A model of the groundwater flowing within a leaky aquifer using the concept of local variable order derivative

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Abstract
One of the big problems we encounter in groundwater modeling is to provide a correct model that can be used to describe the movement of water via a particular geological formation. In this work, in order to further enhance the model of groundwater flow in a leaky aquifer, we made use of a new derivative called the local variable order derivative. The derivative includes into mathematical formula the complexity of the leaky aquifer, which is for instance the variation of the aquifer, or the heterogeneity of the leaky aquifer. The modified equation was solved using the concept of iterative method. We presented in detail the stability and the uniqueness of the special solution.

Keywords: Leaky aquifer; variable order derivative; stability and uniqueness analysis, special solution.

1. Introduction
For the first time, Pythagoras realized that mathematics tools can be used to describe the pattern of real physical problems. Later on Sir Diophantus of Alexandria realized that, these natural patterns can be described via mathematical equations [1]. The notion has been used intensively in the circle of mathematics; however the idea of motion was not already introduced by that time. In the year 18th, Sir Newton and Leibniz independently introduced the concept of motion leading to the concept of derivative [2-4]. Since then, this concept has been used in almost all the branches of sciences to model real world problems. It is perhaps important to note that, the big challenge in this process is to include into mathematical formula all the detail surrounding the physical problem under observation. It happens to appear that, the Newtonian concept of derivative cannot satisfy all the complexity of the natural occurrences. For instance, how, do we explain accurately the movement of water within the leaky aquifer? An attempt to answer this question, Hantush has proposed and equation based on the model proposed by Theis in [5-8].
Although this model has being used by many hydro-geologists, it is worth noting that, the model does not take into account all the details surrounding the movement of water through a leaky geological formation. A first attempt to enhance model, was to introduce the concept of derivative with fractional order [9]. This model has improved the description of this physical problem at a certain extend. Nonetheless, to be accurate, when dealing complexes systems, even the concept of fractional order derivative has some limitations, for instance it is not possible to accurately model the trap of water under matrices rocks. We shall mention that, a mathematical model will be considered accurate if and only if the numerical representation of the mathematic solution is in good agreement with the observed facts. If not there are two questions that need to be answer: the first one is to know if the experimental data were accurately measured. The second one will be to know if the mathematical equation is accurately implemented. If the second question appears to be negative, then, the model needs to be revised. In the case of leaky aquifer model with non-integer and integer order derivatives have failed to do the job. The aim of our paper is to revise this model by introducing the concept of variable order derivative, which so far appears to be the best concept for complexes systems.

2. **Groundwater water flow equation using the local variable order derivative**

The initial proposed groundwater equation within the leaky aquifer that was proposed by Hantush is given by:

\[
\frac{\partial^2 S(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r,t)}{\partial r} - \frac{S(r,t)}{B^2} = \frac{S}{T} \frac{\partial S(r,t)}{\partial t}.
\]

The above equation then modified by Atangana [9] as follows

\[
\frac{\partial^2 S(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r,t)}{\partial r} - \frac{S(r,t)}{B^2} = \frac{S}{T} \frac{\partial^\alpha S(r,t)}{\partial t^\alpha},
\]

\[
\frac{\partial^\alpha S(r,t)}{\partial r^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{1-\alpha} \frac{\partial S(r,x)}{\partial r} dx, \quad 0 < \alpha \leq 1.
\]

As we said before, the above model was also unable to describe accurately the complexity of the geological formation. Therefore in order to further include into mathematical formula the
complexity of the aquifer through which the flow take place, we shall proposed the following version

\[
\frac{\partial^2 S(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r, t)}{\partial r} - \frac{S(r, t)}{B^2} = \frac{S}{T} \partial\!\!\!\!\!D_\alpha^T(S(r, t)),
\]

\[
\partial\!\!\!\!\!D_\alpha^T(S(r, t)) = \lim_{\varepsilon \to 0} \frac{S\left(r, t + \varepsilon \left(t + \frac{1}{\Gamma(1 - \mu(r, t))}\right)^{1-\mu(r, t)}\right) - S(r, t)}{\varepsilon}
\]

Here the function \(l(r, t)\) accounts for the complexity associated with the leaky aquifer, \(S(r, t)\) is the change of level of water, \(S\) is the storativity, \(T\) is the transmissivity and \(B\) is the factor that account for the leakage. We shall show some useful properties of the local variable order derivative.

