be normal it is necessary and sufficient that Y_+ is normal and X_+ satisfies the condition: there exist a constant $M > 0$ such that for each $x \in X$ there are two sequences (u_n) and (v_n) such that $u_n, v_n \in X_+$ with $\|u_n\|, \|v_n\| \leq M \|x\|$ and $x = \lim(u_n - v_n)$. In particular, if X is a Banach space then the cone $L_+(X, Y)$ is normal if and only if the cone X_+ is reproducing and Y_+ is normal.

3. If $\text{int}(X_+) \neq \emptyset$ and Y_+ is normal and regular, then $L_+(X, Y)$ is normal and regular.

4. Let $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ be ordered normed spaces. For the wedge $L_+(X, Y)$ to have nonempty interior it is necessary and sufficient that the cone X_+ allows plastering and the cone Y_+ has interior points.

DEFINITION 2.2. Let be $A \in L(X, Y)$. An operator $T \in L(Y, X)$ is called the inverse of A if $AT = I_Y$ and $TA = I_X$, where I_X, I_Y are the identity operators in X and Y , respectively. T is uniquely defined and denoted by A^{-1} .

If the operator $A \in L(X, Y)$ is invertible then the continuity of A^{-1} follows either from Banach's theorem (if X and Y are Banach spaces) or, if $A^{-1} \geq 0$ is known, from continuity theorems for positive operators³ (see [1, 24]).

For an operator $A \in L(X)$ the spectrum is denoted by $\sigma(A)$ and the spectral radius is defined as $r(A) = \sup\{|\lambda|: \lambda \in \sigma(A)\}$. For the spectral radius there holds the Gelfand formula

$$r(A) = \lim_{n \rightarrow \infty} \sqrt[n]{\|A^n\|} = \inf_n \sqrt[n]{\|A^n\|}.$$

It is well known (see e. g. [12, chapter V]) that $r(A) < 1$ for an operator $A \in L(X)$ is a sufficient condition for the convergence of the series

$$I + A + A^2 + \dots + A^n + \dots \tag{1}$$

The spectral radius of an invertible operator A is positive. Indeed $1 = r(I) \leq r(A)r(A^{-1})$ implies $r(A) > 0$ and $r(A^{-1}) > 0$. If an operator A is positively invertible then $r(A^{-1}) \in \sigma(A^{-1})$ (see [15, Theorem 8.1]). For two operators A, B with $-B \leq A \leq B$ in a space X with a reproducing and normal cone one has $r(A) \leq r(B)$ (see [15, Theorem 16.5]). The last inequality, in particular, holds for positive operators with $0 \leq A \leq B$.

The next theorem provides some well known conditions on an operator C in a Banach space X to guarantee the invertibility of the operator $I - C$ in $L(X)$.

Theorem 2.3 [27, Theorem 2.1]. *Let X be a Banach space and $C: X \rightarrow X$ be a continuous linear operator on X . Consider the following properties:*

- (i) *the spectral radius of C satisfies $r(C) < 1$;*
- (ii) *C is quasinilpotent, i. e. $\lim_{n \rightarrow \infty} \|C^n\| = 0$;*
- (iii) *there exists the inverse operator $(I - C)^{-1}$, where*

$$(I - C)^{-1} = I + C + C^2 + \dots + C^n + \dots \tag{2}$$

Then there hold the implications: (i) \implies (ii) \implies (iii).

If $X = (X, X_+, \|\cdot\|)$ is an ordered Banach space and the operator C is positive, i. e. $C \in L_+(X)$, then from formula (2) it is easy to see, that (i) implies even $(I - C)^{-1} \geq 0$.

³ If X is an ordered Banach space such that each positive linear functional on X is continuous and Y is an ordered Banach space with a closed cone then any linear positive operator $A: X \rightarrow Y$ is continuous (Lozanovski's Theorem). The condition for X holds e. g. if $\text{int}(X_+) \neq \emptyset$ or if X_+ is closed and reproducing. The norm-completeness of Y can be removed if the cone Y_+ in the ordered normed space Y is assumed to be normal. Of course, in case of normed lattices the situation is much simpler: If X is a Banach lattice and Y any normed lattice then each positive operator is continuous.

