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C-class functions and fixed point theorems for generalized α-η-ψ-φ-F-contraction type mappings in α-η-complete metric spaces

Arslan Hojat Ansari[a], Anchalee Kaewcharoen[b,*]

[a]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, 31485-313, Iran.

[b]Research Center for Academic Excellence in Mathematics, Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.

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Abstract

In this paper, we introduce the concept of generalized α-η-ψ-φ-F-contraction type mappings where ψ is the altering distance function and φ is the ultra altering distance function. The unique fixed point theorems for such mappings in the setting of α-η-complete metric spaces are proven. We also assure the fixed point theorems in partially ordered metric spaces. Moreover, the solution of the integral equation is obtained using our main result.

Keywords: α-η-complete metric spaces, α-η-continuous mappings, triangular α-orbital admissible mappings with respect to η, C-class functions, generalized α-η-ψ-φ-F-contraction type mappings.
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1. Introduction and Preliminaries

The Banach contraction principle introduced by Banach [3] is one of the most important results in fixed point theory. Many authors extended and generalized the Banach contraction principle in several directions (see [2, 4, 5, 7–17] and references contained therein). In 2014, Ansari [1] introduced the concept of $\mathcal{C}$-class functions and proved the unique fixed point theorems for certain contractive mappings with respect to the $\mathcal{C}$-class functions.

*Corresponding author
Email addresses: analsisamirmath2@gmail.com (Arslan Hojat Ansari), anchaleeka@nu.ac.th (Anchalee Kaewcharoen)

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In this paper, we introduce the definition of generalized α-η-ψ-φ-F-contraction type mappings where ψ is the altering distance function and φ is the ultra altering distance function. The unique fixed point theorems for such mappings in the setting of α-η-complete metric spaces are proven. We also assure the fixed point theorems in partially ordered metric spaces. Moreover, the solution of the integral equation is obtained using our main result.

Samet et al. [17] introduced the notion of α-admissible mappings as the following.

Definition 1.1 ([17])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is α-admissible if

$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(Tx, Ty) \geq 1.$$

Karapinar et al. [12] introduced the concept of triangular α-admissible mappings.

Definition 1.2 ([12])**.** Let $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is triangular α-admissible if

(a) T is α-admissible;
(b) $\alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1 \quad \text{imply} \quad \alpha(x, y) \geq 1.$

In 2014, Popescu [16] gave the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings.

Definition 1.3 ([16])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \quad \text{implies} \quad \alpha(Tx, T^2x) \geq 1.$$

Definition 1.4 ([16])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is triangular α-orbital admissible if

(a) T is α-orbital admissible;
(b) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

Recently, Chuadchawna et al. [6] introduced the notions of α-orbital admissible mappings with respect to η and triangular α-orbital admissible mappings with respect to η.

Definition 1.5 ([6])**.** Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$. Then T is α-orbital admissible with respect to η if

$$\alpha(x, Tx) \geq \eta(x, Tx) \text{ implies } \alpha(Tx, T^2x) \geq \eta(Tx, T^2x).$$

Definition 1.6 ([6])**.** Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$. Then T is triangular α-orbital admissible with respect to η if

(a) T is α-orbital admissible with respect to η;
(b) $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply $\alpha(x, Ty) \geq \eta(x, Ty)$.

The following lemma will be used for proving our main results.

Lemma 1.7 ([6])**.** *Let $T : X \to X$ be a triangular α-orbital admissible with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Recently, Karapinar [11] introduced the concept of α-ψ-Geraghty contraction type mappings in complete metric spaces.

Let Ψ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;

(c) $\psi(t) = 0$ if and only if $t = 0$;
(d) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$.

Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Definition 1.8 ([11])**.** Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$. A mapping $T : X \to X$ is said to be a generalized α-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y),\, d(x,Tx),\, d(y,Ty)\} \text{ and } \psi \in \Psi.$$

Remark 1.9. We now present some properties of elements in $\mathcal{F}$.

