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H-space structure on pointed mapping spaces

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Abstract We investigate the existence of an H-space structure on the function space, $\mathcal{F}_*(X, Y, *)$, of based maps in the component of the trivial map between two pointed connected CW-complexes X and Y. For that, we introduce the notion of $H(n)$-space and prove that we have an H-space structure on $\mathcal{F}_*(X, Y, *)$ if Y is an $H(n)$-space and X is of Lusternik-Schnirelmann category less than or equal to n. When we consider the rational homotopy type of nilpotent finite type CW-complexes, the existence of an $H(n)$-space structure can be easily detected on the minimal model and coincides with the differential length considered by Y. Kotani. When X is finite, using the Haefliger model for function spaces, we can prove that the rational cohomology of $\mathcal{F}_*(X, Y, *)$ is free commutative if the rational cup length of X is strictly less than the differential length of Y, generalizing a recent result of Y. Kotani.

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1 Introduction

Let X and Y be pointed connected CW-complexes. We study the occurrence of an H-space structure on the function space, $\mathcal{F}_*(X, Y, *)$, of based maps in the component of the trivial map. Of course when X is a co-H-space or Y is an H-space this mapping space is an H-space. Here, we are considering weaker conditions, both on X and Y, which guarantee the existence of an H-space structure on the function space. In Definition 3, we introduce the notion of $H(n)$-space designed for this purpose and prove:

Proposition 1 *Let Y be an $H(n)$-space and X be a space of Lusternik-Schnirelmann category less than or equal to n. Then the space $\mathcal{F}_*(X, Y, *)$ is an H-space.*

The existence of an $H(n)$-structure and the Lusternik-Schnirelmann category (LS-category in short) are hard to determine. We first study some properties of $H(n)$-spaces and give some examples. Concerning the second hypothesis, we are interested in replacing $\mathrm{cat}(X) \leq n$ by an upper bound on an approximation of the LS-category (see [5, Chapter 2]). We succeed in Proposition 7 with an hypothesis on the dimension of X but the most interesting replacement is obtained in the rational setting which constitutes the second part of this paper.

We use Sullivan minimal models for which we refer to [6]. We recall here that each finite type nilpotent CW-complex X has a unique minimal model $(\wedge V, d)$ that characterises all the rational homotopy type of X. We first prove that the existence of an $H(n)$-structure on a rational space X_0 can be easily detected from its minimal model. It corresponds to a valuation of the differential of this model, introduced by Y. Kotani in [11]: The differential d of the minimal model $(\wedge V, d)$ can be written as $d = d_1 + d_2 + \cdots$ where d_i increases the word length by i. The *differential length of* $(\wedge V, d)$, denoted $\mathrm{dl}(X)$, is the least integer n such that d_{n-1} is non zero. As a minimal model of X is defined up to isomorphism, the differential length is a rational homotopy type invariant of X, see [11, Theorem 1.1]. Proposition 8 establishes a relation between $\mathrm{dl}(X)$ and the existence of an $H(n)$-structure on the rationalisation of X.

Finally, recall that *the rational cup-length* $\mathrm{cup}_0(X)$ *of* X is the maximal length of a nonzero product in $H^{>0}(X;\mathbb{Q})$. In [11], by using this cup-length and the invariant $\mathrm{dl}(Y)$, Y. Kotani gives a necessary and sufficient condition for the rational cohomology of $\mathcal{F}_*(X,Y,*)$ to be free commutative when X is a rational formal space and when the dimension of X is less than the connectivity of Y. We show here that a large part of the Kotani criterium remains valid, without hypothesis of formality and dimension. We prove:

Theorem 2 *Let X and Y be nilpotent finite type CW-complexes, with X finite.*

(1) *The cohomology algebra $H^*(\mathcal{F}_*(X,Y,*);\mathbb{Q})$ is free commutative if* $\mathrm{cup}_0(X) < \mathrm{dl}(Y)$.

