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UNSOLVED NONSTANDARD PROBLEMS

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In the fall of 1990 a small colloquium on nonstandard analysis was arranged at the request of a group of graduate and postgraduate students of Novosibirsk State University. At the meetings many unsolved problems were formulated stemming from various branches of analysis and seemingly deserving attention of the novices of nonstandard analysis. In 1994 some discussion took place on combining nonstandard methods at the international conference «Interaction Between Functional Analysis, Harmonic Analysis and Probability» (Missouri University, Columbia USA). The same topics were submitted to the international conference «Analysis and Logic» held in Belgium in 1997. In 1998 an INTAS research project was submitted. The problems raised in the framework of these projects are the core of this article. The list of the problems contains not only simple questions for drill but also topics for serious research intended mostly at the graduate and post graduate level. Some problems need creative thought to clarify and specify them.

In the fall of 1990 we arranged a small colloquium on nonstandard analysis at the request of a group of graduate and postgraduate students of Novosibirsk State University. E. I. Gordon and A. G. Kachurovski also took part in discussions. At the meetings many unsolved problems were formulated stemming from various branches of analysis and seemingly deserving attention of the novices of nonstandard analysis. The list of problems was written down, polished slightly, and presented to the readership in [46].

In 1994 some discussion took place on combining nonstandard methods at the international conference «Interaction Between Functional Analysis, Harmonic Analysis and Probability» (Missouri University, Columbia USA). The most principle problems were collected in [47]. The same topics were submitted to the international conference «Analysis and Logic» held in Belgium in 1997 (see [48]).

In 1998 an INTAS research project was submitted with the participation of A. Wickstead, E. I. Gordon M. Wolff A. G. Kusraev, and S. S. Kutateladze. Among the problems raised in the framework of this project we list Problems 17–28 which were posed by E. I. Gordon.

The problem in [42, 44–47, 49] are the core of this chapter. Please note that the list of the problems below contains not only simple questions for drill but also topics for serious research intended mostly at the graduate and post graduate level. Some problems need creative thought to clarify and specify them. In a word, this selection is rather haphazard, appearing *in statu nascendi*. Our list reflects our personal tastes the directions of our research in functional and nonstandard analysis we are engrossed in the recent two decades.

1. Nonstandard Hulls and Loeb Measures

1.1. The concept of nonstandard hull, stemming from the seminal works by W. A. J. Luxemburg, is an topical object of intensive study. Many interesting facts are now in stock on the structure of the nonstandard hulls of Banach spaces and topological vector spaces (cf. [7, 30, 66]). However, much is still unravelled in the interaction of the main constructions and concepts of Banach space theory and the various instances of nonstandard hulls. There is no detailed description for the nonstandard hulls of many function and operator spaces we deal with in functional analysis. We will give a few relevant statements. By $\widehat{X}$ we denote the nonstandard hull of a Banach space X. The prerequisites from the theory of normed spaces, vector measures, and operator ideals may be found in [10, 64].

Problem 1. *Find conditions for $\widehat{X}$ to possess the Kreĭn–Milman property.*

Look at [44] for a bunch of problems close to the Kreĭn–Milman Theorem and its abstraction to K-spaces.

Problem 2. *Find conditions for $\widehat{X}$ to possess the Radon–Nikodỳm property.*

Problem 3. *Study other geometric properties of the nonstandard hull of a Banach space such as smoothness, rotundity, the Asplund property, etc.*

Problem 4. *Describe the nonstandard hull of the projective tensor product of Banach spaces.*

Problem 5. *Describe the nonstandard hulls of various classes of bounded linear operators such as Radon–Nikodỳm operators, radonifying operators, order summing, p-summing and similar operators, etc.*

1.2. The vector space $M(\nu)$ of cosets of measurable functions on a finite measure space $(\Omega, \mathcal{B}, \nu)$ possesses the metric

$$\rho(f,g) = \int\limits_{\Omega} \tfrac{|f-g|}{1+|f-g|}\, d\nu.$$

Furnished with the metric topology, $M(\nu)$ becomes a topological vector space.