1. There exists a continuous function which is not in $\mathcal{F}$. Indeed, if we let $\beta(t) = \frac{t}{1+t}$ for all $t \in [0,\infty)$ and $t_n = n$ for all $n \in \mathbb{N}$, then we have $\lim\limits_{n\to\infty} \dfrac{t_n}{1+t_n} = 1$ but $\lim\limits_{n\to\infty} t_n \neq 0$. Therefore $\beta \notin \mathcal{F}$.
2. There exists a function in $\mathcal{F}$ which is not continuous. Indeed, if we define a function $\beta : [0,\infty) \to [0,1)$ by
$$\beta(t) = \begin{cases} \frac{1}{1+t} & ,t>0; \\ 0 & ,t=0, \end{cases}$$
then $\beta \in \mathcal{F}$ but it is not continuous from the right at $x = 0$.

Theorem 1.10 ([11])**.** *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) *T is a generalized α-ψ-Geraghty contraction type mapping;*
(ii) *T is a triangular α-admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping.*

Then, T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Ansari [1] considered the concept of C-class functions as the following:

Definition 1.11 ([1])**.** A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and for all $s,t \in [0,\infty)$,

(a) $F(s,t) \leq s$;
(b) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

We denote $\mathcal{C}$ as the family of all C-class functions.

Example 1.12. The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements in $\mathcal{C}$.

(1) $F(s,t) = s - t$ for all $s,t \in [0,\infty)$;

(2) $F(s,t) = ms$ for all $s,t \in [0,\infty)$ where $0<m<1$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$ for all $s,t \in [0,\infty)$ where $r \in (0,\infty)$;

(4) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$ for all $s,t \in [0,\infty)$ where $l > 1, r \in (0,\infty)$;

(5) $F(s,t) = s \log_{t+a} a$ for all $s,t \in [0,\infty)$ where $a > 1$;

(6) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$ for all $s,t \in [0,\infty)$;

(7) $F(s,t) = s\beta(s)$ for all $s,t \in [0,\infty)$ where $\beta : [0,\infty) \to [0,1)$ and is continuous;

(8) $F(s,t) = s - \varphi(s)$ for all $s,t \in [0,\infty)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;

(9) $F(s,t) = sh(s,t)$ for all $s,t \in [0,\infty)$ where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $s,t \in [0,\infty)$;

(10) $F(s,t) = s - (\frac{2+t}{1+t})t$ for all $s,t \in [0,\infty)$;

(11) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$ for all $s,t \in [0,\infty)$.

We now drop the subadditivity of $\psi \in \Psi$ by considering the following definition.

Definition 1.13 ([13])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(a) ψ is nondecreasing and continuous;
(b) $\psi(t) = 0$ if and only if $t = 0$.

The family of all altering distance functions is denoted by Φ.

Definition 1.14. A function $\varphi : [0,\infty) \to [0,\infty)$ is called an ultra altering distance function if the following properties are satisfied:

(a) φ is continuous;
(b) $\varphi(t) > 0$ for all $t > 0$.

We denote Φ_u the family of all ultra altering distance functions.

Lemma 1.15 ([6])**.** *Suppose that (X,d) is a metric space and $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$.

On the other hand, Hussain et al. [9] introduced the concepts of α-η-complete metric spaces and α-η-continuous functions.

Definition 1.16 ([9])**.** Let (X,d) be a metric space and $\alpha,\eta : X \times X \to [0,+\infty)$. Then, X is said to be α-η-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ converges in X.

Example 1.17 ([6])**.** Let $X = (0,\infty)$ and define a metric on X by $d(x,y) = |x-y|$ for all $x,y \in X$. Therefore X is not complete. Let Y be a closed subset of X. Define $\alpha,\eta : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} (x+y)^3, & \text{if } x,y \in Y \\ 0, & \text{otherwise,} \end{cases}$$
$$\eta(x,y) = 3x^2y.$$

We obtain that (X,d) is an α-η-complete metric space.

Definition 1.18 ([9]). Let (X,d) be a metric space and $\alpha,\eta : X\times X \to [0,+\infty)$. A mapping $T : X \to X$ is said to be an α-η-continuous mapping if each sequence $\{x_n\}$ in X with $x_n \to x$ as $n\to\infty$ and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$ implies $Tx_n \to Tx$ as $n\to\infty$.