**Theorem 1:** Assume that \(f(x, y)\) is function which \(\partial_x^{l(x)}\left[\partial_y^{\alpha(x)}(f(x, y))\right]\) and \(\partial_y^{\alpha(y)}\left[\partial_x^{l(x)}(f(x, y))\right]\) exist and is continuous over the domain \(D \subset \mathbb{R}_2\) then [10]

\[
\partial_x^{l(x)}\left[\partial_y^{\alpha(y)}(f(x, y))\right] = \partial_y^{\alpha(y)}\left[\partial_x^{l(x)}(f(x, y))\right].
\]

**Theorem 2:** Assuming that, a given function says \(f : [a, \infty) \to \mathbb{R}\) is \(o(x)\) —differentiable at a given point says \(x_0 \geq a\), then, \(f\) is also continuous at \(x_0\) [10]

Proof. Assuming that \(f\) is \(o(x)\) —differentiable then

\[
\partial_x^{o(x)}(f(x_0)) = \lim_{\varepsilon \to 0} \frac{f\left(x_0 + \varepsilon \left(x_0 + \frac{1}{\Gamma(1 - o(x))}\right)^{1-o(x)}\right) - f(x_0)}{\varepsilon}
\]

**Theorem 3:** Assuming that \(f\) is \(o(x)\) —differentiable on an open interval \((a, b)\) then [10]

1. If \(\partial_x^{o(x)}(f(x)) < 0\) for all \(x \in (a, b)\) then \(f\) is decreasing there
2. If \(\partial_x^{o(x)}(f(x)) > 0\) for all \(x \in (a, b)\) then \(f\) is increasing there
3. If \(\partial_x^{o(x)}(f(x)) = 0\) for all \(x \in (a, b)\) then \(f\) is constant there

**Definition 1:** Let \(f : [a, \infty) \to \mathbb{R}\) is given function, then we propose that the anti-variable derivative of \(f\) is
\[ A^0_{x,t} (f(x)) = \int_a^x \left( t + \frac{1}{\Gamma(1 - o(x,t))} \right)^{o(x,t)-1} f(t) \, dt \]

The above operator is the inverse operator of the proposed fractional derivative. We shall present to underpin this statement by the following theorem [10].

**Theorem 4:** Fundamental theorem of local variable calculus: \[ A_0^0_{x,t} \left[ A^0_{x,t} f(x) \right] = f(x) \] for all \( x \geq a \) with \( f \) a given continuous and differentiable function [10].

### 3. Construction of a possible special solution

The aim of this section is to construct a possible solution of the novel groundwater flow within a leaky aquifer. To construct a solution to the new equation, we employ the \( o(x,t) \)-Laplace operator defined as

**Definition 2:** Let \( g \) be a function defined in \((0, \infty)\), then, we defined the \( o(x,t) \)-Laplace transform of \( f \) as

\[ L_{o(x)}(f(x))(s) = \int_0^\infty \left( t + \frac{1}{\Gamma(1 - o(x,t))} \right)^{o(x,t)-1} e^{-st} f(t) \, dt \]

We shall give some properties of the above operator. The above operator satisfies the following properties, \( F(s) \) is the Laplace of \( f(t) \)

\[ L_{o(x)} \left( A_0^0_{x,t} \left( \frac{df(x)}{dx} \right) \right)(s) = s^2 F(s) - sf(0) - f(0) \]

The proposed operator satisfies the following properties, \( F(s) \) is the Laplace of \( f(t) \)

1. **Linearity**
   \[ L_{o(x)}(af(x) + bg(x))(s) = aL_{o(x)}(f(x))(s) + bL_{o(x)}(g(x))(s) \]

2. **Time delay**
   \[ L_{o(x)} \left( A_0^0_{x,t} \{ f(x - a) \cdot \delta(x - a) \} \right)(s) = se^{-sa} F(s) \]