(2) *If $\dim(X) \leq \mathrm{conn}(Y)$, then the cohomology algebra $H^*(\mathcal{F}_*(X,Y,*);\mathbb{Q})$ is free commutative if, and only if,* $\mathrm{cup}_0(X) < \mathrm{dl}(Y)$.

As an application, we describe in Theorem 12 the Postnikov tower of the rationalisation of $\mathcal{F}_*(X,Y,*)$ where X is a finite nilpotent space and Y a finite type CW-complex whose connectivity is greater than the dimension of X. Our description implies the solvability of the rational Pontrjagin algebra of $\Omega(\mathcal{F}_*(X,Y,*))$.

Section 2 contains the topological setting and the proof of Proposition 1. The link with rational models is done in Section 3. Our proof of Theorem 2 uses the Haefliger model for mapping spaces. In order to be self-contained, we recall briefly Haefliger's construction in Section 4. The proof of Theorem 2 is contained in Section 5. Finally, Section 6 is devoted to the description of the Postnikov tower.

In this text, all spaces are supposed of the homotopy type of connected pointed CW-complexes and we will use cdga for *commutative differential graded algebra.* A *quasi-isomorphism* is a morphism of cdga's which induces an isomorphism in cohomology.

2 Structure of $H(n)$-space

First we recall the construction of Ganea fibrations, $p_n^X\colon G_n(X) \to X$.

- Let $F_0(X) \overset{i_0}{\to} G_0(X) \overset{p_0^X}{\to} X$ denote the path fibration on X, $\Omega X \to PX \to X$.
- Suppose a fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n^X} X$ has been constructed. We extend p_n^X to a map $q_n\colon G_n(X) \cup C(F_n(X)) \to X$, defined on the mapping cone of i_n, by setting $q_n(x) = p_n^X(x)$ for $x \in G_n(X)$ and $q_n([y,t]) = *$ for $[y,t] \in C(F_n(X))$.
- Now convert q_n into a fibration $p_{n+1}^X\colon G_{n+1}(X) \to X$.

This construction is functorial and the space $G_n(X)$ has the homotopy type of the n^{th}-classifying space of Milnor [12]. We quote also from [8] that the direct limit $G_\infty(X)$ of the maps $G_n(X) \to G_{n+1}(X)$ has the homotopy type of X. As spaces are pointed, one has two canonical applications $\iota_n^l\colon G_n(X) \to G_n(X \times X)$ and $\iota_n^r\colon G_n(X) \to G_n(X \times X)$ obtained from maps $X \to X \times X$ defined respectively by $x \mapsto (x, *)$ and $x \mapsto (*, x)$.

Definition 3 A space X is *an $H(n)$-space* if there exists a map $\mu_n\colon G_n(X \times X) \to X$ such that $\mu_n \circ \iota_n^l = \mu_n \circ \iota_n^r = p_n^X : G_n(X) \to X$.

Directly from the definition, we see that an $H(\infty)$-space is an H-space and that any space is a $H(1)$-space. Recall also that any co-H-space is of LS-category 1. Then, Proposition 1 contains the trivial cases of a co-H-space X and of an H-space Y.

Proof of Proposition 1 From the hypothesis, we have a section $\sigma\colon X \to G_n(X)$ of the Ganea fibration p_n^X and a map $\mu_n\colon G_n(Y\times Y) \to Y$ extending the Ganea fibration p_n^Y, as in Definition 3. If f and g are elements of $\mathcal{F}_*(X,Y,*)$, we set

$$f \bullet g = \mu_n \circ G_n(f\times g)\circ G_n(\Delta_X)\circ \sigma,$$

where Δ_X denotes the diagonal map of X. One checks easily that $f\bullet * \simeq * \bullet f \simeq f$. □