Example 1.19 ([6]). Let $X=[0,\infty)$ and define a metric on X by $d(x,y)=|x-y|$ for all $x,y\in X$. Assume that $T : X\to X$ and $\alpha,\eta : X\times X\to[0,+\infty)$ are defined by

$$Tx = \begin{cases} x^4 & \text{if } x\in[0,1] \\ \cos\pi x+3 & \text{if } x\in(1,\infty), \end{cases}$$

$$\alpha(x,y) = \begin{cases} x^3+y^3+1 & \text{if } x,y\in[0,1] \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta(x,y)=x^3.$$

Therefore T is an α-η-continuous mapping but T is not continuous.

2. Main results

We now introduce the concept of generalized α-η-ψ-φ-F-contraction type mappings and prove the fixed point theorems for such mappings.

Definition 2.1. Let (X,d) be a metric space and $\alpha,\eta : X\times X\to[0,\infty)$. A mapping $T : X\to X$ is said to be a generalized α-η-ψ-φ-F-contraction type mapping if $\alpha(x,y)\geq\eta(x,y)$ implies

$$\psi(d(Tx,Ty)) \leq F(\psi(M(x,y)),\varphi(M(x,y))), \tag{2.1}$$

where

$$M(x,y)=\max\{d(x,y),\, d(x,Tx),\, d(y,Ty)\},\ \psi\in\Phi, \varphi\in\Phi_u \text{ and } F\in\mathcal{C}.$$

Remark 2.2. In Definition 2.1, if we define $F(s,t)=s\beta(s)$ where $\beta : [0,\infty)\to[0,1)$ is continuous, then, (2.1) reduces to the contraction

$$\psi(d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y))$$

if $\alpha(x,y)\geq\eta(x,y)$.

We now assure the fixed point theorems for generalized α-η-ψ-φ-F-contraction type mappings in the setting of α-η-complete metric spaces.

Theorem 2.3. *Let (X,d) be a metric space. Assume that $\alpha,\eta : X\times X\to[0,\infty)$ and $T : X\to X$. Suppose that the following conditions are satisfied:*

(i) *(X,d) is an α-η-complete metric space;*
(ii) *T is a generalized α-η-ψ-φ-F-contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq\eta(x_1,Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^*\in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Define a sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If $x_{n_0}=x_{n_0+1}$ for some $n_0\in\mathbb{N}$, then T has a fixed point. Suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. By Lemma 1.7, we have $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$. Since T is a generalized α-η-ψ-φ-F-contraction type mapping, we have

$$\begin{aligned}\psi(d(x_{n+1},x_{n+2})) &= \psi(d(Tx_n,Tx_{n+1}))\\ &\leq F(\psi(M(x_n,x_{n+1})),\varphi(M(x_n,x_{n+1})))\\ &< \psi(M(x_n,x_{n+1}))\end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$ where

$$\begin{aligned}M(x_n,x_{n+1}) &= \max\{d(x_n,x_{n+1}),\, d(x_n,Tx_n),\, d(x_{n+1},Tx_{n+1})\}\\ &= \max\{d(x_n,x_{n+1}),\, d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\}\\ &= \max\{d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\}.\end{aligned}$$

If $\max\{d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\} = d(x_{n+1},x_{n+2})$, then

$$\begin{aligned}\psi(d(x_{n+1},x_{n+2})) &\leq F(\psi(M(x_n,x_{n+1})),\varphi(M(x_n,x_{n+1})))\\ &< \psi(M(x_n,x_{n+1}))\\ &= \psi(d(x_{n+1},x_{n+2})),\end{aligned}$$

which is a contradiction. Thus we conclude that

$$\max\{d(x_n,x_{n+1}),\, d(x_{n+1},x_{n+2})\} = d(x_n,x_{n+1}).$$

By (2.2), we get that $\psi(d(x_{n+1},x_{n+2})) < \psi(d(x_n,x_{n+1}))$ for all $n \in \mathbb{N}$. Since ψ is nondecreasing, we have $d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. It follows that the sequence $\{d(x_n,x_{n+1})\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. We claim that $r = 0$. Using (2.2), we have

$$\psi(d(x_{n+1},x_{n+2})) \leq F(\psi(d(x_n,x_{n+1})),\varphi(d(x_n,x_{n+1}))).$$

