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# INEQUALITIES FOR THE SCHWARZIAN DERIVATIVE FOR SUBCLASSES OF CONVEX FUNCTIONS IN THE UNIT DISC[1]

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Nehari norm of the Schwarzian derivative of an analytic function is closely related to its univalence. The famous Nehari–Kraus theorem ([3], [4]) and Ahlfors–Weill theorem [1] are of fundamental importance in this direction. For a non-constant meromorphic function $f$ on $D$ the unite disc, the Schwarzian derivative $S_f$ of $f$ by is holomorphic at $z_0 \in D$ if and only if $f$ is locally univalent at $z_0$. The aim of this paper is to give sharp estimates of the Nehari norm for the subclasses of convex functions in the unit disc.

## 1. Introduction

Let $\phi(z)$ be an analytic function defined in $D = \{z : |z| < 1\}$. If $\phi(z)$ satisfies the condition $|\phi(z)| \leqslant 1$ for all $z \in D$, then it is called a Schwarzian function. The class of Schwarzian functions is denoted by $\Omega^*$.

Next, let $\Omega$ be the family of functions $w(z)$ regular in $D$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in D$.

Further, for arbitrary fixed complex numbers $A$ and $B$ denote by $P(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1}$$

regular in $D$ and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2}$$

for some $w(z) \in \Omega$ and every $z \in D$. This class was introduced by W. Janowski [2].

Finally, let $C(A, B)$ denote the family of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{3}$$

regular in $D$ such that $f(z) \in C(A, B)$ if and only if

$$1 + z\frac{f''(z)}{f'(z)} = p(z) \tag{4}$$

for some $p(z) \in P(A, B)$ and all $z$ in $D$.



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## 2. Schwarzian Derivative Inequality For the Class $C(A, B)$

The following lemma which is due to Caratheodory [1] is fundamental for our present aim.

**Lemma 1.** *If $\phi \in \Omega^*$ then*

$$|\phi'(z)| \leqslant \frac{1 - |\phi(z)|^2}{1 - |z|^2} \tag{5}$$

*for some complex number $B$ such that $|B| < 1$.*

**Lemma 2.** *The class $\Omega^*$ is invariant under rotations.*

$\lhd$ It follows easily from the definition of $\Omega^*$ that the function $\varphi$ defined by $\varphi(z) := e^{-i\alpha}\phi(e^{i\alpha}z)$, $\phi(z) \in \Omega^*$, $z \in D$, $0 \leqslant \alpha \leqslant 2\pi$, satisfies

$$|\varphi(z)| = |e^{-i\alpha}\phi(e^{i\alpha}z)| = |e^{-i\alpha}||\phi(e^{i\alpha}z)| \leqslant 1. \quad \rhd$$

**Lemma 3.** *If $\phi(z)$ is an element of $\Omega^*$, then*

$$|\phi'(z)| \leqslant \frac{1 - |\phi(z)|^2}{1 - |B|^2|z|^2}.$$

$\lhd$ Using Lemma 1 and Lemma 2, and after simple calculations,we get

$$|B| < 1,\ 0 \leqslant \alpha \leqslant 2\pi \quad \Rightarrow \quad e^{i\alpha}Bz \in D \Rightarrow$$

$$|\phi'(e^{i\alpha}Bz)| \leqslant \frac{1 - |\phi(e^{i\alpha}Bz)|^2}{1 - |e^{i\alpha}Bz|^2} \quad \Rightarrow \quad |\phi'(e^{i\alpha}Bz)| \leqslant \frac{1 - |\phi(e^{i\alpha}Bz)|^2}{1 - |B|^2|z|^2}. \tag{6}$$

On the other hand, since $e^{i\alpha}Bz \in D$, the inequality (6) can be written in the form

$$|\phi'(z)| \leqslant \frac{1 - |\phi(z)|^2}{1 - |B|^2|z|^2} \quad \Leftrightarrow \quad (1 - |B|^2|z|^2)|\phi'(z)| + |\phi(z)|^2 \leqslant 1. \quad \rhd \tag{7}$$

**Theorem 1.** *If $f$ belongs to the class $C(A, B)$ then*

$$|S_f| \leqslant \begin{cases} \frac{(A-B)|z|^2}{(1-B^2|z|^2)^2} - \frac{|A+B|(A-B)|z|^2}{(1+B|z|)^2}, & B \neq 0, \\ (1 - A)A|z|^2, & B = 0. \end{cases} \tag{8}$$

$\lhd$ Since $f \in C(A, B)$, we can write

$$1 + z\frac{f''(z)}{f'(z)} = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{9}$$