In the rest of this section, we are interested in the existence of $H(n)$-structures on a given space. For the detection of an $H(n)$-space structure, one may replace the Ganea fibrations p_n^X by any functorial construction of fibrations $\hat{p}_n\colon \hat{G}_n(X) \to X$ such that one has a functorial commutative diagram,

$$\begin{array}{ccc} \hat{G}_n(X) & \rightleftarrows & G_n(X) \\ & {\scriptstyle \hat{p}_n}\searrow \quad \swarrow{\scriptstyle p_n^X} & \\ & X. & \end{array}$$

Such maps $\hat{p}_n$ are called fibrations à la Ganea in [13] and substitutes to Ganea fibrations here. Moreover, as we are interested in product spaces, the following filtration of the space $G_\infty(X)\times G_\infty(Y)$ plays an important role:

$$(G(X)\times G(Y))_n = \cup_{i+j=n} G_i(X)\times G_j(Y)\,.$$

In [10], N. Iwase proved the existence of a commutative diagram

$$\begin{array}{ccc} (G(X)\times G(Y))_n & \rightleftarrows & G_n(X\times Y) \\ & {\scriptstyle \cup(p_i^X\times p_j^Y)}\searrow \quad \swarrow{\scriptstyle p_n^{X\times Y}} & \\ & X\times Y & \end{array}$$

and used it to settle a counter-example to the Ganea conjecture. Therefore, in Definition 3, we are allowed to replace the Ganea space $G_n(X\times X)$ by $(G(X)\times G(X))_n$. Moreover, if $\hat{p}_n\colon \hat{G}_n(X) \to X$ are substitutes to Ganea fibrations as above, we may also replace $G_n(X\times X)$ by

$$(\hat{G}(X)\times \hat{G}(Y))_n = \cup_{i+j=n} \hat{G}_i(X)\times \hat{G}_j(Y)\,.$$

We will use this possibility in the rational setting.

In the case $n=2$, we have a cofibration sequence,

$$\Sigma(G_1(X)\wedge G_1(X)) \xrightarrow{Wh} G_1(X)\vee G_1(X) \longrightarrow G_1(X)\times G_1(X),$$

coming from the Arkowitz generalisation of a Whitehead bracket, [2]. Therefore, the existence of an $H(2)$-structure on a space X is equivalent to the triviality of $(p_1^X \vee p_1^X) \circ Wh$. As the loop Ωp_1^X of the Ganea fibration $p_1^X \colon G_1(X) \to X$ admits a section, we get the following *necessary condition:*

– if there is an $H(2)$-structure on X, then the homotopy Lie algebra of X is abelian, i.e. all Whitehead products vanish.

Example 4 In the case X is a sphere S^n, the existence of an $H(2)$ structure on S^n implies $n = 1$, 3 or 7, [1]. Therefore, only the spheres which are already H-spaces endow a structure of $H(2)$ space. One can also observe that, in general, if a space X is both of category n and an $H(2n)$-space, then it is an H-space. The law is given by

$$X \times X \xrightarrow{\sigma} G_{2n}(X \times X) \xrightarrow{\mu_{2n}} X,$$

where the existence of the section σ to $p_{2n}^{X\times X}$ comes from $\mathrm{cat}(X \times X) \leq 2\,\mathrm{cat}(X)$.

Example 5 If we restrict to spaces whose loop space is a product of spheres or of loop spaces on a sphere, the previous necessary condition becomes a criterion. For instance, it is proved in [3] that all Whitehead products are zero in the complex projective 3-space. This implies that $\mathbb{C}P^3$ is an $H(2)$-space. (Observe that $\mathbb{C}P^3$ is not an H-space.) From [3], we know also that the homotopy Lie algebra of $\mathbb{C}P^2$ is not abelian. Therefore $\mathbb{C}P^2$ is not an $H(2)$-space.

The following example shows that we can find $H(n)$-spaces, for any $n > 1$.