Taking $n \to \infty$, we obtain that

$$\psi(r) \leq F(\psi(r),\varphi(r)).$$

This implies that $\psi(r) = 0$ or $\varphi(r) = 0$ which yields

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = r = 0. \tag{2.3}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. By Lemma 1.15, there exist an $\varepsilon > 0$ and two subsequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)},x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1},x_{n(k)}) < \varepsilon$ and

$$\lim_{k\to\infty} d\left(x_{n(k)},x_{m(k)}\right) = \lim_{k\to\infty} d\left(x_{n(k)-1},x_{m(k)}\right) = \lim_{k\to\infty} d\left(x_{m(k)-1},x_{n(k)}\right) = \varepsilon. \tag{2.4}$$

By Lemma 1.7, we have $\alpha(x_{n(k)-1},x_{m(k)-1}) \geq \eta(x_{n(k)-1},x_{m(k)-1})$. Thus we have

$$\begin{aligned}\psi(d(x_{n(k)},x_{m(k)})) &= \psi(d(Tx_{n(k)-1},Tx_{m(k)-1}))\\ &\leq F(\psi(M(x_{n(k)-1},x_{m(k)-1})),\varphi(M(x_{n(k)-1},x_{m(k)-1}))),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}M(x_{n(k)-1},x_{m(k)-1}) &= \max\{d(x_{n(k)-1},x_{m(k)-1}),\, d(x_{n(k)-1},Tx_{n(k)-1}),\, d(x_{m(k)-1},Tx_{m(k)-1})\}\\ &= \max\{d(x_{n(k)-1},x_{m(k)-1}),\, d(x_{n(k)-1},x_{n(k)}),\, d(x_{m(k)-1},x_{m(k)})\}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} M(x_{n(k)-1},x_{m(k)-1}) = \varepsilon. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)).$$

It follows that $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. This implies that $\varepsilon = 0$ which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is an α-η-complete metric space and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, there is $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is α-η-continuous, we get $\lim_{n\to\infty} Tx_n = Tx^*$ and so $x^* = Tx^*$. Hence T has a fixed point. □

In the following theorem, we replace the continuity of T in Theorem 2.3 by some suitable conditions.

Theorem 2.4. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is a generalized α-η-ψ-φ-F-contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. By the analogous proof as in Theorem 2.3, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\ &\leq F(\psi(M(x_{n(k)}, x^*)), \varphi(M(x_{n(k)}, x^*))), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M(x_{n(k)}, x^*) &= \max\{d(x_{n(k)}, x^*),\ d(x_{n(k)}, Tx_{n(k)}),\ d(x^*, Tx^*)\} \\ &= \max\{d(x_{n(k)}, x^*),\ d(x_{n(k)}, x_{n(k)+1}),\ d(x^*, Tx^*)\}. \end{aligned}$$

It follows that

$$\lim_{k\to\infty} M(x_{n(k)}, x^*) = d(x^*, Tx^*).$$

From (2.7), letting $k \to \infty$ in the above inequality, we have

$$\psi(d(x^*, Tx^*)) \leq F(\psi(d(x^*, Tx^*)), \varphi(d(x^*, Tx^*))).$$

We obtain that $\psi(d(x^*, Tx^*)) = 0$ or $\varphi(d(x^*, Tx^*)) = 0$. This implies that $d(x^*, Tx^*) = 0$. It follows that $Tx^* = x^*$. □

For the uniqueness of a fixed point of a generalized α-η-ψ-φ-F-contraction type mapping, we assume the suitable condition introduced by Popescu [16].

