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Important inequalities for preinvex functions

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Abstract

The paper deals with fundamental inequalities for preinvex functions. The result relating to preinvex functions on the invex set that satisfies condition C shows that such functions are convex on every generated line segment. As an effect of that convexity, the paper provides symmetric forms of the most important inequalities which can be applied to preinvex functions.

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1. Introduction

In this work, we investigate widely applicable integral inequalities that do not use the function derivative. The aim of the paper is to establish the basic inequalities that can be used within the framework of preinvex functions. With this intention, we will explore the Jensen and Hermite-Hadamard inequality for convex functions on the line segment.

The basic structure that we use in this research is the real vector space $\mathbb{R}^k$, and in particular its line segments.

Combining preinvex and convex functions, main results such as Theorem 3.1, Theorem 4.4 and Theorem 5.4 are distributed in several sections.

Drawing on the experience with convex sets, the paper offers the notion of the invex combination and invex hull. By linking a preinvex function with this notation, the cumulative result arises in formula (3.6). Relying on free vectors, a geometric visualization of the invex line segment is presented in Lemma 5.1 and Corollary 5.2. The actual Jensen's inequality for preinvex functions is established in formula (5.9), and the Hermite-Hadamard inequality for preinvex functions is determined in formula (5.14).

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2. Invex set and preinvex function

A notion of preinvex function was introduced in [10] and [11], and came from the notion of invex function. Some prominent properties of preinvex functions can be found in [15]. We briefly present the concept of preinvexity (referring to a preinvex function on the invex set) notifying two definitions and two examples.

Definition 2.1. A set $K \subseteq \mathbb{R}^k$ is said to be invex respecting a vector function $v : K \times K \to \mathbb{R}^k$ if the inclusion

$$x + t\, v(y, x) \in K \tag{2.1}$$

holds for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

The invex set K contains the line segment between points x and $x + v(y, x)$ for every pair of points x and y of K, because

$$x + tv(y, x) = (1 - t)x + t\,(x + v(y, x))\,. \tag{2.2}$$

Any subset $K \subseteq \mathbb{R}^k$ is invex respecting the vector function v identically equal to null vector. Besides the term vector function, we will also use the term mapping.

Every convex set K is invex respecting the mapping $v(y, x) = y - x$. The following example demonstrates that the reverse statement is not true.

Example 2.2. The set $K = (-\infty, -a] \cup [a, +\infty) \subset \mathbb{R}$, where $a \geq 0$, is invex respecting the mapping $v(y, x) = x$ because it contains the combinations

$$x + tv(y, x) = (1 + t)x \tag{2.3}$$

for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

Definition 2.3. Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a vector function $v : K \times K \to \mathbb{R}^k$. A function $f : K \to \mathbb{R}$ is said to be preinvex respecting v if the inequality

$$f\,(x + tv(y, x)) \leq (1 - t)f(x) + tf(y) \tag{2.4}$$

holds for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

Every convex function f on the convex set K is preinvex respecting the mapping $v(y, x) = y - x$. As the following example (see [11]) shows, the converse is not true.

Example 2.4. The function $f(x) = -|x|$ observed on the set $K = \mathbb{R}$ is preinvex respecting the mapping

$$v(y, x) = \begin{cases} y - x, & xy \geq 0, \\ x - y, & xy < 0. \end{cases} \tag{2.5}$$

In the case $xy \geq 0$, we obtain formula (2.4) with the sign of equality. In the case $xy < 0$, we obtain formula (2.4).

3. Inequality for invex combinations

Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping $v : K \times K \to \mathbb{R}^k$, let $x_1, x_2, x_3 \in K$ be points, and let $t_1, t_2 \in [0, 1]$ be coefficients. Then the expression

$$(x_1, x_2, x_3; t_1, t_2)_v = x_1 + t_1 v(x_2, x_1) + t_2 v\,(x_3, x_1 + t_1 v(x_2, x_1)) \tag{3.1}$$

can be called the invex combination respecting v of points x_1, x_2, x_3 and coefficients t_1, t_2. The related set of invex combinations,

$$\operatorname{inv}(x_1, x_2, x_3)_v = \{x_1 + t_1 v(x_2, x_1) + t_2 v\,(x_3, x_1 + t_1 v(x_2, x_1)) : t_1, t_2 \in [0, 1]\}\,, \tag{3.2}$$

can be called the invex hull respecting v of points x_1, x_2, x_3. The combination in (3.1) is in K because its subcombination $x_1 + t_1 v(x_2, x_1)$ is in K. So, the invex hull in (3.2) is a subset of K.

Visual presentation of invex combinations of points x_1, x_2, x_3 can be seen in Figure 1. Combinations are marked by pairs of coefficients t_1 and t_2, thus the pair $(0, 1)$ represents the combination $x_1 + v(x_3, x_1)$.

Doing a double application of formula (2.4) to the invex combination in (3.1), we obtain the inequality

$$\begin{aligned} f\left((x_1, x_2, x_3; t_1, t_2)_v\right) &= f\left(x_1 + t_1 v(x_2, x_1) + t_2 v\left(x_3, x_1 + t_1 v(x_2, x_1)\right)\right) \\ &\le (1-t_1)(1-t_2) f(x_1) + t_1(1-t_2) f(x_2) + t_2 f(x_3). \end{aligned} \tag{3.3}$$

The sum of coefficients of the last term of formula (3.3) is equal to 1, so the last term is the convex combination of function values $f(x_1)$, $f(x_2)$ and $f(x_3)$.