The implication (iii) \Rightarrow (i) holds if the cone X_+ in the ordered Banach space has additional properties. Moreover, if the operator C is compact and satisfies $r(C) > 0$ then the number $r(C)$ turns out to be an eigenvalue of C possessing a positive eigenvector⁴. More precisely, one has the following result (see [15, Theorem 9.2 and § 25]).

Theorem 2.4. *Let $(X, X_+, \|\cdot\|)$ be an ordered Banach space and $C: X \rightarrow X$ a continuous linear operator on X such that $C \geq 0$.*

(i) *Then $r(C) < 1$ implies $(I - C)^{-1} \geq 0$ and, if the cone X_+ is normal and reproducing, also vice versa, i. e. the existence of $(I - C)^{-1}$ and $(I - C)^{-1} \geq 0$ imply $r(C) < 1$.*

(ii) *Let the cone X_+ satisfy the condition $\overline{X_+ - X_+} = X$. If C is compact and $r(C) > 0$ then $r(C)$ is an eigenvalue of C possessing a positive eigenvector. If additionally $(I - C)^{-1} \geq 0$ then $r(C) < 1$.*

\triangleleft (i): If $r(C) < 1$ and $C \geq 0$ then by Theorem 2.3 the operator $(I - C)^{-1}$ exists and $(I - C)^{-1} \geq 0$ immediately follows from (2). The converse statement is Theorem 25.1 of [15].

(ii): The compactness of the positive operator C implies the existence of a positive eigenvector to the eigenvalue $r(C)$, so there exists a nonzero vector $x_0 \in X_+$ such that $Cx_0 = r(C)x_0$. If there would be $r(C) = 1$ then $(I - C)x_0 = 0$ contradicts the invertibility of $I - C$. Consequently, $r(C) \neq 1$ and $(I - C)x_0 = (1 - r(C))x_0$ implies $\frac{1}{1 - r(C)}x_0 = (I - C)^{-1}x_0$. Since $x_0 > 0$ and $(I - C)^{-1} \geq 0$ there must hold $r(C) < 1$. \triangleright

Let again X and Y be ordered normed spaces. For two operators $B, C \in L(X, Y)$ with $B \leq C$ the (order) interval $[B, C]$ is defined as the subset of all operators “between” B and C , i. e.

$$[B, C] = \{A \in L(X, Y) : B \leq A \leq C\}.$$

An operator interval $\tau = [B, C]$ is said to be *invertible (positively invertible)*, if each operator $A \in \tau$ is invertible (positively invertible).

In the theory of positive operators the following question arises in several situations and is studied by many authors: Assume the “endpoints” of an operator interval have some property (P). Do all operators $A \in [B, C]$ share this property too? If $0 \leq A \leq C$ and the property (P) holds for C then the “domination problem for positive operators” is equivalent to the question: does A satisfy (P)? It turns out that under appropriate conditions on the spaces X and Y the answer for many properties (P) is affirmative. In this regard, the following result is pertinent and will also be used later.

Theorem 2.5 ([14, Theorem 1] and [15, Theorem 25.4]). *Let be $X = (X, X_+, \|\cdot\|)$ an ordered normed space and $Y = (Y, Y_+, \|\cdot\|)$ an ordered Banach space with a normal cone Y_+ that satisfies the condition $\text{int}(Y_+) \neq \emptyset$. Let be $B, C: X \rightarrow Y$ two linear continuous operators, where $B \leq C$ and C is positively invertible. Then B is positively invertible if and only if $B(X_+) \cap \text{int}(Y_+) \neq \emptyset$.*

3. Splittings and positive invertibility

In what follows $X = (X, X_+, \|\cdot\|)$ and $Y = (Y, Y_+, \|\cdot\|)$ will be two ordered normed spaces and $A \in L(X, Y)$ is an arbitrary operator represented as a difference of two operators, i. e., $A = U - V$, where $U, V \in L(X, Y)$. We consider the following splittings⁵ of the operator A .

⁴ Such eigenvalues are called *Perron-Frobenius-eigenvalues*.

⁵ In former papers, e. g. in [27], splittings were called decompositions.