Theorem 2.5. *Suppose all assumptions of Theorem 2.3 (respectively Theorem 2.4) hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x, v) \geq \eta(x, v)$, $\alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*, v) \geq \eta(x^*, v)$, $\alpha(y^*, v) \geq \eta(y^*, v)$, and $\alpha(v, Tv) \geq \eta(v, Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*, T^n v) \geq \eta(x^*, T^n v) \quad \text{and} \quad \alpha(y^*, T^n v) \geq \eta(y^*, T^n v),$$

for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} \psi(d(x^*, T^{n+1}v)) &= \psi(d(Tx^*, TT^n v)) \\ &\leq F(\psi(M(x^*, T^n v)), \varphi(M(x^*, T^n v))), \end{aligned} \tag{2.8}$$

for all $n \in \mathbb{N}$ where

$$\begin{aligned} M(x^*, T^n v) &= \max\{d(x^*, T^n v),\, d(x^*, Tx^*),\, d(T^n v, T^{n+1}v)\} \\ &= \max\{d(x^*, T^n v),\, d(T^n v, T^{n+1}v)\}. \end{aligned}$$

By Theorem 2.3, we deduce that $\{T^n v\}$ converges to a fixed point z^* of T. This implies that

$$\lim_{n\to\infty} M_T(x^*, T^n v) = d(x^*, z^*).$$

Taking $n \to \infty$ in (2.8), we have

$$\psi(d(x^*, z^*)) \leq F(\psi(d(x^*, z^*)), \varphi(d(x^*, z^*))).$$

It follows that $\psi(d(x^*, z^*)) = 0$ or $\varphi(d(x^*, z^*)) = 0$. Therefore $d(x^*, z^*) = 0$. Hence $x^* = z^*$. Similarly, we can prove that $y^* = z^*$. Hence T has a unique fixed point. □

In Theorem 2.3 and Theorem 2.4, if we put $F(s, t) = s\beta(s)$ where $\beta : [0, \infty) \to [0, 1)$ is continuous, $\eta(x, y) = 1$ and $\varphi(t) = t$, then we obtain the following result.

Corollary 2.6. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *for all $x, y \in X$, $\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$ where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$, $\psi \in \Phi$ and $\beta : [0, \infty) \to [0, 1)$ is continuous;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Using Example 1.12 (3), Theorem 2.3, and Theorem 2.4, we immediately obtain the following corollary.

Corollary 2.7. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *for all $x, y \in X$, $\alpha(x, y)\psi(d(Tx, Ty)) \leq \frac{\psi(M(x,y))}{(1+\varphi(M(x,y))^r}$ where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$, $\psi \in \Phi, \varphi \in \Phi_u$ and $r \in (0, \infty)$;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

3. Consequences

Definition 3.1. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be an α-η-ψ-φ-F-contraction type mapping if $\alpha(x, y) \geq \eta(x, y)$ implies

$$\psi(d(Tx, Ty)) \leq F(\psi(d(x, y)), \varphi((d(x, y))),$$

where $\psi \in \Phi, \varphi \in \Phi_u,$ and $F \in \mathcal{C}$.

Theorem 3.2. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-φ-F-contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. As in the proof of Theorem 2.3, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since T is α-η-continuous, we have

$$x_{n+1} = Tx_n \to Tx^* \text{ as } n \to \infty.$$

Hence T has a fixed point . □

Theorem 3.3. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-φ-F-contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. As in the proof of Theorem 2.3, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
\psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\
&\leq F(\psi(d(x_{n(k)}, x^*)), \varphi(d(x_{n(k)}, x^*))) \\
&\leq \psi(d(x_{n(k)}, x^*)).
\end{aligned}$$

Letting $k \to \infty$ in above inequality, we obtain that

$$\psi(d(x^*, Tx^*)) \leq \psi(0) = 0.$$

Thus $\psi(d(x^*, Tx^*)) = 0$. This implies that $d(x^*, Tx^*) = 0$. Hence $x^* = Tx^*$. □