Example 6 Denote by $\varphi_r \colon K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2r)$ the map corresponding to the class $x^r \in H^{2r}(K(\mathbb{Z}, 2); \mathbb{Z})$, where x is the generator of $H^2(K(\mathbb{Z}, 2); \mathbb{Z})$. Let E be the homotopy fibre of φ_r. We prove below that *E is an $H(r-1)$-space.*

First we derive, from the homotopy long exact sequence associated to the map φ_r, that ΩE has the homotopy type of $S^1 \times K(\mathbb{Z}, 2r-2)$. Therefore, the only obstruction to extend $G_{r-1}(E) \vee G_{r-1}(E) \to E$ to $(G(E) \times G(E))_{r-1} = \cup_{i+j=r-1} G_i(E) \times G_j(E)$ lies in

$$\mathrm{Hom}(H_{2r}((G(E) \times G(E))_{r-1}; \mathbb{Z}), \pi_{2r-2}(E)).$$

If A and B are CW-complexes, we denote by $A \sim_n B$ the fact that A and B have the same n-skeleton. If we look at the Ganea total spaces and fibres, we get:

$$\Sigma\Omega E \sim_{2r} S^2 \vee S^{2r-1} \vee S^{2r},\ F_1(E) = \Omega E * \Omega E \sim_{2r} S^3 \vee S^{2r} \vee S^{2r},$$

and more generally, $F_s(E) \sim_{2r} S^{2s+1}$, for any s, $2 \le s \le r-1$. Observe also that $H_{2r}(F_2(E);\mathbb{Z}) \to H_{2r}(G_1(E);\mathbb{Z})$ is onto. (As we have only spherical classes in this degree, this comes from the homotopy long exact sequence.)

As a conclusion, we have no cell in degree $2r$ in $(G(E)\times G(E))_{r-1}$ and E is an $H(r-1)$-space.

We end this section with a reduction to a more computable invariant than the LS-category. Consider $\rho_n^X \colon X \to G_{[n]}(X)$ the homotopy cofibre of the Ganea fibration p_n^X. Recall that, by definition, $\mathrm{wcat}_{\mathrm{G}}(X) \le n$ if the map ρ_n^X is homotopically trivial. Observe that we always have $\mathrm{wcat}_{\mathrm{G}}(X) \le \mathrm{cat}(X)$, see [5, Section 2.6] for more details on this invariant.

Proposition 7 *Let X be a CW-complex of dimension k and Y be a CW-complex $(c-1)$-connected with $k \le c-1$. If Y is an $H(n)$-space such that $\mathrm{wcat}_{\mathrm{G}}(X) \le n$, then $\mathcal{F}_*(X,Y,*)$ is an H-space.*

Proof Let f and g be elements of $\mathcal{F}_*(X,Y,*)$. Denote by $\tilde{\iota}_n^X \colon \tilde{F}_n(X) \to X$ the homotopy fibre of $\rho_n^X \colon X \to G_{[n]}(X)$. This construction is functorial and the map $(f,g)\colon X \to Y\times Y$ induces a map $\tilde{F}_n(f,g)\colon \tilde{F}_n(X) \to \tilde{F}_n(Y\times Y)$ such that $\tilde{\iota}_n^{Y\times Y} \circ \tilde{F}_n(f,g) = (f,g)\circ \tilde{\iota}_n^X$.

By hypothesis, we have a homotopy section $\tilde{\sigma}\colon X \to \tilde{F}_n(X)$ of $\tilde{\iota}_n^X$. Therefore, one gets a map $X \to \tilde{F}_n(Y\times Y)$ as $\tilde{F}_n(f,g)\circ\tilde{\sigma}$.