	$U \geq 0$	$V \geq 0$	$U^{-1} \geq 0$	$U^{-1}V \geq 0$	$VU^{-1} \geq 0$	other condition
positive	×	×				
regular		×	×			
positively regular	×	×	×			
weak regular			×	×		
B -splitting	×	×			×	(B)

where condition (B) is the following implication: $\left. \begin{matrix} Ax \geq 0 \\ Ux \geq 0 \end{matrix} \right\} \implies x \geq 0$.

These splittings are generalizations (to the infinite-dimensional case) of the corresponding splittings known for square matrices, where $X = Y = \mathbb{R}^n$ (see [16, 18, 22, 27]).

As was mentioned in [27], the condition $VU^{-1} \geq 0$ is equivalent to the implication $Ux \geq 0 \implies Vx \geq 0$.

If an operator A has a positive splitting $A = U - V$ such that U^{-1} exists, then it can be represented as

$$A = U - V = (I_Y - VU^{-1})U, \quad \text{with } U, V \geq 0. \tag{3}$$

Our first main result to follow is to prove a generalization of a result of Varga ([22, Theorem 3.37]) to the operator case.

Theorem 3.1. *Let be X an ordered Banach space and Y an ordered normed space such that the cone $L_+(Y, X)$ is regular⁶. Suppose an operator $A \in L(X, Y)$ has a weak regular splitting $A = U - V$, i. e., $U^{-1} \geq 0$ and $U^{-1}V \geq 0$. Then the operator A is positively invertible if and only if $r(U^{-1}V) < 1$.*

◁ Let $A = U - V$ be the given weak regular splitting of A . Set $C = U^{-1}V: X \rightarrow X$. Then $C \geq 0$. Also

$$A = U(I_X - U^{-1}V) = U(I_X - C), \quad \text{so that } U^{-1}A = U^{-1}(U - V) = I_X - C. \tag{4}$$

If $r(C) < 1$ then by Theorem 2.4 (i) the operators $I_X - C$ and A are positively invertible, and $A^{-1} = (I_X - C)^{-1}U^{-1}$.

Conversely, if $A^{-1} \geq 0$ then the representation of $U^{-1}A$ in (4) gives $U^{-1} = (I_X - C)A^{-1}$ and $(I_X + C)U^{-1} = (I_X + C)(I_X - C)A^{-1} = (I_X - C^2)A^{-1}$. For any integer k define $B_k = (I_X + C + C^2 + \dots + C^k)U^{-1}$. Then $B_k \geq 0$, $B_k = (I_X - C^{k+1})A^{-1}$ and $(B_{k+1} - B_k)U = C^{k+1}$. By taking into consideration $C^{k+1}A^{-1} \geq 0$, it then follows that $0 \leq B_k \leq A^{-1}$. Consequently, (B_k) is an order bounded increasing sequence in the ordered Banach space $L(Y, X)$ with a regular cone $L_+(Y, X)$. So, the sequence (B_k) converges in $L(Y, X)$. Due to $\|B_{k+1}U - B_kU\| \leq \|B_{k+1} - B_k\| \|U\|$ the sequence (B_kU) converges in $L(X)$. In particular, $\|B_{k+1}U - B_kU\| = \|C^{k+1}\| \xrightarrow[k]{\rightarrow} 0$. Since $\|C^k\| < 1$ for sufficiently large k , the Gelfand formula shows that $r(C) = \inf_n \sqrt[n]{\|C^n\|} \leq \sqrt[k]{\|C^k\|} < 1$. ▷

The next two theorems are generalizations to the case $Y \neq X$ of some results of [27]. Since the proofs are similar, they are omitted. In particular, the first could be viewed as an extension of well-known characterizations of M -matrices (see [8, 22]).