Theorem 3.4. *Suppose all assumptions of Theorem 3.2 (respectively Theorem 3.3) hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq \eta(x,v)$, $\alpha(y,v) \geq \eta(y,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Then, T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*,v) \geq \eta(x^*,v)$, $\alpha(y^*,v) \geq \eta(y^*,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*,T^nv) \geq \eta(x^*,T^nv) \quad \text{and} \quad \alpha(y^*,T^nv) \geq \eta(y^*,T^nv)$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \psi(d(x^*,T^{n+1}v)) &= \psi(d(Tx^*,TT^nv)) \\ &\leq F(\psi(d(x^*,T^nv)),\varphi(d(x^*,T^nv))) \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. Since $\alpha(v,Tv) \geq \eta(v,Tv)$, we obtain that $\{T^nv\}$ converges to a fixed point z^* of T. By (3.1) letting limit $n \to \infty$, we have

$$\psi(d(x^*,z^*)) \leq F(\psi(d(x^*,z^*)),\varphi(d(x^*,z^*))).$$

This implies that so $\psi(d(x^*,z^*)) = 0$ or $\varphi(d(x^*,z^*)) = 0$. Therefore, $x^* = z^*$. Similarly, we can prove that $y^* = z^*$. Hence $x^* = y^*$. □

Corollary 3.5. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that $T : X \to X$ and assume that the following conditions are satisfied:*

(i) *there exists $F \in \mathcal{C}$ such that*

$$\psi(d(Tx,Ty)) \leq F(\psi(d(x,y)),\varphi(d(x,y)))$$

for all $x,y \in X$ with $x \preceq y$ where $\psi \in \Phi$ and $\varphi \in \Phi_u$;

(ii) *there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;*

(iii) *T is nondecreasing with respect to $\preceq$;*

(iv) *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^nx_1\}$ converges to x^*. Further if for all $x \neq y \in X$, there exists $v \in X$ such that $x \preceq v$, $y \preceq v$ and $v \preceq Tv$, then T has a unique fixed point.

Proof. Define functions $\alpha,\eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y \\ \frac{1}{4}, & \text{otherwise} \end{cases},$$

and

$$\eta(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x \preceq y \\ 2, & \text{otherwise.} \end{cases}$$

Let $x,y \in X$ with $\alpha(x,y) \geq \eta(x,y)$. By (i), we have

$$\psi(d(Tx,Ty)) \leq F(\psi(d(x,y)),\varphi(d(x,y))).$$

This implies that T is an α-η-ψ-φ-F-contraction type mapping. Since X is a complete metric space, we have X is an α-η-complete metric space. By (ii), there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$. Let $\alpha(x,Tx) \geq \eta(x,Tx)$, we have $x \preceq Tx$. Since T is nondecreasing, we obtain that $Tx \preceq T(Tx)$. Then $\alpha(Tx,T^2x) \geq \eta(Tx,T^2x)$. Let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, so we have $x \preceq y$ and $y \preceq Ty$. It follows that $x \preceq Ty$. Then $\alpha(x,Ty) \geq \eta(x,Ty)$. Thus, all conditions of Theorem 3.2 and Theorem 3.3 are satisfied. Hence, T has a fixed point. □

We now give an example for supporting Theorem 3.2.

Example 3.6. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $F(s, t) = \frac{s}{1+2s}$ for all $s, t \in [0, \infty)$. Let $\psi(t) = \frac{t}{4}, \varphi(t) = t^2$ and a mapping $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{2}x, & \text{if } 0 \leq x \leq 1 \\ 3x, & \text{if } x > 1. \end{cases}$$

Define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta(x, y) = \begin{cases} \frac{1}{5}, & \text{if } 0 \leq x, y \leq 1 \\ 3, & \text{otherwise.} \end{cases}$$

First, we will prove that (X, d) is an α-η-complete metric space. If $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0, 1]$. Since $([0, 1], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[0, 1] \subseteq X$. Let $\alpha(x, Tx) \geq \eta(x, Tx)$. Thus, $x \in [0, 1]$ and $Tx \in [0, 1]$ and so $T^2x = T(Tx) \in [0, 1]$. Then, $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Thus, T is α-orbital admissible with respect to η. Let $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$. We have $x, y, Ty \in [0, 1]$. This implies that $\alpha(x, Ty) \geq \eta(x, Ty)$. Hence, T is triangular α-orbital admissible with respect to η. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Then, $\{x_n\} \subseteq [0, 1]$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{1}{2}x_n = \frac{1}{2}x = Tx$. That is T is α-η-continuous. It is clear that condition (iv) of Theorem 3.2 is satisfied with $x_1 = 1$ since $\alpha(1, T(1)) = \alpha(1, \frac{1}{2}) = 1 > \frac{1}{5} = \eta(1, \frac{1}{2}) = \eta(1, T(1))$. Finally, we will prove that T is an α-η-ψ-φ-F-contraction type mapping. Let $\alpha(x, y) \geq \eta(x, y)$. Therefore, $x, y \in [0, 1]$. It follows that