Recall now that, if $A \to B \to C$ is a cofibration with A $(a-1)$-connected and C $(c-1)$-connected, then the canonical map $A \to F$ in the homotopy fibre of $B \to C$ is an $(a+c-2)$-equivalence. We apply it in the following situation:

$$\begin{array}{ccccc} G_n(Y\times Y) & \xrightarrow{p_n^{Y\times Y}} & Y\times Y & \xrightarrow{\rho_n^{Y\times Y}} & G_{[n]}(Y\times Y) \\ \big\downarrow{\scriptstyle j_n^{Y\times Y}} & \nearrow{\scriptstyle \tilde{\iota}_n^{Y\times Y}} & & & \\ \tilde{F}_n(Y\times Y) & & & & \end{array}$$

The space $G_n(Y\times Y)$ is $(c-1)$-connected and $G_{[n]}(Y\times Y)$ is c-connected. Therefore the map $j_n^{Y\times Y}$ is $(2c-1)$-connected. From the hypothesis, we get $k \le c-1 < 2c-1$ and the map $j_n^{Y\times Y}$ induces a bijection

$$[X, G_n(Y\times Y)] \xrightarrow[\cong]{} [X, \tilde{F}_n(Y\times Y)].$$

Denote by $g_n \colon X \to G_n(Y\times Y)$ the unique lifting of $\tilde{F}_n(f,g)\circ\tilde{\sigma}$. The composition $g\bullet f$ is defined as $\mu_n \circ g_n$ where μ_n is the $H(n)$-structure on Y.

If we set $g = *$, then $\tilde{F}_n(f, g)$ is obtained as the composite of $\tilde{F}_n(f)$ with the map $\tilde{F}_n(Y) \to \tilde{F}_n(Y \times Y)$ induced by $y \mapsto (y, *)$. As before, one has an isomorphism

$$[X, G_n(Y)] \xrightarrow[\cong]{} [X, \tilde{F}_n(Y)].$$

A chase in the following diagram shows that $f \bullet * = f$ as expected,

$$\begin{array}{ccccc} & & G_n(Y) & \longrightarrow & G_n(Y \times Y) \\ & & \downarrow & & \downarrow \\ \tilde{F}_n(X) & \xrightarrow{\tilde{F}_n(f)} & \tilde{F}_n(Y) & \longrightarrow & \tilde{F}_n(Y \times Y) \\ {\scriptstyle \tilde{\sigma}}\uparrow & & \downarrow{\scriptstyle \tilde{\iota}_n^Y} & & \\ X & \xrightarrow{f} & Y. & & \end{array}$$

□

3 Rational characterisation of $H(n)$-spaces

Define $m_H(X)$ as the greatest integer n such that X admits an $H(n)$-structure and denote by X_0 the rationalisation of a nilpotent finite type CW-complex X. Recall that $\mathrm{dl}(X)$ is the valuation of the differential of the minimal model of X, already defined in the introduction.

Proposition 8 *Let X be a nilpotent finite type CW-complex of rationalisation X_0. Then we have:*

$$m_H(X_0) + 1 = \mathrm{dl}(X).$$

Proof Let $(\wedge V, d)$ be the minimal model of X. Recall from [7] that a model of the Ganea fibration p_n^X is given by the following composition,

$$(\wedge V, d) \to (\wedge V / \wedge^{>n} V, \bar{d}) \hookrightarrow (\wedge V / \wedge^{>n} V, \bar{d}) \oplus S,$$

where the first map is the natural projection and the second one the canonical injection together with $S \cdot S = S \cdot V = 0$ and $d(S) = 0$. As the first map is functorial and the second one admits a left inverse over $(\wedge V, d)$, we may use the realisation of $(\wedge V, d) \to (\wedge V / \wedge^{>n} V, d)$ as substitute for the Ganea fibration.

Suppose $\mathrm{dl}(X) = r$. We consider the cdga $(\wedge V', d') \otimes (\wedge V'', d'')/I_r$ where $(\wedge V', d')$ and $(\wedge V'', d'')$ are copies of $(\wedge V, d)$ and where I_r is the ideal $I_r = \oplus_{i+j \geq r} \wedge^i V' \otimes \wedge^j V''$. Observe that this cdga has a zero differential and that the morphism

$$\varphi : (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_r$$

defined by $\varphi(v) = v' + v''$ satisfies $\varphi(dv) = 0$. Therefore φ is a morphism of cdga's and its realisation induces an $H(n)$-structure on the rationalisation X_0. That shows: $m_H(X_0) + 1 \geq \mathrm{dl}(X)$.