Theorem 3.2 ([27, Theorem 3.4] and [25, Theorem 3 generalized]). *Let $X = (X, X_+, \|\cdot\|)$ and $Y = (Y, Y_+, \|\cdot\|)$ be ordered Banach spaces, where X_+ and Y_+ are normal cones and Y_+*

⁶ According to Theorem 2.1 this holds, e.g. if $\text{int}(Y_+) \neq \emptyset$ and X_+ is normal and regular.

satisfies the condition $\text{int}(Y_+) \neq \emptyset$. Let $A: X \rightarrow Y$ be a linear continuous operator. Consider the conditions:

- (i) A is positively invertible,
- (ii) $Y_+ \subset A(X_+)$,
- (iii) there exists some $x_0 \in X_+$ such that $Ax_0 \in \text{int}(Y_+)$.

Then we have the implications (i) \Rightarrow (ii) \Rightarrow (iii).

Consider the following condition under the assumption that A possesses a B -splitting:

- (iv) $r(VU^{-1}) < 1$.

Then \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i), i. e., in this case all conditions are equivalent.

The question of existence of a B -splitting is also settled in the affirmative in the next result.

Theorem 3.3 ([27, Theorem 3.7 generalized]). *Let $X = (X, X_+, \|\cdot\|)$ and $Y = (Y, Y_+, \|\cdot\|)$ be ordered Banach spaces, where the cone X_+ allows plastering and Y_+ satisfies the condition $\text{int}(Y_+) \neq \emptyset$. If the operator $A \in L(X, Y)$ is positively invertible then A possesses a B -splitting $A = U - V$ such that $r(VU^{-1}) < 1$.*

REMARK. 1. If the operator A is invertible then (ii) is a sufficient condition for the positivity of its inverse. Indeed, apply A^{-1} to the inclusion (ii).

2. Only the proof of the implication (iv) \Rightarrow (i) requires the condition (B) of a B -splitting.

3. If $A: X \rightarrow Y$ is an invertible operator then $A^{-1} \geq 0$ if and only if $Y_+ \subset A(X_+)$. Indeed, $A^{-1} \geq 0$ means $A^{-1}(Y_+) \subset X_+$. After applying A the required inclusion follows. If A^{-1} exists, the application of the operator A^{-1} to $Y_+ \subset A(X_+)$ yields $A^{-1}(Y_+) \subset X_+$, i. e. $A^{-1} \geq 0$.

In an ordered Hilbert space for a selfadjoint operator with a positive regular splitting the first three conditions of Theorem 3.2 are equivalent to another spectral condition (vi):

Theorem 3.4. *Let $H = (H, H_+, \|\cdot\|)$ be an ordered Hilbert spaces, where H_+ is a closed, normal cone such that $\text{int}(H_+) \neq \emptyset$. Let $A: H \rightarrow H$ be a linear continuous selfadjoint operator. Consider the conditions:*

- (i) A is positively invertible,
- (ii) $H_+ \subset A(H_+)$,
- (iii) there exists some $x_0 \in H_+$ such that $Ax_0 \in \text{int}(H_+)$.

Then we have the implications (i) \Rightarrow (ii) \Rightarrow (iii). Consider the following condition under the assumption that A possesses a positive regular splitting: $A = U - V$ with $U \geq 0$, $V \geq 0$, $U^{-1} \geq 0$

- (vi) $r(U^{-1}V) < 1$.

Then (iii) \Rightarrow (vi) \Rightarrow (i), i. e., in this case all four conditions are equivalent.

\triangleleft The proof of the implications (i) \Rightarrow (ii) \Rightarrow (iii) follows as a particular case of the corresponding parts of Theorem 3.2. We prove only the implications (iii) \Rightarrow (vi) and (vi) \Rightarrow (i). Let $A = U - V$ be a splitting of A such that $U \geq 0$, $V \geq 0$, $U^{-1} \geq 0$. Set $C = U^{-1}V$. Then $C \geq 0$ and, similar to (4), $A = U(I - C)$. Moreover, $A^* = (I - C^*)U^*$.

(iii) \Rightarrow (vi): From $C \geq 0$ it follows that $C^* \geq 0$. So, $I - C^* \leq I$. By assumption there is a vector $x_0 \in H_+$ with $Ax_0 = A^*x_0 \in \text{int}(H_+)$, i. e. $(I - C^*)U^*x_0 \in \text{int}(H_+)$. Since $U^*x_0 \in H_+$, the positive invertibility of $I - C^*$ is now guaranteed by Theorem 2.5. Using (i) of Theorem 2.4, it follows that $r(C^*) < 1$, i. e. $r(C) < 1$.