Consider the nonstandard hull $M(\nu)\widehat{}:= M(\nu)_{\rm fin}/\mu_\rho(0)$, with $\mu_\rho(0) := \{f \in M(\nu) : \rho(f,0) \approx 0\}$ and $M(\nu)_{\rm fin} := \{f \in M(\nu) : \varepsilon f \in \mu_\rho(0) \text{ for } \varepsilon \approx 0\}$. Let $(\Omega, \mathcal{B}_L, \nu_L)$ stand for the corresponding Loeb space. Then $M(\nu)\widehat{}$ and $M(\nu_L)$ are isometric spaces. Problems 8–10 were formulated by E. I. Gordon.

Problem 6. *What is the matter with the space of measurable vector-functions $M(\nu, X)$?*

Let E be an order ideal in $M(\nu)$; i.e., E is a subspace of $M(\nu)$ and, given $f \in M(\nu)$ and $g \in E$, the inequality $|f| \le |g|$ implies that $f \in E$. Denote by $E(X)$ the space of $f \in M(\nu, X)$ such that the function $v(f) : t \mapsto \|f(t)\|$ $(t \in Q)$ belongs to E, implying identification of equivalent functions. If E is a Banach lattice then $E(X)$ is a Banach space under the mixed norm $|||f||| = \|v(f)\|_E$.

Problem 7. *Describe the nonstandard hull of $E(X)$.*

1.3. Assume that (X, Σ, μ) is a finite measure space. Consider a hyperfinite set $M \subset {}^*X$, satisfying $\mu(A) = \frac{|A\cap M|}{|M|}$. Let (M, S_L, ν_L) — stand for the corresponding Loeb space.

Problem 8. *Is true that under a suitable embedding $\varphi : \Sigma/\mu \to S_L/\nu_L$ the regular subalgebra $\varphi(\Sigma/\mu)$ splits into a factor? If this is do, describe the internal sets that corresponds to the complementary factor (which present so to say ¡¡purely nonstandard¿¿ members of S_L/ν_L).*

Problem 9. *The same problem for embedding an interval with Lebesgue measure in the Loeb space.*

Problem 10. *The same problem for the spaces in Problems 70 and 71 below.*

1.4. Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \nu)$ be standard finite measure spaces. A function $\mu : \mathcal{A} \times Y \to \mathbb{R}$ is a *random measure* if

(1) the function $\mu(A, \cdot)$ is $\mathcal{B}$-measurable for all $A \in \mathcal{A}$;

(2) the function $\mu(\cdot, y)$ is a finite positive measure on $\mathcal{A}$ for ν-almost all $y \in Y$.

Problem 11. *Suggest a definition of a random Loeb measure μ_L so that it serve as a random measure on the pair $(X, \mathcal{A}_L, \lambda_L)$, $(Y, \mathcal{B}_L, \nu_L)$.*

Problem 12. *What is the connection between the integral operators*

$$\int f(x)\, d\mu(x,\cdot) \text{ and } \int f(x)\, d\mu_L(x,\cdot)?$$

What is the analog of S-integrability in this case?

A variant of solution to Problems 11 and 12 one can find in [79].

Problem 13. *Suggest a definition of a Loeb measure with values in a vector lattice (in the absence of any topology). This is to be done so that the random Loeb measure of Problem 11 correlates with the concept of Loeb measure for the vector measure* $A \mapsto \mu(A, \cdot)$.

1.5. The next three problems are invoked by the article [3], belonging to the theory of spaces of differentiable functions (cf. [8, 16, 17, 58, 59]).

Problem 14. *Suggest a nonlinear potential theory by using Lebesgue–Loeb measure.*

Problem 15. *Suggest a nonstandard capacity theory.*

Problem 16. *Define and study the spaces of differential forms by using Lebesgue–Loeb measure* (*cf.* [16, 17]).

2. Hyperfinite Approximation and Spectral Theory

2.1. The articles [19, 22] reveal an approach to approximation of topological groups by finite groups on using nonstandard analysis. The main object here is a hyperfinite approximant G to a topological group $\mathfrak{G}$, i.e., a hyperfinite subgroup $G \subset \mathfrak{G}$ «nicely-located» in $\mathfrak{G}$. It turns out that each locally compact Abelian group admits hyperfinite approximation; moreover, this approximation agrees with the Pontryagin–van Campen duality and the Fourier transform for the original group is approximated the discrete Fourier transform on an approximant. This explains interest in studying the case of noncommutative groups. We come to a new class of «approximable» locally compact groups. This class seemingly includes amenable groups; however, no precise description is known for this class of groups. (The class of approximable discrete groups was investigated in [80], where the approximability of nilpotent Lie groups was also proved.) Problems 17–28 were suggested by E. I. Gordon.