From the equality (9) we have the following:

$$\begin{cases} \frac{f''}{f'} = \frac{p(z)-1}{z} \Rightarrow \left(\frac{f''}{f'}\right)' = \frac{zp'(z)-(p(z)-1)}{z^2}, \\ \left(\frac{f''}{f'}\right)' = \frac{A-B}{(1+Bw)^2}\left[\frac{zw'-w}{z^2}\right] - \frac{B(A-B)}{(1+Bw)^2}\left(\frac{w^2}{z^2}\right), \end{cases} \tag{10}$$

$$\frac{zw' - w}{z^2} = \frac{(A - B)zp' - (A - Bp)(p - 1)}{z^2(A - Bp)^2}, \tag{11}$$

$$\frac{w^2}{z^2} = \frac{(p - 1)^2}{z^2(A - Bp)^2}, \tag{12}$$

$$\frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}\frac{(p-1)^2}{z^2}. \tag{13}$$

Hence,

$$S_f = \frac{1}{A-B}\Big[(A-B)zp' - (A-Bp)(p-1)\Big] - \frac{A+B}{A-B}|p-1|^2. \tag{14}$$

On the other hand, we have

$$|p(z)-1|^2 - |z|^2|A-p(z)|^2 = (1-B^2|z|^2)\left|p(z) - \frac{1-AB|z|^2}{1-B^2|z|^2}\right| - \frac{(A-B)^2|z|^2}{(1-B^2|z|^2)}, \tag{15}$$

and

$$\begin{cases} w(z)\in\Omega,\ \phi(z)\in\Omega^* \ \Rightarrow\ w(z)=z\phi(z) \ \Rightarrow\ \phi'(z)=\frac{zw'(z)-w(z)}{z^2}, \\ (1-B^2|z|^2)|\phi'(z)| + |\phi(z)|^2 \leqslant 1 \ \Rightarrow\ (1-B^2|z|^2)\left|\frac{zw'(z)-w(z)}{z^2}\right| + \left|\frac{(w(z))^2}{z^2}\right| \leqslant 1. \end{cases} \tag{16}$$

Considering (10)–(16) together yields (8). ▷

## 3. Special Cases

For special values of $A$ and $B$, we obtain the following inequalities.

(1) From Lemma 2 and the equality (11) we have

$$\big|(A-B)zp'(z) - (p(z)-1)(A-Bp(z))\big| \leqslant \frac{(A-B)|z|^2}{(1-B^2|z|^2)^2}.$$

In this case we have the following inequalities.

(1a) $A=1$, $B=-1$:

$$\big|2zp'(z) + 1 - (p(z))^2\big| \leqslant \frac{4|z|^2}{(1-|z|^2)^2}.$$

This inequalities was proved by M. S. Robertson ([5]).

(1b) $A = 1-2\alpha$ $(0\leqslant\alpha<1)$, $B=-1$:

$$\big|2(1-\alpha)zp'(z) - (p(z)-1)((1-2\alpha)+p(z))\big| \leqslant \frac{4(1-\alpha)^2|z|^2}{(1-|z|^2)^2}.$$

(1c) $A=1$, $B=\frac{1}{M}-1$ $(M>\frac{1}{2})$:

$$\left|\left(2-\frac{1}{M}\right)zp'(z) - (p(z)-1)\left(1-\left(1-\frac{1}{M}\right)p(z)\right)\right| \leqslant \frac{\left(2-\frac{1}{M}\right)^2|z|^2}{\left(1-\left(\frac{1}{M}-1\right)|z|^2\right)^2}.$$

(1d) $A=\beta$, $B=-\beta$ $(0\leqslant\beta<1)$:

$$\big|zp'(z) + 1 - (p(z))^2\big| \leqslant \frac{4\beta|z|^2}{(1-\beta^2|z|^2)^2}.$$

(2) $A=1$, $B=-1$:

$$|S_f| \leqslant \frac{4|z|^2}{(1-|z|^2)^2}.$$

This equality was obtained by M. S. Robertson [5].
(3) $A = 1 - 2\alpha$, $B = -1$:

$$|S_f| \leqslant \frac{4(1-\alpha)|z|^2}{(1-|z|^2)^2} - \frac{4\alpha(1-\alpha)|z|^2}{(1-|z|)^2}.$$

(4) $A = 1$, $B = \frac{1}{M} - 1$ $(M > \frac{1}{2})$:

$$|S_f| \leqslant \frac{\left(2-\frac{1}{M}\right)^2|z|^2}{\left(1-\left(\frac{1}{M}-1\right)^2|z|^2\right)^2} - \frac{\frac{1}{M}\left(2-\frac{1}{M}\right)|z|^2}{\left(1+\left(\frac{1}{M}-1\right)|z|\right)^2}.$$

(5) $A = \beta$, $B = -\beta$ $(0 \leqslant \beta < 1)$:

$$|S_f| \leqslant \frac{4\beta^2}{(1-\beta^2|z|^2)^2}.$$

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