(vi) \Rightarrow (i): Since any positively regular splitting is weak regular, the proof follows from Theorem 3.1 provided the cone $L_+(H)$ is regular. The last condition is guaranteed by Theorem 2.1 (3), since the cone H_+ is closed and normal, and therefore, by [17] it is regular. \triangleright

REMARK. Due to the previous Remark 2 under the assumptions of the Theorem the condition (iv) $r(VU^{-1}) < 1$ also holds.

4. On operator intervals

For operator intervals the following theorem is well known.

Theorem 4.1 ([13, Theorem 3], [14, Theorem 4]). *Let $(X, X_+, \|\cdot\|) = X$ be ordered normed space and $(Y, Y_+, \|\cdot\|) = Y$ an ordered Banach space, with closed cones X_+ and Y_+ , respectively. Let the cone Y_+ be generating and normal. Let $B, C \in L(X, Y)$ such that $B \leq C$. Then each operator $A \in [B, C]$ is positively invertible if and only if the operators B, C are both positively invertible.*

This result was reproved for interval matrices later in [19], where the condition $B^{-1} \geq 0$ has been equivalently replaced either by a condition on the spectral radius $r(C^{-1}(C - B))$ or the condition that all matrices of the interval $[B, C]$ are invertible.

We prove now our second main result, the operator version of Rohn's result, namely

Theorem 4.2. *Let $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ be ordered normed spaces, where (Y, Y_+) is Dedekind complete. Assume the cones X_+ and Y_+ satisfy the conditions $\text{int}(X_+) \neq \emptyset$, X_+ is normal, $\text{int}(Y_+) \neq \emptyset$, Y_+ is normal and regular. Let be $\tau = [B, C]$ the operator interval defined by two given operators $B, C \in L(X, Y)$ with $B \leq C$. Then the following assertions are equivalent*

- (a) τ is positively invertible,
- (b) $C^{-1} \geq 0$ and $B^{-1} \geq 0$,
- (c) $C^{-1} \geq 0$ and $r(C^{-1}(C - B)) < 1$.

\triangleleft (a) \implies (b) is trivial.

(b) \implies (c): Put $U = C$ and $V = U - B$. Then obviously, $U^{-1}V \geq 0$. Notice that $B = U - V$ is a weak regular splitting. By Theorem 3.1 the inequality $B^{-1} \geq 0$ implies $r(U^{-1}V) = r(C^{-1}(C - B)) < 1$.

(c) \implies (a): Let A be an arbitrary operator in τ represented as $A = U - V$, where $U = C$ and $V = C - A$. Then $U^{-1} = C^{-1} \geq 0$ and $V \geq 0$ so that $U^{-1}V \geq 0$. We also have $r(U^{-1}V) = r(C^{-1}(C - A)) \leq r(C^{-1}(C - B)) < 1$. By Theorem 3.1, it now follows that A is positively invertible. \triangleright

Corollary 4.3. *Denote by (d) the statement $C^{-1} \geq 0$ and assume that τ is invertible, i. e., A^{-1} exists for each $A \in \tau$. Under the conditions of Theorem 4.2 suppose that the operator $C - B$ is compact. Then (d) \implies (c) and, so all conditions (a)–(d) are equivalent.*

\triangleleft We use the same argument as in the finite dimensional setting (see [19]). If the identity operator in X is denoted by I then $C^{-1}(C - B) = I - C^{-1}B$. Suppose the contrary, i. e. $r = r(I - C^{-1}B) \geq 1$. Since the operator $I - C^{-1}B$ is positive and compact (and obviously, $r > 0$) by (ii) of Theorem 2.4 we have that r is an eigenvalue of the operator $I - C^{-1}B$. Consequently, for some nonzero vector x_0 one has then $C^{-1}(C - B)x_0 = rx_0$, i. e. $((1 - \frac{1}{r})C + \frac{1}{r}B)x_0 = 0$. Set $\lambda = \frac{1}{r}$ and $D = \lambda B + (1 - \lambda)C$. Then $0 < \lambda \leq 1$ and hence D is a convex combination of the operators B and C . So, $D \in \tau$. But $Dx_0 = 0$ is a contradiction, as τ is invertible. \triangleright

Corollary 4.4. *Under the conditions of Theorem 4.2 suppose that one of the conditions (a)–(c) holds. Then for $A \in \tau$ we have the following representation of its inverse:*

$$A^{-1} = \left(\sum_{k=0}^{\infty} (C^{-1}(C - A))^k \right) C^{-1}.$$

It follows that $A^{-1} \geq 0$.