3. **First derivative**
   \[ L_{o(x)} \left( A_0^0_{x,t} \left( \frac{df(x)}{dx} \right) \right)(s) = s^2 F(s) - sf(0) - f(0) \]

4. **N order derivative**
   \[ L_{o(x)} \left( A_0^0_{x,t} \left( \frac{d^n f(x)}{dx^n} \right) \right)(s) = s^{n+1} F(s) - \sum_{j=0}^{n} s^j f^{(n-1)}(0) \]
5. Fractional derivative Caputo type
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha f(x)}{\partial x^\alpha} \right) (s) = L_{o(x)} \left( \frac{d^\alpha f(x)}{dx^\alpha} \right) (s)
= s^{\alpha + 1} F(s) - \sum_{j=0}^{n} s^{\alpha - j} f^{(n-1)}(0), n - 1 < \alpha \leq n
\]

6. Integral
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha \left( \int_0^x f(t) dt \right)}{\partial x^\alpha} \right) (s) = F(s)
\]

7. Convolution
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha (f * g(x))}{\partial x^\alpha} \right) (s) = sF(s)G(s)
\]

8. Multiplication by distance
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha \{xf(x)\}}{\partial x^\alpha} \right) (s) = -sF'(s)
\]

9. Complex shift
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha \{e^{-ax} f(x)\}}{\partial x^\alpha} \right) (s) = sF(s + a) - f(0)
\]

10. Distance Scaling
\[
L_{o(x)} \left( \frac{\partial D_x^\alpha \{f(ax)\}}{\partial x^\alpha} \right) (s) = \frac{s}{a} F \left( \frac{s}{a} \right) - f(0)
\]

Therefore, applying the above operator on both sides of equation (3), we obtain the following equation
\[
L_{o(x,t)} \left\{ \frac{\partial^2 S(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r,t)}{\partial r} - \frac{S(r,t)}{B^2} \right\} (u) = \frac{s}{T} \left( u^2 S(r,u) - uS(r,0) - S(r,0) \right)
\]

The above equation can be rearranged as follow
\[
S(r,u) = \frac{T}{Su^2} L_{o(x,t)} \left\{ \frac{\partial^2 S(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r,t)}{\partial r} - \frac{S(r,t)}{B^2} \right\} (u) + \left\{ \frac{1}{u} + \frac{1}{u^2} \right\} S(r,0)
\]

We next employ the inverse Laplace transform operator on both sides of the above equation to obtain
\[
S(r,t) = L^{-1} \left\{ \frac{T}{Su^2} L_{o(x,t)} \left\{ \frac{\partial^2 S(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r,t)}{\partial r} - \frac{S(r,t)}{B^2} \right\} (u) \right\} + L^{-1} \left\{ \frac{1}{u} + \frac{1}{u^2} \right\} S(r,0)
\]

For simplicity, we put
\[
g(r,t) = L^{-1} \left\{ \frac{1}{u} + \frac{1}{u^2} \right\} S(r,0)
\]
Then, from equation (10) one can construct a recursive formula that will be used to generate the special solution of equation (3). The recursive formula associate to equation (10) is

$$S_{n+1}(r, t) = L^{-1} \left\{ \frac{T}{Su^2} \left\{ L_{o(x,t)} \left( \frac{\partial^2 S_n(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial S_n(r, t)}{\partial r} - \frac{S_n(r, t)}{B^2} \right) (u) \right\} \right\} \text{ for } n \geq 1$$

$$S_0(r, t) = g(r, t)$$

Our next step is to prove the stability of used iteration method.