Problem 17. *Given a locally compact (not necessarily Abelian) group* G, *construct ¡¡hyperfinite approximants¿¿ to bounded linear operators in the space* $L_2(G)$ [19, 22, 24]. *Such an approximant is a linear operator in the vector space of hyperfinite dimension taken as approximant to the original space.*

2.2. E. I. Gordon propounded a theory of hyperfinite approximation for locally compact abelian groups in [19, 22, 24]. This theory allows us to construct hyperfinite approximants to pseudodifferential operators in the Hilbert space of square integrable functions on a locally compact abelian group.

This was done in [2], were the convergence of spectra and eigenfuctions for this approximations was proved for Schrodiger type operators with a positive potential increasing on infinity and the Hilbert-Schmidt operators in the case of the groups with compact open subgroups. Another approach was pursued in [63, 65, 82]. The latter is more general since it is confined to the spaces of functions on a locally compact group. However, the former leads to more refined results. Therefore, interplay between the two approaches seems promising. The intriguing problem arises of abstracting available results to other pseudodifferential operators on a locally compact group and constructing analogous approximants to operators in function spaces over other approximable groups, for instance, such as in [36, 37].

Another bunch of problems consists in studying the convergence properties of spectra and eigenvalues of hyperfinite approximants to a pseudodifferential operator on a locally compact group.

Problem 18. *Prove convergence of the spectrum and eigenvalues of a hyperfinite-rank approximant to a Schrödinger-type operator with positive potential growing at infinity. the same for a Hilbert–Schmidt operator.*

Problem 19. *The same problem as in 18 for a Schrödinger with periodic potential.*

Problem 20. *Study interconnection between hyperfinite approximation to a locally compact abelian group and its Bohr compactification.*

Problem 21. *Construct approximants to a Schrödinger-type operator with almost periodic potential making use of Problem 19 and study their convergence.*

Problem 22. *Study convergence of the spectra of approximants in a boundary value problem for the Schrödinger operator in a rectangle in finite-dimensional space.*

2.3. We now list a bunch of problems relating to approximants to operators in function spaces over an approximable noncommutative locally compact groups and convergence of these approximants.

Problem 23. *Study approximation to irreducible representations of the Heisenberg group by using representations of approximating finite groups.*

Problem 24. *Given a Hilbert function space of the Heisenberg group, find approximants to the operators in the algebra spanned over multiplications by the matrix elements of irreducible representations and shifts.*

Problem 25. *The same as in Problem 24 for other approximable nilpotent groups and suitable matrix groups over local fields.*

Problem 26. *Study the approximation problem for simple Lie groups.*

Problem 27. *Study methods for summation of divergent series over an approximable discrete group basing on approximation of this group by finite groups.*

Problem 28. *Study interplay between nonstandard summation methods of divergent series with nonstandard extensions of a densely defined operator.*

2.4. Construction of hyperfinite approximants is not always determined from hyperfinite approximation to a locally compact group. Moreover, if the domain of the operator under study is a space of functions over a domain other than a group the above method of hyperfinite approximation is not applicable in general. However, using the specifics od the domain of an operator, we may construct hyperfinite approximants. We list a few relevant problems. Observe that Problems 30 and 31 are formulated jointly with V. T. Pliev.

Problem 29. *Suggest a theory of Fredholm determinants on using appropriate hyperfinite approximation.*

Problem 30. *Prove the Lidskiĭ Theorem of coincidence of the matrix and spectral traces of a trace-class operators by hyperfinite approximation.*

Problem 31. *Use nonstandard discretization methods for studying the spectral propertied of operator pencils. In particular, find an analog of the Keldysh Theorem on completeness of the derived chains of operator pencils* (*cf.* [39]).

Problem 32. *Construct a hyperfinite analog of the Radon transform* [27] *in the spirit of* [19, 22, 24].

Problem 33. *Apply hyperfinite approximants to the Radon transform in analysis of the discrete scanning schemes in computer tomography* [60].