\triangleleft Denote the identity operator in X by I and represent A as $A = C(I - C^{-1}(C - A))$. Then, by the hypotheses and by what has been mentioned in §2 for the spectral radii of positive operators, one has $r(C^{-1}(C - A)) \leq r(C^{-1}(C - B)) < 1$. Therefore,

$$(I - C^{-1}(C - A))^{-1} \geq 0 \quad \text{and} \quad (I - C^{-1}(C - A))^{-1} = \sum_{k=0}^{\infty} (C^{-1}(C - A))^k.$$

It follows immediately that $A^{-1} = \left(\sum_{k=0}^{\infty} (C^{-1}(C - A))^k \right) C^{-1} \geq 0$. \triangleright

REMARK. 1. From (b) of the theorem also follows that $r(I_Y - BC^{-1}) < 1$. Indeed,⁸ notice that the positive operator $CB^{-1} = (C - B)B^{-1} + I_Y$ is the inverse of BC^{-1} . Define the operator A by $A = I_Y - BC^{-1}$ then $A = (C - B)C^{-1}$, what yields $A \geq 0$. Since $I_Y - A = BC^{-1}$, by (i) of Theorem 2.4 the positive invertibility of $I_Y - A$ implies $r(A) < 1$, i. e. $r(I_Y - BC^{-1}) < 1$.

2. If for some operator $B: X \rightarrow Y$ there exist a positively invertible operator C such that $B \leq C$, the interval $[B, C]$ is invertible and the operator $C - B$ is compact, then $A^{-1} \geq 0$ for any operator $A \in [B, C]$. This immediately follows from the implications (d) \implies (c) \implies (a).

3. Other conditions for a positive operator to possess non-zero positive eigenvalues are studied in [4, 6, 7]. They also might be used to prove sufficient conditions for the positive invertibility of appropriated operator intervals.

4. If $B \leq C$ and $C^{-1} \geq 0$, $B^{-1} \geq 0$ then $C^{-1} \leq B^{-1}$. Indeed, $B^{-1} \geq 0$ implies $BB^{-1} \leq CB^{-1}$, i. e. $I_Y \leq CB^{-1}$. So $C^{-1} \geq 0$ gives $C^{-1} \leq C^{-1}CB^{-1}$, i. e. $C^{-1} \leq B^{-1}$.

In general, even for matrices, the conditions $B \leq C$, $C^{-1} \geq 0$ and the existence of B^{-1} do not imply $C^{-1} \leq B^{-1}$.

Consider

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then $B \leq C$, $C^{-1} \geq 0$ and $B^{-1} \leq C^{-1}$. The result $C^{-1} \leq B^{-1}$ holds for invertible M -matrices B, C with $B \leq C$ (see [22, §3.5]).

5. The theorem remains true if Y is a Banach space and Y_+ is assumed to be a closed regular cone, since as was already mentioned, in this case the cone Y_+ is normal.

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⁸ This statement is Lemma 25.1 in [15].

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О ПОЛОЖИТЕЛЬНОЙ ОБРАТИМОСТИ И РАЗЛОЖЕНИИ ОПЕРАТОРОВ В УПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ БАНАХА

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Положительная обратимость операторов, действующих между упорядоченными пространствами Банаха, характеризуется с помощью разложений операторов в разность двух операторов с подходящими спектральными свойствами. Некоторые результаты, известные до сих пор только для матриц, обобщаются на операторный случай и на операторные интервалы.

Ключевые слова: упорядоченные пространства Банаха, конуса в упорядоченных пространствах, положительная обратимость операторов, разложение операторов, операторные интервалы.