Hyers–Ulam stability of nth order linear differential equations

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Abstract: For nth order linear homogenous and nonhomogenous differential equations with nonconstant coefficients, we prove Hyers–Ulam stability by using open mapping theorem. The generalized Hyers–Ulam stability is also investigated.
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1 Introduction

The theory of stability is an important branch of the qualitative theory of differential equations. In 1940, Ulam [24] raised a problem when can we assert that the solutions of an inequality are close to one of the exact solutions of the corresponding equation. Since then, this question has attracted the attention of many researchers. We refer the reader to [1–23, 25–28] and the references cited therein. Note that the first solution to this question was given by Hyers [10] in 1941. Thereafter, Aoki [2], Bourgin [3], and Rassias [21] improved the result reported in [10].

To the best of our knowledge, papers by Obłoza [17, 18] were among the first contributions dealing with Hyers–Ulam stability of differential equations. Alsina and Ger [1] proved Hyers–Ulam stability of differential equation $y'(x) = y(x)$. Later, Takahasi et al. [23] extended results of [1] to the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (Hyers–Ulam stability holds true for this equation). Using direct method, iteration method, fixed point method, and open mapping theorem, Huang and Li [8] investigated the Hyers–Ulam stability of some classes of functional differential equations. In the paper by Zada et al. [26], the concepts of Hyers–Ulam stability are generalized for a class of nonautonomous linear differential systems. Very recently, Choi and Jung [6] studied the generalized Hyers–Ulam stability of the linear differential equation

$$y''(x) + f(x)y'(x) + g(x)y(x) = r(x).$$

It should be noted that research in this paper was strongly motivated by the recent contributions of Choi and Jung [6] and Huang and Li [8]. Our principal goal is to prove Hyers–Ulam stability of

$$\sum_{i=0}^{n} \beta_{n-i}(x)y^{(n-i)}(x) = \alpha(x) \tag{1.1}$$

and the generalized Hyers–Ulam stability of

$$y^{(n)}(x) + \sum_{i=1}^{n} Q_{n-i}(x)y^{(n-i)}(x) = f(x) \tag{1.2}$$

by using different approaches.

2 Preliminaries

We present the following definitions to make this paper self-dependent.

Definition 2.1 (see [12, 13]) *Let $\mathcal{W}$ be a normed space and I be an open interval. Then, equation* (1.1) *is said to have Hyers–Ulam stability on I if, for any function $g : I \to \mathcal{W}$ satisfying*

$$\left\| \sum_{i=0}^{n} \beta_{n-i}(x)g^{(n-i)}(x) - \alpha(x) \right\| \leq \varepsilon$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a function $g_0 : I \to \mathcal{W}$ such that

$$\sum_{i=0}^{n} \beta_{n-i}(x) g_0^{(n-i)}(x) = \alpha(x)$$

and $\|g - g_0\| \leq K(\varepsilon)$ for all $x \in I$, where $K(\varepsilon)$ is an expression for ε only.

If the above statement is also true when we replace ε and $K(\varepsilon)$ with $\phi(x)$ and $\Phi(x)$, where $\phi, \Phi : I \to [0, \infty)$ are functions not depending on g and g_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability (or the Hyers–Ulam–Rassias stability).

Definition 2.2 (see [8]) *Let $\Upsilon : A \to B$ be an operator from a space A to another space B. We say that Υ has Hyers–Ulam stability if, for any $g \in \Upsilon(A)$, $\varepsilon \geq 0$, and $f \in A$ satisfying $\|\Upsilon f - g\| \leq \varepsilon$, there exist a constant K and an $f_0 \in A$ such that $\Upsilon f_0 = g$ and $\|f - f_0\| \leq K\varepsilon$, where K is called a Hyers–Ulam stability constant of operator Υ.*

In order to establish our main results, we need the following auxiliary lemmas.

Lemma 2.3 (see [8, Theorem 2.4]) *Let A and B be Banach spaces and $\Upsilon : A \to B$ be a bounded linear operator. Then the following statements are equivalent:*