### 3.1. Uniqueness of the solution

Let assume by contradiction that, there exist two different special solutions $S_{sp1}(r, t)$ and $S_{sp2}(r, t)$. let

$$G(S) = \frac{\partial}{\partial r} \left( S(r, t) \right) = \frac{\partial^2 S(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial S(r, t)}{\partial r} - \frac{S(r, t)}{B^2}$$

The aim of our proof is to show that using the inner product that:

$$\|S_{sp1} - S_{sp2}\| \ll \epsilon$$

To achieve this, we evaluate $(G(S_{sp1}) - G(S_{sp2}), w)$ for $w \in H = \{u, v/ \int uv < \infty\}$

However, 

$$G(S_{sp1}) - G(S_{sp2})$$

$$= \frac{\partial^2}{\partial r^2} \left\{ S_{exp2}(r, t) - S_{exp1}(r, t) \right\} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ S_{exp2}(r, t) - S_{exp1}(r, t) \right\}$$

$$+ \frac{1}{B^2} \left\{ S_{exp1}(r, t) - S_{exp2}(r, t) \right\}$$

Thus, 

$$(G(S_{sp1}) - G(S_{sp2}), w)$$

$$= \left( \frac{\partial^2}{\partial r^2} \left\{ S_{exp2}(r, t) - S_{exp1}(r, t) \right\}, w \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} \left\{ S_{exp2}(r, t) - S_{exp1}(r, t) \right\}, w \right)$$

$$+ \left( \frac{1}{B^2} \left\{ S_{exp1}(r, t) - S_{exp2}(r, t) \right\}, w \right)$$

We shall evaluate the first component 

$$\left( \frac{\partial^2}{\partial r^2} \left\{ S_{exp2}(r, t) - S_{exp1}(r, t) \right\}, w \right)$$
In the real world problem, the level is bounded, that $S_{sp1}, S_{sp2}$ are bounded, therefore we can find a positive constant $M$ such that, $(S_{sp1}, S_{sp1}) < M^2$. It follows by the use of Schwartz inequality that

$$\left( \frac{\partial^2}{\partial r^2} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) \leq \left\| \frac{\partial^2}{\partial r^2} \{S_{exp2}(r, t) - S_{exp1}(r, t)\} \right\| \ \| w \|$$

However, we can find a positive constant $\omega_1, \omega_2$ such that

$$\left\| \frac{\partial^2}{\partial r^2} \{S_{exp2}(r, t) - S_{exp1}(r, t)\} \right\| \leq \omega_1 \omega_2 \| S_{exp2}(r, t) - S_{exp1}(r, t) \|$$

$$\left( \frac{\partial^2}{\partial r^2} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) \leq \omega_1 \omega_2 \| S_{exp2}(r, t) - S_{exp1}(r, t) \| \| w \|$$

We next evaluate,

$$\left( \frac{1}{r} \frac{\partial}{\partial r} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right)$$

It follows by the use of Schwartz inequality that,

$$\left( \frac{1}{r} \frac{\partial}{\partial r} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) \leq \left\| \frac{1}{r} \frac{\partial}{\partial r} \{S_{exp2}(r, t) - S_{exp1}(r, t)\} \right\| \ | w |$$

However, we can find a positive constant $O_1$ such that

$$\left( \frac{1}{r} \frac{\partial}{\partial r} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) \leq \frac{O_1}{r_1} \| S_{exp2}(r, t) - S_{exp1}(r, t) \| \ | w |$$

$$\left( \frac{\partial^2}{\partial r^2} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} \{S_{exp2}(r, t) - S_{exp1}(r, t)\}, w \right) + \left( \frac{1}{B^2} \{S_{exp1}(r, t) - S_{exp2}(r, t)\}, w \right)$$

$$\leq \left( \frac{O_1}{r_1} + \omega_1 \omega_2 + \frac{1}{B^2} \right) \| S_{exp2}(r, t) - S_{exp1}(r, t) \| \ | w |$$

Subsequently $S(r, t)$ the exact solution converges to $S_{sp1}, S_{sp2}$ then we can find two large number $N, M$ such that
\[ \|S - S_{sp1}\| \ll \frac{\epsilon}{2 \left( \frac{Q_1}{r_1} + \omega_1 \omega_2 + \frac{1}{B^2} \right) \|w\|} \quad \text{for} \ N \ \text{and} \ \|S - S_{sp2}\| \ll \frac{\epsilon}{2 \left( \frac{Q_t}{r_1} + \omega_1 \omega_2 + \frac{1}{B^2} \right) \|w\|} \quad \text{for} \ M \]