3. Combining Nonstandard Methods

3.1. We have mentioned elsewhere that there are various ways of combining nonstandard methods: we may proceed with infinitesimal construction inside a Boolean valued universe or we may seek for Boolean valued interpretation in the framework of some theory of internal or external sets; cf. [42]. However, serious difficulties arise and it is not always clear how to obviate them. At the same time, successive application of nonstandard methods leads often to a success. Other examples are available of this style of using nonstandard analysis; cf. [47–49].

Problem 34. *Develop a combined ¡¡scalarization-discretization¿¿ technique to unify successive application of nonstandard methods.*

Problem 35. *Suggest a Boolean valued version of the Loeb measure and the relevant integration theory. Study the respective classes of operators. In particular, construct a Loeb measure with target a Kantorovich space.*

Problem 36. *Give Boolean valued interpretations of available nonstandard hulls. Study the corresponding ¡¡descended¿¿ nonstandard hulls.*

Problem 37. *Using various nonstandard methods, derive a combined transfer principle from finite-dimensional normed algebras to relevant classes of Banach algebras.*

Problem 38. *Using a combined technique of ¡¡scalarization-discretization,¿¿ construct hyperfinite Hilbert space approximants to representations of locally compact groups.*

3.2. Substituting the laws of intuitionistic logic for the logical part of ZF (cf. [14, 25]), set theory $\mathrm{ZF_I}$. We may construct models for $\mathrm{ZF_I}$ by using a similar scheme. Namely, if Ω is a complete Heyting lattice then the universe $\mathbf{V}^{(\Omega)}$ serves as a Heyting valued model of $\mathrm{ZF_I}$, on defining the appropriate truth values $[\![\cdot \in \cdot]\!]$ and $[\![\cdot = \cdot]\!]$ from $\mathbf{V}^{(\Omega)} \times \mathbf{V}^{(\Omega)}$ to $\mathbf{V}^{(\Omega)}$. Details are in [14, 25, 75, 77]. Other approaches to modelling intuitionistic logic lead to toposes and sheaf categories; cf. [15, 18, 33].

Problem 39. *Study numeric systems inside Heyting valued models and the corresponding algebraic structures; cf.* [15, 18, 33].

Problem 40. *Study classical Banach spaces inside Heyting valued models; cf.* [5].

Problem 41. *Does interpretation of Hilbert space theory inside Heyting valued models lead to a meaningful theory of Hilbert modules?*

3.3. Consider the following claim:

Let X and Y be normed spaces. Assume given X_0 a subspace of X and T_0 a bounded linear operator from X_0 to Y. Then, to each $0 < \varepsilon \in \mathbb{R}$, there is a bounded linear extension T of T_0 to the whole of X such that $\|T\| \le (1+\varepsilon)\|T_0\|$.

The Hahn–Banach Theorem fails in constructive mathematics. However (cf. [4]), it is well known that the above claim holds for functional with located kernels, defined in a separable Banach space (i.e., in the case X separable and $Y = \mathbb{R}$). Consequently, this claim is valid inside every Heyting valued model for functionals with located kernels in separable Banach spaces. Concerning Heyting valued models see [75, 77].

The same claim holds in the classical sense, i.e., in the von Neumann universe for compact operators ranging in the space $C(Q)$ of continuous function on a compact space Q ([55]).

Problem 42. *Does the affinity of the two extension theorems for a functional and a compact operator ensue from some transfer principle for Heyting valued models?*

Problem 43. *For which objects and problems of functional analysis and operator theory is there an effective transfer principle resting on the technique of Heyting valued models? Toposes? Sheaves? (cf.* [28] *and the entire collection* [13]).

3.4. Let B be a quantum logic (cf. [50]). If we define the functions $[\![\cdot \in \cdot]\!]$ and $[\![\cdot = \cdot]\!]$ by the formulas of Section 2.1.4 in [50] and introduce the same truth values as in Section 2.1.7 in [50] then all Axioms $\mathrm{ZF}_2-\mathrm{ZF}_6$ and AC become valid inside the universe $\mathbf{V}^{(B)}$. Therefore, we may practize set theory inside $\mathbf{V}^{(B)}$. In particular, the reals inside $\mathbf{V}^{(B)}$ correspond to observables in a mathematical model of a quantum mechanical system (cf. [74]). In [74] there is shown that if B is a quantum logic ([50]) then $\mathbf{V}^{(B)}$ serves for a certain quantum set theory. Studying quantum theories as logical systems is a challenging topic as well as constructing quantum set theory and developing the corresponding quantum mathematics. However, this direction of research still leaves much to be discovered. Adequate mathematical tools and signposts reveal; themselves most likely in the theory of von Neumann algebras and various «noncommutative» branches stemming from it such as noncommutative probability theory, noncommutative integration, etc.