(*a*) *Υ has Hyers–Ulam stability;*

(*b*) *The range of Υ is closed.*

Lemma 2.4 (see [28]) *Assume that $c_1, c_2, \ldots, c_n$ are arbitrary real-valued constants. If $y_c = \sum_{i=1}^{n} c_i y_i$ is the complementary solution of equation (1.2) on a common interval I, where $Q_j, f \in C(I, \mathbb{R})$, $j = 0, 1, \ldots, n-1$ and $f(x) \neq 0$ for all $x \in I$, then its general solution $y \in C^n(I, \mathbb{R})$ is given by*

$$y(x) = y_c(x) + \sum_{i=1}^{n} y_i(x) \int_{x_i}^{x} \frac{P_i(t)}{W(t)} f(t) dt,$$

where $x, x_1, x_2, \ldots, x_n \in I$, W is the Wronskian of $y_1, y_2, \ldots, y_n$, $P_i = W_i/f$, and W_i, $i = 1, 2, \ldots, n$ are the Wronskian with ith column replaced with the *vector* $(0,0,0,\ldots,f(x))$, *i.e.,*

$$W_i = \begin{vmatrix} y_1 & y_2 & y_3 & \ldots & 0 & \ldots & y_n \\ y_1' & y_2' & y_3' & \ldots & 0 & \ldots & y_n' \\ y_1'' & y_2'' & y_3'' & \ldots & 0 & \ldots & y_n'' \\ . & . & . & \ldots & 0 & \ldots & . \\ . & . & . & \ldots & 0 & \ldots & . \\ . & . & . & \ldots & 0 & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \ldots & f & \ldots & y_n^{(n-1)} \end{vmatrix}.$$

Remark 2.5 *P_i, $i = 1,2,\ldots,n$ in Lemma 2.4 can be written in the form*

$$W_i = \begin{vmatrix} y_1 & y_2 & y_3 & \ldots & 0 & \ldots & y_n \\ y_1' & y_2' & y_3' & \ldots & 0 & \ldots & y_n' \\ y_1'' & y_2'' & y_3'' & \ldots & 0 & \ldots & y_n'' \\ . & . & . & \ldots & 0 & \ldots & . \\ . & . & . & \ldots & 0 & \ldots & . \\ . & . & . & \ldots & 0 & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \ldots & 1 & \ldots & y_n^{(n-1)} \end{vmatrix}.$$

3 Hyers–Ulam Stability of equation (1.1)

In what follows, $C[a,b]$ denotes the set of all continuous functions on $[a,b]$ and $C^n[a,b]$ stands for the set containing all n times continuously differentiable functions on $[a,b]$.

Theorem 3.1 *If $\beta_i \in C[a,b]$ for $i = 0,1,\ldots,n$, then the homogeneous nth order linear differential equation*

$$\sum_{i=0}^{n} \beta_{n-i}(x) y^{(n-i)}(x) = 0 \tag{3.1}$$

has Hyers–Ulam stability on $[a,b]$.

Proof. For every $y \in C^n[a,b]$, define a norm in $C^n[a,b]$ by

$$||y|| := \max\left(\max_{1\le i\le n} \sup_{x\in[a,b]} \left|y^{(i)}(x)\right|, \sup_{x\in[a,b]} |y(x)| \right).$$

It is not difficult to verify that $C[a,b]$ and $(C^n[a,b], \|\cdot\|)$ are Banach spaces (see Huang et al. [9] for details). For all $g \in C^n[a,b]$, define now an operator $\Upsilon : C^n[a,b] \to C[a,b]$ by

$$(\Upsilon g)(x) := \sum_{i=0}^{n} \beta_{n-i}(x) g^{(n-i)}(x).$$

Clearly, Υ is well-defined and is a linear operator. By virtue of

$$\begin{aligned}
||\Upsilon|| &= \sup_{\|g\|=1} \|\Upsilon g\| \\
&= \sup_{\|g\|=1} \sup_{x\in[a,b]} \left| \sum_{i=0}^{n} \beta_{n-i}(x) g^{(n-i)}(x) \right| \\
&\leq \sup_{\|g\|=1} \sup_{x\in[a,b]} \sum_{i=0}^{n} \left| \beta_{n-i}(x) g^{(n-i)}(x) \right| \\
&= \sup_{\|g\|=1} \sup_{x\in[a,b]} \sum_{i=0}^{n} |\beta_{n-i}(x)| \left| g^{(n-i)}(x) \right| \\
&\leq \sup_{x\in[a,b]} \sum_{i=0}^{n} |\beta_{n-i}(x)|
\end{aligned}$$