$$\begin{aligned} F(\psi(d(x, y)), \varphi(d(x, y))) - \psi(d(Tx, Ty)) &= \psi(d(x, y)) \cdot \frac{1}{1 + 2\psi(d(x, y))} - \frac{1}{4}d(Tx, Ty) \\ &= \frac{1}{4}|x - y| \cdot \frac{1}{1 + \frac{1}{2}|x - y|} - \frac{1}{4}|\frac{1}{2}x - \frac{1}{2}y| \\ &= \frac{\frac{1}{4}|x - y|}{1 + \frac{1}{2}|x - y|} - \frac{1}{8}|x - y| \\ &= \frac{|x - y|(4 - 2 - |x - y|)}{8(2 + |x - y|)} \\ &\geq 0. \end{aligned}$$

Then, we have $\psi(d(Tx, Ty)) \leq F(\psi(d(x, y)), \varphi((d(x, y)))$. Thus, all assumptions of Theorem 3.2 are satisfied. Hence, T has a fixed point $x^* = 0$.

4. Applications to ordinary differential equations

The following ordinary differential equation is taken from Karapinar [11] and Chaudchawna et al. [6] :

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), & t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The Green function associated to (4.1) is defined by

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1 \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Assume that $C(I)$ is the set of all continuous functions defined on I where $I = [0,1]$. Suppose that $d(x,y) = \|x-y\|_\infty = \sup_{t\in I}|x(t)-y(t)|$ for all $x,y \in C(I)$. Therefore, $(C(I),d)$ is a complete metric space.

Suppose that the following conditions hold:

(i) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $a,b \in \mathbb{R}$ with $\xi(a,b) \geq 0$, we have $|f(t,a)-f(t,b)| \leq 8\ln(|a-b|+1)$ for all $t \in I$;

(ii) there exists $x_1 \in C(I)$ such that for all $t \in I$,
$$\xi(x_1(t), \int_0^1 G(t,s)f(s,x_1(s))ds) \geq 0;$$

(iii) for all $t \in I$ and for all $x,y,z \in C(I)$,
$$\xi(x(t),y(t)) \geq 0 \text{ and } \xi(y(t),z(t)) \geq 0 \text{ imply } \xi(x(t),z(t)) \geq 0;$$

(iv) for all $t \in I$ and for all $x,y \in C(I)$,
$$\xi(x(t),y(t)) \geq 0 \text{ implies } \xi(\int_0^1 G(t,s)f(s,x(s))ds, \int_0^1 G(t,s)f(s,y(s))ds) \geq 0;$$

(v) if $\{x_n\}$ is a sequence in $C([0,1])$ such that $x_n \to x \in C([0,1])$ and $\xi(x_n(t),x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t),x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

We now prove the existence of a solution of the above second order differential equation.

Theorem 4.1. *Suppose that conditions (i)–(v) are satisfied. Then,* (4.1) *has at least one solution* $x^* \in C^2(I)$.

Proof. We know that $x^* \in C^2(I)$ is a solution of (4.1) if and only if $x^* \in C(I)$ is a solution of the integral equation (see [11])
$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds \quad \text{for all } t \in I.$$

Define a mapping $T : C(I) \to C(I)$ by
$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds \quad \text{for all } t \in I.$$