And then,
\[ \|(S_{sp1} - S_{sp2})\| \leq \|S - S_{sp2}\| + \|S - S_{sp1}\| \]
Considering \( m = \max(N, M) \), then
\[ \|(S_{sp1} - S_{sp2})\| \ll \frac{\epsilon}{\left( \frac{Q_t}{r_1} + \omega_1 \omega_2 + \frac{1}{B^2} \right) \|w\|} \]

Replacing the above in (16), we arrive at
\[ (G(S_{sp1}) - G(S_{sp2}), w) \ll \epsilon \]

Now with \( \epsilon \) extremely very small, we have that,
\[ \|(S_{sp1} - S_{sp2})\| = 0 \implies S_{sp1} = S_{sp2} \]

This completes the proof. We shall next present the stability of the method

3.2. Stability analysis of the used method

The stability of method for solving an equation is very important component of analysis since it shows the strength of the method for solving that equation. To achieve this, we need to make use of the inner product and the operator \( G \)
\[ (G(S) - G(S_1), S - S_1) \]

for any \( u, v \in H \) constructed in (12). In particular we aim to show that, we can find a positive number \( k \) such that
\[ (G(S) - G(S_1), S - S_1) \leq L \|S - S_1\|^2 \]

Proof

First we have that, \( (G(S) - G(S_1), S - S_1) = \left( \frac{\partial^2}{\partial t^2} \{S(r, t) - S_1(r, t)\}, S - S_1 \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} \{S(r, t) - S_1(r, t)\}, S - S_1 \right) + \left( \frac{1}{B^2} \{S(r, t) - S_1(r, t)\}, S - S_1 \right) \)

We shall evaluate the first component
\[ \left( \frac{\partial^2}{\partial t^2} \{S(r, t) - S_1(r, t)\}, S(r, t) - S_1(r, t) \right) \]

It follows by the use of Schwartz inequality that
\[ (18) \]
However, we can find a positive constant $f_1 f_2$ such that

$$\left\| \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)] \right\| \leq f_1 f_2 \|S(r, t) - S_1(r, t)\|$$

$$\left( \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)] \right) \leq f_1 f_2 \|S(r, t) - S_1(r, t)\|$$

We next evaluate,

$$\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right)$$

It follows by the use of Schwartz inequality that,

$$\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right) \leq \left\| \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right\| \|S(r, t) - S_1(r, t)\|$$

However, we can find a positive constant $g_1$ such that

$$\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right) \leq g_1 \|S(r, t) - S_1(r, t)\| \|S(r, t) - S_1(r, t)\|$$

$$\left( \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)] \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right)$$

$$+ \left( \frac{1}{B^2} [S(r, t) - S_1(r, t)] \right) \|S(r, t) - S_1(r, t)\| \|S(r, t) - S_1(r, t)\|$$

Thus,

$$(G(S) - G(S_1), S - S_1) \leq \left( \frac{g_1}{r_1} + f_1 f_2 + \frac{1}{B^2} \right) \|S(r, t) - S_1(r, t)\| \|S(r, t) - S_1(r, t)\|$$