Problem 44. *Is there any reasonable version of the transfer principle from measure (integral) theory to noncommutative measure (integral) theory resting on the model* $\mathbf{V}^{(B)}$ *of quantum set theory?*

Problem 45. *Suggest a noncommutative theory for the Loeb measure; i.e., apply the construction of the Loeb measure to a measure on a quantum logic.*

Problem 46. *Suggest a theory of noncommutative vector (center-valued) integration on a von Neumann algebra (*AW^**-algebra) and study the relevant spaces of measurable and integrable elements by the method of Boolean valued realization.*

Problem 47. *What properties of the quantum complex numbers (i.e., the complex numbers inside* $\mathbf{V}^{(B)}$ *for a quantum logic* B*) correspond to meaningful properties of a von Neumann algebra (*AW^**-algebra)?*

3.5. Let E and F be vector lattices, with F Dedekind complete. An operator T from E to an arbitrary vector space is called *disjointly additive* if $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in E$, $x_1 \perp x_2$. We denote by $\mathcal{U}(E, F)$ the set of all disjointly additive order bounded operators from E to F. The members of $\mathcal{U}(E, F)$ are *abstract Urysohn operators* (see [57]). As demonstrated in [57], we make the space $\mathcal{U}(E, F)$ into a Dedekind complete vector lattice by furnishing it with the following order: $S \geq 0$ if and only if $S(x) \geq 0$ for all $x \in E$, with $S_1 \geq S_2$ implying that $S_1 - S_2 \geq 0$.

A disjointly additive operator in a K-space which commutes with each band projection we call an *abstract Nemytskiĭ operator*.

Problem 48. *Apply the ¡¡scalarization-discretization¿¿ method to nonlinear integral Urysohn operators as well as to their abstract analogs, i.e., bounded disjointly additive operators.*

Problem 49. *Give a Boolean valued interpretation of disjointly additive functional and study the corresponding class of nonlinear operators.*

Problem 50. *Leaning on Problem 49, describe the band generated by a positive disjointly additive operator.*

Problem 51. *Suggest a Boolean valued realization for an abstract Nemytskiĭ operator and find its function representation.*

3.6. The next problem resembles a species of convex analysis. However, it reflect the principal difficulty that stems from nonuniqueness of the standard part operation and related infinitesimal constructions inside a Boolean valued universe.

Problem 52. *Considering a standard K-space, describe the subdifferential of the operator $p(e) := \inf{}^*\{f \in E : f \geq e\}$.*

Some related result can be found in [12, 26].

4. Convex Analysis and Extremal Problems

4.1. We start with problems on extreme points

Problem 53. *Study the points infinitely close to extreme points of a subdifferential.*

Problem 54. *Find the Boolean valued status of the o-extreme points of a subdifferential* [43].

Problem 55. *Describe external equivalences that are kept invariant under the Young–Fenchel transform* (*cf.* [43]).

4.2. Assume that (Q, Σ, μ) is a measure space, X is a Banach space, and E is a Banach lattice. Let Y stand for some space of measurable vector functions $u : Q \to X$, with identification of equivalent functions. Suppose that $f : Q \times X \to E \cup \{+\infty\}$ is a convex mapping in the second variable $x \in X$ for almost all $t \in Q$, with the composite $t \mapsto f\big(t, u(t)\big)$ measurable for all $u \in Y$. We may then define some integral operator I_f over Y by the formula

$$I_f(u) = \int\limits_Q f\big(t, u(t)\big)\, d\mu(t) \qquad (u \in Y).$$

We agree that $I_f(u) = +\infty$ if the vector function $f\big(\cdot, u(\cdot)\big)$ fails to be summable. Clearly, $I_f : Y \mapsto E \cup \{+\infty\}$ is a convex operator. Convex analysis pays much attention to operators of this sort. In particular, the problems are topical of describing the subdifferential $\partial I_f(u_0)$ and the Young–Fenchel transform $(I_f)^*$ also called the conjugate of I_f. As regards the general properties of convex operators, see [43, 54]; about integral convex functionals (in the case of $E = \mathbb{R}$), see [6, 11, 54].