Suppose now that $m_H(X_0) + 1 > \mathrm{dl}(X) = r$. By hypothesis, we have a morphism of cdga's

$$\varphi : (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_{r+1}\,.$$

By construction, in this quotient, a cocycle of wedge degree r cannot be a coboundary. Since the composition of φ with the projection on the two factors is the natural projection, we have $\varphi(v) - v' - v'' \in \wedge^+ V' \otimes \wedge^+ V''$. Now let $v \in V$, of lowest degree with $d_r(v) \neq 0$. From $d_r(v) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} v_{i_1} v_{i_2} \ldots v_{i_r}$, we get

$$\varphi(dv) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} (v'_{i_1} + v''_{i_1}) \cdot (v'_{i_2} + v''_{i_2}) \cdots (v'_{i_r} + v''_{i_r})\,.$$

This expression cannot be a coboundary and the equation $d\varphi(x) = \varphi(dx)$ is impossible. We get a contradiction, therefore one has $m_H(X_0) + 1 = \mathrm{dl}(X)$. □

4 The Haefliger model

Let X and Y be finite type nilpotent CW-complexes with X of finite dimension. Let $(\wedge V, d)$ be the minimal model of Y and (A, d_A) be a finite dimensional model for X, which means that (A, d_A) is a finite dimensional cdga equipped with a quasi-isomorphism ψ from the minimal model of X into (A, d_A). Denote by $A^\vee$ the dual vector space of A, graded by

$$(A^\vee)^{-n} = \mathrm{Hom}(A^n, \mathbb{Q})\,.$$

We set $A^+ = \oplus_{i=1}^\infty A^i$, and we fix an homogeneous basis $(a_1, \cdots, a_p)$ of A^+. The dual basis $(a^s)_{1 \leq s \leq p}$ is a basis of $B = (A^+)^\vee$ defined by $\langle a^s; a_t \rangle = \delta_{st}$.

We construct now a morphism of algebras

$$\varphi : \wedge V \to A \otimes \wedge(B \otimes V)$$

by

$$\varphi(v) = \sum_{s=1}^{p} a_s \otimes (a^s \otimes v)\,.$$

In [9] Haefliger proves that there is a unique differential D on $\wedge(B \otimes V)$ such that φ is a morphism of cdga's, i.e. $(d_A \otimes D) \circ \varphi = \varphi \circ d$.

In general, the cdga $(\wedge(B\otimes V),D)$ is not positively graded. Denote by $D_0\colon B\otimes V\to B\otimes V$ the linear part of the differential D. We define a cdga $(\wedge Z,D)$ by constructing Z as the quotient of $B\otimes V$ by $\oplus_{j\leq 0}(B\otimes V)^j$ and their image by D_0. Haefliger proves:

Theorem 9 [9] *The commutative differential graded algebra $(\wedge Z,D)$ is a model of the mapping space $\mathcal{F}_*(X,Y,*)$.*

5 Proof of Theorem 2

Proof We start with an explicit description of the Haefliger model, keeping the notation of Section 4. The cdga $(\wedge V,d)$ is a minimal model of Y and we choose for V a basis (v_k), indexed by a well-ordered set and satisfying $d(v_k)\in\wedge(v_r)_{r<k}$ for all k. As homogeneous basis $(a_s)_{1\leq s\leq p}$ of A, we choose elements h_i, e_j and b_j such that:

– the elements h_i are cocycles and their classes $[h_i]$ form a linear basis of the reduced cohomology of A;

– the elements e_j form a linear basis of a supplement of the vector space of cocycles in A, and $b_j=d_A(e_j)$.