and the facts that β_i are continuous on $[a,b]$ for $i = 0, 1, \ldots, n$, we deduce that Υ is bounded. Next, we prove that the range of Υ is closed. Obviously, $\Upsilon g \in C[a,b]$ for every $g \in C^n[a,b]$. Conversely, using the basic theory of differential equations (see [27]), we conclude that, for every $y \in C[a,b]$, there are some $f \in C^n[a,b]$ such that $\Upsilon f = y$. Moreover, $C[a,b]$ is a Banach space, and so the range of Υ is closed. It follows now from Lemma 2.3 that Υ has Hyers–Ulam stability. Noticing that $0 \in \Upsilon(C^n[a,b])$ and using Definition 2.2, we obtain that, for any $\varepsilon \geq 0$ and $f \in C^n[a,b]$ satisfying

$$\left\| \sum_{i=0}^{n} \beta_{n-i}(x) f^{(n-i)}(x) \right\| \leq \varepsilon$$

for all $x \in [a,b]$, there exist a constant K and a solution $f_0 \in C^n[a,b]$ to (3.1) with the property

$$\|f - f_0\| \leq K\varepsilon.$$

By virtue of Definition 2.1, equation (3.1) has Hyers–Ulam stability on $[a,b]$. The proof is complete. □

Similar as in the proof of Theorem 3.1, one can obtain the following result.

Corollary 3.2 *If $\beta_i, \alpha \in C[a,b]$ for $i = 0, 1, \ldots, n$, then the nth order nonhomogeneous linear differential equation* (1.1) *has Hyers–Ulam stability on* $[a,b]$.

4 The generalized Hyers–Ulam stability of equation (1.2)

In this section, we prove the generalized Hyers–Ulam stability of an nth order nonhomogenous linear differential equation (1.2) on a nondegenerate interval I.

Theorem 4.1 *Assume that $Q_j, f \in C(I, \mathbb{R})$, $j = 0, 1, \ldots, n-1$ and $f(x) \neq 0$ for all $x \in I$, and let $y_c = \sum_{i=1}^{n} c_i y_i$ be the complementary solution of equation* (1.2) *on I, where c_i are arbitrary real-valued constants. If a function $y \in C^n(I, \mathbb{R})$ satisfies the inequality*

$$\left| y^{(n)}(x) + \sum_{i=1}^{n} Q_{n-i}(x) y^{(n-i)}(x) - f(x) \right| \leq \phi(x) \tag{4.1}$$

for all $x \in I$, where $\phi : I \to [0, \infty)$ is given such that the following integrals exist, then there is a solution $y_0 \in C^n(I, \mathbb{R})$ of (1.2) *satisfying*

$$|y(x) - y_0(x)| \leq \sum_{i=1}^{n} |y_i(x)| \int_{x_i}^{x} \left| \frac{P_i(t)}{W(t)} \right| \phi(t) dt,$$

where $x, x_1, x_2, \ldots, x_n \in I$, W, W_i, and P_i are defined as in Lemma 2.4.

Proof. Let $y \in C^n(I, \mathbb{R})$ be such that (4.1) holds. Define a function $r \in C(I, \mathbb{R})$ by

$$r(x) := y^{(n)}(x) + \sum_{i=1}^{n} Q_{n-i}(x) y^{(n-i)}(x). \tag{4.2}$$

It follows from (4.1) that, for all $x \in I$,

$$|r(x) - f(x)| \leq \phi(x). \tag{4.3}$$

By virtue of Lemma 2.4, the general solution of (4.2) is given by

$$y(x) = y_c(x) + \sum_{i=1}^{n} y_i(x) \int_{x_i}^{x} \frac{P_i(t)}{W(t)} r(t) dt,$$

where $W \neq 0$ for all $t \in I$ due to the fact that $y_1, y_2, \ldots, y_n$ are linearly independent on I. For every $x \in I$, define a new function $y_0 : I \to R$ by

$$y_0(x) := y_c(x) + \sum_{i=1}^{n} y_i(x) \int_{x_i}^{x} \frac{P_i(t)}{W(t)} f(t)dt.$$

It follows now from Lemma 2.4 that y_0 is the solution of (1.2). An application of (4.3) implies that

$$\begin{aligned} |y(x) - y_0(x)| &= \left| \sum_{i=1}^{n} y_i(x) \int_{x_i}^{x} \frac{P_i(t)}{W(t)} r(t)dt - \sum_{i=1}^{n} y_i(x) \int_{x_i}^{x} \frac{P_i(t)}{W(t)} f(t)dt \right| \\ &\leq \sum_{i=1}^{n} |y_i(x)| \int_{x_i}^{x} \left| \frac{P_i(t)}{W(t)} \right| \phi(t)dt, \end{aligned}$$

which completes the proof. □

Remark 4.2 *Theorem 4.1 reduces to [6, Theorem 3.2] in the case where $n = 2$.*

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