E. I. Gordon demonstrated in [21] that there are a real Δ and a hyperfinite set $\{t_1, \dots, t_N\} \subset Q$ such that

$$\int\limits_Q \varphi(t)\, d\mu(t) = {}^\circ \left(\Delta \sum_{k=1}^{N} \varphi(t_k) \right)$$

for each standard measurable function φ. Consequently, we may represent the integral functional I_f as follows

$$I_f(u) = {}^\circ \left(\Delta \sum_{k=1}^{N} f\big(t_k, u(t_k)\big) \right) \qquad (u \in Y).$$

Problem 56. *Study the convex integral functional I_f by means of the above representation. In particular, derive formulas for calculating the subdifferential $\partial I_f(u_0)$.*

Problem 57. *Study convex and nonconvex integrands and corresponding integral functionals by infinitesimal discretization,*

4.3. Various selection theorems are listed among powerful tools for studying functionals like I_f. We now state two available results precisely (cf. [6, 11, 54]).

Assume that Q is a topological (measurable) space, and X is a Banach space. A correspondence , $\subset Q \times X$ is called *lower semicontinuous* (*measurabl*) provided that , $^{-1}(G)$ is open (measurable) for all open $G \subset X$. A mapping $\gamma : \mathrm{dom} f \to X$ is a *selection* from , provided that $\gamma(q) \in$, (q) for all $q \in \mathrm{dom}$, .

Michael Continuous Selection Theorem. *Suppose that Q is a paracompact space, , is a lower semicontinuous correspondence, and , (q) is a nonempty closed convex set for all $q \in Q$. Then there is a continuous selection from , .*

Rokhlin–Kuratowski–Ryll-Nardzewski Theorem. *Suppose that Q is a measurable space, X is a Polish space, i.e. a complete separable metric space, and , $\subset Q \times X$ is a measurable correspondence, with , (q) closed for all $q \in Q$. Then there is a measurable selection from , .*

Problem 58. *Carry out discretization of a paracompact space and suggest a nonstandard proof of the Michael Theorem.*

Problem 59. *Find a nonstandard approach to the measurable selection problem and, in particular, suggest a nonstandard proof for the Rokhlin–Kuratowski–Ryll-Nardzewski Theorem.*

4.4. S. S. Kutateladze introduced the concept of infinitesimal optimum in the article [52]. Consider a standard vector space X, a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a convex set $C \subset X$. A member x_0 in C is a. *infinitesimal solution* to the problem $x \in C$, $f(x) \to \inf$ provided that $f(x_0) \leq f(x) + \varepsilon$ for all standard $x \in C$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

The same article [52] contains a proof of the Lagrange principle for infinitesimal optima in convex programs (also cf. [41, 43]). In this connection we come to a series of problems: Find necessary and/or sufficient infinitesimal optimality conditions for various problems of convex and nonconvex constrained optimization.

Problem 60. *Suggest a concept of infinitesimal solution to problems of optimal control and variational calculus.*

Problem 61. *Find a nonstandard extension of an abstract nonlinear extremal problem with operator constraints and study the behaviour of infinitesimal optima.*

Problem 62. *Pursue an infinitesimal approach to relaxation of nonconvex variational problems.*

Problem 63. *Suggest some subdifferential calculus for functions over Boolean algebras and study the extremal problems of optimal choice of a member of a Boolean algebra.*

5. Miscellany

In the subsection we collect a few groups of problems related to various areas of mathematics.

5.1. *Relative Standardness.* The next problem uses the concept of relatively standard set as suggested by E. I. Gordon [20], see also [61–63]. Problems 64 and 66 were formulated by E. I. Gordon.