We denote by h^i, e^j and b^j the corresponding elements of the basis of $B=(A^+)^\vee$. By developing $D_0(\sum_s a_s\otimes(a^s\otimes v))=0$, we get a direct description of the linear part D_0 of the differentiel D of the Haefliger model:

$$D_0(b^j\otimes v)=-(-1)^{|b^j|}e^j\otimes v \text{ and } D_0(h^i\otimes v)=0, \text{ for each } v\in V.$$

A linear basis of the graded vector space Z is therefore given by the elements:

$$\begin{cases} b^j\otimes v_k, & \text{with } |b^j\otimes v_k|\geq 1,\\ e^j\otimes v_k, & \text{with } |e^j\otimes v_k|\geq 2,\\ h^i\otimes v_k, & \text{with } |h^i\otimes v_k|\geq 1.\end{cases}$$

Now, from $\varphi(dv)=(D-D_0)\varphi(v)$ and $d(v)=\sum c_{i_1i_2\cdots i_r}v_{i_1}v_{i_2}\cdots v_{i_r}$, we deduce:

$$(D-D_0)(a^s\otimes v)=$$
$$\pm\sum c_{i_1i_2\cdots i_r}\sum_{a_{i_1},a_{i_2}\cdots,a_{i_r}}\langle a^s;a_{i_1}a_{i_2}\cdots a_{i_r}\rangle\,(a_{i_1}\otimes v_{i_1})\cdot(a_{i_2}\otimes v_{i_2})\cdots(a_{i_r}\otimes v_{i_r})$$

where, as usual, the sign $\pm$ is entirely determined by a strict application of the Koszul rule for a permutation of graded objects.

Suppose first that $\mathrm{cup}_0(X)<\mathrm{dl}(Y)$.

We prove, by induction on k, that in $(\wedge Z, D)$ the ideal I_k generated by the elements

$$\begin{cases} b^j \otimes v_s\,, & s \leq k, \quad \text{with degree at least } 1, \\ e^j \otimes v_s\,, & s \leq k, \quad \text{with degree at least } 2, \end{cases}$$

is a differential ideal and that the elements $h^i \otimes v_s$, with $s \leq k$ and $|h^i \otimes v_s| \geq 1$, are cocycles in the quotient $((\wedge Z)/I_k, \bar{D})$. Note that this ideal is acyclic as shown by the formula given for D_0. Therefore the quotient map $\rho\colon (\wedge Z, D) \to ((\wedge Z)/I_k, D)$ is a quasi-isomorphism. The induction will prove that the differential is zero in the quotient, which is the first assertion of Theorem 2.

Begin with the induction. One has $dv_1 = 0$ which implies $(D - D_0)(a^s \otimes v_1) = 0$. Therefore, we deduce $D(b^j \otimes v_1) = -(-1)^{|b^j|} e^j \otimes v_1$ and $D(h^i \otimes v_1) = 0$. That proves the assertion for $k = 1$.

We suppose now that the induction step is true for the integer k. Taking the quotient by the ideal I_k gives a quasi-isomorphism

$$\rho : (\wedge Z, D) \to (\wedge T, D) := ((\wedge Z)/I_k, D)\,.$$

As the elements $b^j \otimes v_s$ and $e^j \otimes v_s$, $s \leq k$, have disappeared and as $\mathrm{cup}_0(X) < \mathrm{dl}(Y)$, we have $\rho \circ \varphi(dv_{k+1}) = 0$. Therefore $D(b^j \otimes v_{k+1}) = -(-1)^{|b^j|} e^j \otimes v_{k+1}$ and $D(h^i \otimes v_{k+1}) = 0$. The induction is thus proved.

We consider now the case $\mathrm{cup}_0(X) \geq \mathrm{dl}(Y)$ *in the case* $\dim(X) \leq \mathrm{conn}(Y)$.

We choose first in the lowest possible degree q an element $y \in V$ that satisfies $dy = d_r y + \cdots$ with $d_r(y) \neq 0$ and $r \leq \mathrm{cup}_0(X)$. As above we can kill all the elements $e^j \otimes v$ and $b^j \otimes v$ with $|v| < q$ and keep a quasi-isomorphism $\rho\colon (\wedge Z, D) \to (\wedge T, D) := (\wedge Z/I_{q-1}, D)$.