Let $\left( \frac{g_1}{r_1} + f_1 f_2 + \frac{1}{B^2} \right) = L$ and thus

$$G(S) - G(S_1), S - S_1 \leq L \|S(r, t) - S_1(r, t)\|^2$$

The next step is to prove that

$$(G(S) - G(S_1), S - S_1) \leq H \|S - S_1\| \|W\|$$
Thus,\(^{(20)}\)
\[
(G(S) - G(S_1), W) = \left( \frac{\partial^2}{\partial r^2} (S(r, t) - S_1(r, t)), W \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} (S(r, t) - S_1(r, t)), W \right) + \left( \frac{1}{B^2} [S(r, t) - S_1(r, t)], W \right)
\]
We shall evaluate the first component
\[
\left( \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)], W \right)
\]
It follows by the use of Schwartz inequality that,
\[
\left( \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)], W \right) \leq \left\| \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)] \right\| ||W||
\]
However, we can find a positive constant \(m_1, m_2\) such that
\[
\left\| \frac{\partial^2}{\partial r^2} [S_{exp2}(r, t) - S_{exp1}(r, t)] \right\| \leq m_1 m_2 \|S(r, t) - S_1(r, t)\|
\]
\[
\left( \frac{\partial^2}{\partial r^2} [S_{exp2}(r, t) - S_{exp1}(r, t)], W \right) \leq m_1 m_2 \|S(r, t) - S_1(r, t)\|||W||
\]
We next evaluate,
\[
\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)], W \right)
\]
It follows by the use of Schwartz inequality that,\(^{(21)}\)
\[
\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)], W \right) \leq \left\| \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)] \right\| ||W||
\]
However, we can find a positive constant \(N_1\) such that
\[
\left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)], W \right) \leq \frac{N_1}{r_1} \|S(r, t) - S_1(r, t)\|||W||
\]
However putting all these equation together, we obtain the following
\[
\left( \frac{\partial^2}{\partial r^2} [S(r, t) - S_1(r, t)], W \right) + \left( \frac{1}{r} \frac{\partial}{\partial r} [S(r, t) - S_1(r, t)], W \right) + \left( \frac{1}{B^2} [S_1(r, t) - S(r, t)], W \right) \leq \left( \frac{N_1}{r_1} + m_1 m_2 + \frac{1}{B^2} \right) \|S(r, t) - S_1(r, t)\|||W||
\]
Taking \( \left( \frac{N_1}{r_1} + m_1m_2 + \frac{1}{B^2} \right) = H \)

Then

\[
(G(S) - G(S_1), \quad S - S_1) \leq H\|S - S_1\|W
\]

This completes the proof.

4. Numerical simulations

We devote this part to the numerical simulation, to achieve this we first propose an algorithm that will be used for numerical simulations.

Algorithm 1

Input

\[ S_0(r, t) = g(r, t) \]

as preliminary input,

- \( i \) - number terms in the rough calculation
- Output \( S_{Ap}(r, t) \), the approximate solution

Step 1: Put \( S_0(r, t) = g(r, t) \) and \( S_{Ap}(r, t) = S_{Ap}(r, t) \)

Step 2: for \( i = 1 \) to \( n - 1 \) do step 3, step 4 and step 5

\[
S_{n+1}(r, t) = \mathcal{L}^{-1} \left\{ \frac{T}{Su^2} \left\{ L_{0(x,t)} \left( \frac{\partial^2 S_n(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial S_n(r, t)}{\partial r} - \frac{S_n(r, t)}{B^2} \right) \right\} (u) \right\}
\]

Step 3: compute

\[
\beta_{1(n+1)}(r, t) = \beta_{1(n)}(r, t) + S_{Ap}(r, t)
\]

Step 4: Compute:

\[
S_{Ap}(r, t) = S_{Ap}(r, t) + \beta_{1(n+1)}(r, t)
\]

Stop.

The overhead method shall be employed to yield the numerical replication of the physical problem under investigation as indicate in the following figure 1 and 2. The considered equation is subjected to the following conditions. The equation (1.1) is subjected to the following initial and boundary conditions:

\[
\Phi(r, 0) = \Phi_0, \quad \lim_{r \to \infty} \Phi(r, t) = \Phi_0 \quad Q = \frac{2\pi^2}{\Gamma(\frac{n}{2})} \frac{r}{n-1} Kd^{3-n} \partial_r \Phi(r_b, t).
\]
Figure 1: Contour plot of the proposed solution as function of space and time. This shows the wave in change of level of water within the confined aquifer during the pumping test on one side of the well for $a(r, t) = 0.8 \sin(r, t)$.

Figure 2: Contour plot of the proposed solution as function of space and time. This shows the wave in change of level of water within the confined aquifer during the pumping test around the well for $a(r, t) = 0.8 \sin(r, t)$.
5. Conclusion
A new derivative that takes into account the complexity of the physical phenomena was used to enhance the model describing the movement of groundwater flowing within a leaky aquifer. We made use of a new operator called $\sigma(x, t)$-Laplace transform together with the concept of iterative method to solve the new groundwater equation. We have showed in detail stability analysis of the used method together with the uniqueness of the special solution. We presented the numerical simulations.

6. References


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