Problem 64. *Using the Euler broken lines with an infinitesimal mesh relative to an infinitesimal ε in the van der Pol, find a direct proof of existence of ¡¡canards¿¿ — duck-shaped solutions — avoiding change-of variable (passage to the Lenard plane) (cf. [1, 83]).*

Consider another definition of relative standardness:

$$x : \mathrm{st} : y \iff \exists^{\mathrm{st}} f : \big(x = f(y)\big).$$

This definition implies that there is a natural $n : \mathrm{st} : y$ succeeding some naturals nonstandard relative to y. This leads to a model of nonstandard analysis with the «perforated» set of naturals which satisfies the transfer principle and the implication $\Rightarrow$ in the idealization principle. The following problem was formulated by E. I. Gordon and V. G. Kanovej.

Problem 65. *Suggest a reasonable axiomatics for such a nonstandard analysis.*

Assume that y is an admissible set and (X, Σ, μ) is a y-standard space with σ-additive measure μ. An element x in X is called *y-random* provided that $x \notin A$ for every y-standard set $A \in \Sigma$ satisfying $\mu(A) = 0$.

Theorem (Gordon E. I.)**.** *If (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) are standard spaces with finite measures, ξ_1 is a random element in X_1, and ξ_2 is a ξ_1-random element in X_2; then (ξ_1, ξ_2) is a random element in the product $X_1 \times X_2$.*

Problem 66. *Is the converse of this theorem true?*

Problem 67. *Study properties of ¡¡dimensioned¿¿ (¡¡inhomogeneous¿¿) real axis.*

Problem 68. *Is it possible to justify physicists' manipulating fractional dimensions?*

5.2. *Topology and Radon Measures.* Assume that X is an internal hypertfinite set and, $\mathcal{R} \subset X^2$ is an equivalence on X which is the intersection of some family of k internal sets, with k a cardinal. Assume further that the nonstandard universe is k^+-saturated. Furnish $X^\# := X/\mathcal{R}$ with the topology with $\{F^\# : F \subset X;\ F$ is internal$\}$ a base for the collection of closed sets. Then $X^\#$ is compact if and only if to each internal $A \supset \mathcal{R}$ there is a subset K of X of standard finite cardinality such that $X = A(K)$ where $A(K) := \{y \in X : (x, y) \in A$ for some $x \in K\}$. Moreover, every compact set may be presented in this manner. Problems 69–71 were suggested by E. I. Gordon.

Problem 69. *Using these terms, describe connected, simply connected, disconnected and extremally disconnected compact spaces.*

Problem 70. *Is each Radon measure on $X^\#$ induced by some Loeb measure on X? In other words, is it true that to each Radon measure μ on $X^\#$ there is a Loeb measure ν_L on X such that $A \subset X^\#$ is μ-measurable if and only if $\pi^{-1}(A)$ is ν_L-measurable, with $\mu(A) = \nu_L(\pi^{-1}(A))$ (here $\pi : X \to X^\#$ stands for the natural projection)*

It is well known that to each compact space $\mathcal{X}$ there are an internal hyperfinite set X and an internal mapping $\Phi : X \to {}^*\mathcal{X}$ satisfying

$$(\forall^{\mathrm{st}} \xi \in {}^*\mathcal{X})(\exists x \in \mathcal{X})\ \Phi(x) \approx \xi;$$

moreover, if $\mathcal{R} = \{(x, y) : \Phi(x) \approx \Phi(y)\}$ then $X/\mathcal{R}$ is homeomorphic with ${}^*\mathcal{X}$.

Problem 71. *Is it true that to each Radon measure μ on $\mathcal{X}$ there are some Φ satisfying the above conditions (or for all these Φ) and some Loeb measure ν_L on X (induced by an internal function $\nu : X \to {}^*\mathbb{R}$, the measure on the atoms) such that*

$$\int\limits_{\mathcal{X}} f\, d\mu = {}^\circ\left(\sum_{x \in X} {}^*f\big(\Phi(x)\big)\nu(x)\right)$$

for all bounded almost continuous function f?

Problem 72. *Describe other topological properties of $X^\#$ (regularity, local compactness, etc.) in terms of the properties of $\mathcal{R}$. What other types of space may be obtained in the same manner?*

Problem 73. *Study monads that serve as external preorders (i. e., quasiuniform spaces).*

5.3. *Theory of Entire Functions.* The next bunch of problems 74–77 was suggested also by E. I. Gordon.