Next we choose cocycles, $h_1, h_2, \cdots, h_m$, such that the class $[\omega]$, associated to the product $\omega = h_1 \cdot h_2 \cdots h_m$, is not zero. We choose $m \geq r$ and suppose that ω is in the highest degree for such a product. Let ω' be an element in $A^\vee$ such that $\langle \omega'; \omega \rangle = 1$. Then, by the Haefliger formula, the differential D is zero in $\wedge T$ in degrees strictly less than $|\omega' \otimes y|$. Observe that $|\omega' \otimes y| \geq 2$ and that the D_r part of the differential $D(\omega' \otimes y)$ is a nonzero sum. This proves that the cohomology is not free. □

Example 10 Let X be a space with $\mathrm{cup}_0(X) = 1$, which means that all products are zero in the reduced rational cohomology of X. Then, for any nilpotent finite type CW-complex Y, the rational cohomology $H^*(\mathcal{F}_*(X, Y, *); \mathbb{Q})$ is a free commutative graded algebra. For instance, this is the case for the (non-formal) space $X = S^3 \vee S^3 \cup_\omega e^8$, where the cell e^8 is attached along a sum of triple Whitehead products.

Example 11 When the dimension of X is greater than the connectivity of Y, the degrees of the elements have some importance. The cohomology can be commutative free even if $\mathrm{cup}_0(X) \geq \mathrm{dl}(Y)$. For instance, consider $X = S^5 \times S^{11}$ and $Y = S^8$. One has $\mathrm{cup}_0(X) = \mathrm{dl}(Y) = 2$ and the function space $\mathcal{F}_*(X,Y,*)$ is a rational H-space with the rational homotopy type of $K(\mathbb{Q},3)\times K(\mathbb{Q},4)\times K(\mathbb{Q},10)$, as a direct computation with the Haefliger model shows.

6 Rationalisation of $\mathcal{F}_*(X,Y,*)$ when $\dim(X) \leq \mathrm{conn}(Y)$

Let X be a finite nilpotent space with rational LS-category equal to $m-1$ and let Y be a finite type nilpotent CW-complex whose connectivity c is greater than the dimension of X. We set $r = \mathrm{dl}(Y)$ and denote by s the maximal integer such that $m/r^s \geq 1$, i.e. s is the integral part of $\log_r m$.

Theorem 12 *There is a sequence of rational fibrations $K_k \to F_k \to F_{k-1}$, for $k = 1, \ldots, s$, with $F_0 = *$, F_s is the rationalisation of $\mathcal{F}_*(X,Y,*)$ and each space K_k is a product of Eilenberg-MacLane spaces. In particular, the rational loop space homology of $\mathcal{F}_*(X,Y,*)$ is solvable with solvable index less than or equal to s.*

Proof By a result of Cornea [4], the space X admits a finite dimensional model A such that m is the maximal length of a nonzero product of elements of positive degree. We denote by $(\wedge V, d)$ the minimal model of Y.

We consider the ideals $I_k = A^{>m/r^k}$, and the short exact sequences of cdga's

$$I_k/I_{k-1} \to A/I_{k-1} \to A/I_k\,.$$

These short exact sequences realise into cofibrations $T_k \to T_{k-1} \to Z_k$ and the sequences

$$(\wedge((A^+/I_k)^\vee \otimes V), D) \to (\wedge((A^+/I_{k-1})^\vee \otimes V), D) \to (\wedge((I_k/I_{k-1})^\vee \otimes V), D)$$

are relative Sullivan models for the fibrations

$$\mathcal{F}_*(Z_k,Y,*) \to \mathcal{F}_*(T_{k-1},Y,*) \to \mathcal{F}_*(T_k,Y,*).$$

Now since the cup length of the space Z_k is strictly less than r, the function spaces $\mathcal{F}_*(Z_k,Y,*)$ are rational H-spaces, and this proves Theorem 12. ◻

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