Therefore, the problem (4.1) is equivalent to finding $x^* \in C(I)$ that is a fixed point of T. Let $x,y \in C(I)$ such that $\xi(x(t),y(t)) \geq 0$ for all $t \in I$. From (i), we obtain that
$$\begin{aligned}
|Tx(t)-Ty(t)| &= \Big|\int_0^1 G(t,s)[f(s,x(s))-f(s,y(s))]ds\Big| \\
&\leq \int_0^1 G(t,s)\big|f(s,x(s))-f(s,y(s))\big|ds \\
&\leq 8\int_0^1 G(t,s)\ln(|x(s)-y(s)|+1)ds \\
&\leq 8\int_0^1 G(t,s)\ln(d(x,y)+1)ds \\
&\leq 8\ln(d(x,y)+1)\Big(\sup_{t\in I}\int_0^1 G(t,s)ds\Big).
\end{aligned}$$

Since $\int_0^1 G(t,s)ds = -(t^2/2)+t/2$ for all $t \in I$, we have $\sup_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}$. This implies that
$$d(Tx,Ty) \leq \ln(d(x,y)+1).$$

Therefore,

$$\ln(d(Tx,Ty)+1) \le \ln(\ln(d(x,y)+1)+1). \tag{4.2}$$

Define mappings $\psi : [0,\infty) \to [0,\infty)$ and $F : [0,\infty)^2 \to \mathbb{R}$ by

$$\psi(x) = \ln(x+1) \text{ and } F(x,y) = \psi(x).$$

Therefore, $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, $\psi(t) = 0$ if and only if $t = 0$ and also $\psi(x) < x$. If $\varphi \in \Phi_u$, then, by (4.2) we obtain that

$$\psi(d(Tx,Ty)) \le F(\psi(d(x,y)), \varphi(d(x,y)))$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \ge 0$ for all $t \in I$.

Define $\alpha, \eta : C(I) \times C(I) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t),y(t)) \ge 0, t \in I \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta(x,y) = \begin{cases} \frac{1}{2}, & \xi(x(t),y(t)) \ge 0, t \in [0,1] \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ such that $\alpha(x,y) \ge \eta(x,y)$. It follows that $\xi(x(t), y(t)) \ge 0$ for all $t \in I$. This yields

$$\psi(d(Tx,Ty)) \le F(\psi(d(x,y)), \varphi(d(x,y))).$$

Therefore T is an α-η-ψ-φ-F-contraction type mapping. Using (iv), for each $x \in C(I)$ such that $\alpha(x,Tx) \ge \eta(x,Tx)$, we obtain that $\xi(Tx(t), T^2x(t)) \ge 0$. This implies that $\alpha(Tx,T^2x) \ge \eta(Tx,T^2x)$. Let $x, y \in C(I)$ such that $\alpha(x,y) \ge \eta(x,y)$ and $\alpha(y,Ty) \ge \eta(y,Ty)$. Thus,

$$\xi(x(t),y(t)) \ge 0 \text{ and } \xi(y(t),Ty(t)) \ge 0 \text{ for all } t \in I.$$

By applying (iii), we obtain that $\xi(x(t), Ty(t)) \ge 0$ and so $\alpha(x,Ty) \ge \eta(x,Ty)$. It follows that T is triangular α-orbital admissible with respect to η. Using (ii), there exists $x_1 \in C(I)$ such that $\alpha(x_1,Tx_1) \ge \eta(x_1,Tx_1)$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \ge 0$. This implies that $\alpha(x_{n(k)},x) \ge \eta(x_{n(k)},x)$. Therefore, all assumptions in Theorem 3.2 are satisfied. Hence, T has a fixed point in $C(I)$. It follows that there exists $x^* \in C(I)$ such that $Tx^* = x^*$ is a solution of (4.1). □

Corollary 4.2. *Assume that the following conditions hold:*

(i) $f : [0,1] \times \mathbb{R} \to [0,\infty)$ *is continuous and nondecreasing;*

(ii) *for all* $t \in [0,1]$, *for all* $a, b \in \mathbb{R}$ *with* $a \le b$, *we have*

$$|f(t,a) - f(t,b)| \le 8\ln(|a-b|+1);$$

(iii) *there exists* $x_1 \in C([0,1])$ *such that for all* $t \in [0,1]$, *we have*

$$x_1(t) \le \int_0^1 G(t,s) f(s, x_1(s)) ds.$$

Then, (4.1) *has a solution in* $C^2([0,1])$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a,b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

□

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