Problem 74. *Describe the class of nonstandard polynomials whose shadows are entire functions or entire functions of finite degree σ.*

Problem 75. *Interpret the Paley–Wiener Theorem* [67] *in terms of Problem 74.*

Problem 76. *Using the solution to Problem 74, find nonstandard proofs for the Kotelnikov Theorem and other interpolation theorems for entire functions* [40, 53].

Problem 77. *Using expansion of polynomials, derive the theorems on product expansion of entire functions (similar to the Eulerian expansion of* $\sin x$*)* [38, 56].

5.4. *Ergodic Theory.* The bunch of Problems 78–83 is suggested by A. G. Kachurovskiĭ (see [46]).

Let N be unlimited natural number. A numeric sequence $\{x_n\}_{n=0}^N$ is called *microconvergent* if there is some real x^* such that $x_n \approx x^*$ for all unlimited $n \le N$. Assume that a sequence $\{x_n\}_{n=0}^\infty$ converges in the conventional sense. The following three cases determine three types of convergence:

(1) *White convergence*: the sequence $\{x_n\}_{n=0}^N$ microconverges for all unlimited N;

(2) *Color convergence*: there are two unlimited naturals N and M such that the sequence $\{x_n\}_{n=0}^N$ is microconvergence whereas the sequence $\{x_n\}_{n=0}^M$ is not;

(3) *Black convergence*: the sequence $\{x_n\}_{n=0}^N$ is not microconvergent for every unlimited N.

von Neumann Statistical Ergodic Theorem. *Let U be an isometry of a complex Hilbert space H and let H_U by the invariant subspace of U, i. e., $H_U = \{f \in H : Uf = f\}$. Denote the orthoprojection to H_U by P_U. Then*

$$\lim_{n\to\infty} \left\| \frac{1}{n+1} \sum_{k=0}^{n} U^k f - P_U f \right\|_H = 0$$

for all $f \in H$.

Corollary. *Assume that (Ω, λ) is a finite measure space, T is an automorphism of this space, and $f \in L_2(\Omega)$. Then the sequence $\left\{\frac{1}{n+1}\sum_{k=0}^{n} f(T^k x)\right\}_{n=0}^{\infty}$ converges in the norm of $L_2(\Omega)$.*

Let $\hat{L}_1(\Omega)$ stand for the external set of such elements $f \in L_1(\Omega)$ that $\|f\|_1 \ll \infty$ and $\lambda(E) \approx 0$ implies $\int_E f\,d\lambda \approx 0$ for all $E \subset \Omega$. We also put $\hat{L}_2(\Omega) = \{f \in L_2(\Omega) : f^2 \in \hat{L}_1(\Omega)\}$. The next result is established in [34, 35]; see [3, 32] for related questions:

Theorem of Bounded Fluctuation. *If f belongs to $\hat{L}_2(\Omega)$ then the sequence of averages has bounded fluctuation (and consequently, its convergence is white or color, that is, nonblack).*

Problem 78. *Find other (possibly, weakest) sufficient conditions implying of averages in the above Corollary.*

Problem 79. *Find necessary conditions implying bounded fluctuation and nonblack convergence for a sequence in the above Corollary which are as close as possible to the necessary conditions of Problem 78.*

Problem 80. *The question of Problem 78 for the von Neumann Statistical Ergodic Theorem.*

Problem 81. *The question of Problem 79 for the von Neumann Statistical Ergodic Theorem.*

Problem 82. *The question of Problem 78 for the Birkhoff–Khinchin Ergodic Theorem.*

Problem 83. *The question of Problem 79 for the Birkhoff–Khinchin Ergodic Theorem.*

5.5. We now list a few problems belonging to none of the above bunches.

Problem 84. *Find criteria for nearstandardness and prenearstandardness for the elements of concrete classical normed spaces.*

Problem 85. *Develop the theory of bornological spaces resting on the monad of a bornology* [29].

Problem 86. *Find comparison tests for finite sums with infinitely large number of terms.*

Problem 87. *Construct approximation schemata for general algebraic (Boolean valued) systems.*

Let X be a Banach space, Denote by $B(X)$ a completion of the metric space $X^\wedge$, the standard name of X inside $\mathbf{V}^{(B)}$.

Problem 88. *Find Banach spaces X and Boolean algebras B satisfying $\mathbf{V}^{(B)} \models B(X') = B(X)'$.*

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