The generalization of the inequality in formula (3.3) can be achieved in the following manner. If we introduce the abbreviation

$$y_n = (x_1, \dots, x_n; t_1, \dots, t_{n-1})_v, \tag{3.4}$$

then we have the recursive formula

$$y_1 = x_1, \;\; y_n = y_{n-1} + t_{n-1} v(x_n, y_{n-1}) \text{ for } n \ge 2. \tag{3.5}$$

Relying on the above recursive formula, we can prove the following generalization.

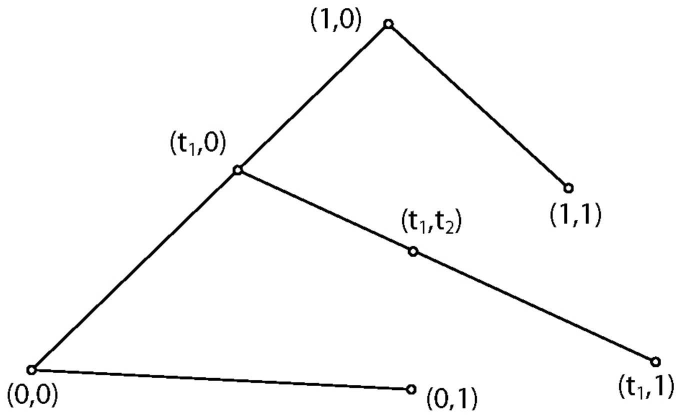

Figure 1: Invex combinations of three points

Theorem 3.1. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping $v : K \times K \to \mathbb{R}$, and let $f : K \to \mathbb{R}$ be a preinvex function respecting v.*

Then the inequality

$$f\left((x_1, \dots, x_n; t_1, \dots, t_{n-1})_v\right) \le \sum_{i=1}^{n} t_{i-1} \prod_{j=i}^{n-1} (1 - t_j) f(x_i) \tag{3.6}$$

holds for each n-tuple of points $x_1, \dots, x_n \in K$, the initial coefficient $t_0 = 1$, and each $(n-1)$-tuple of coefficients $t_1, \dots, t_{n-1} \in [0, 1]$.

Proof. The proof can be carried out by applying mathematical induction to the positive integer n as the number of points x_i.

If $n = 1$, the inequality in formula (3.6) is reduced to the trivial inequality $f(x_1) \le f(x_1)$. So, the basis step is confirmed.

To prove the inductive step, we suppose that the inequality in formula (3.6) is true for all positive integers that are less than or equal to $n-1$. Regarding the integer $n \geq 2$, we have

$$\begin{aligned} f(y_n) &= f\left(y_{n-1} + t_{n-1} v(x_n, y_{n-1})\right) \\ &\leq (1-t_{n-1}) f(y_{n-1}) + t_{n-1} f(x_n) \\ &\leq (1-t_{n-1}) \sum_{i=1}^{n-1} t_{i-1} \textstyle\prod_{j=i}^{n-2} (1-t_j) f(x_i) + t_{n-1} f(x_n) \\ &= \sum_{i=1}^{n} t_{i-1} \textstyle\prod_{j=i}^{n-1} (1-t_j) f(x_i) \end{aligned} \tag{3.7}$$

concluding the inductive step. □

The right term of formula (3.6) is the convex combination of all function values $f(x_1), \ldots, f(x_n)$.

4. Symmetric forms of basic inequalities

We deal with the two most significant inequalities (the Jensen and Hermite-Hadamard) in order to make them applicable to preinvex functions. More specifically, we will present the Jensen (see [6]), Jensen-Mercer (see [7]) and Hermite-Hadamard (see [2, 3]) inequality concerning a convex function on the line segment in $\mathbb{R}^n$. An interesting historical story about the Hermite-Hadamard inequality can be read in [9]. Some generalizations and applications concerning the Hermite-Hadamard inequality can be found in [5, 12, 13] and [14].

Let $a \neq b$ be a pair of points in $\mathbb{R}^k$. The line segment between points a and b will be written as the convex hull

$$\mathrm{conv}\{a,b\} = \{\alpha a + \beta b : \alpha, \beta \in [0,1],\ \alpha + \beta = 1\}. \tag{4.1}$$

Each point $x \in \mathrm{conv}\{a,b\}$ can be presented by the unique binomial convex combination

$$x = \alpha a + \beta b, \tag{4.2}$$

where (using the norm $\| \, \|$)

$$\alpha = \frac{\|b-x\|}{\|b-a\|},\ \ \beta = \frac{\|x-a\|}{\|b-a\|}. \tag{4.3}$$

Regarding the Jensen and Jensen-Mercer inequality, we use convex combinations of points $x_i \in \mathrm{conv}\{a,b\}$, that is, sums $\sum_{i=1}^n \lambda_i x_i$ where coefficients λ_i are nonnegative and their sum is equal to 1.

Lemma 4.1. *Let $a \neq b$ be a pair of points in $\mathbb{R}^k$, and let $\sum_{i=1}^n \lambda_i x_i$ be a convex combination of points $x_i \in \mathrm{conv}\{a,b\}$.*

Then every convex function $f : \mathrm{conv}\{a,b\} \to \mathbb{R}$ satisfies the inequalities

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i) \tag{4.4}$$

and

$$f\left(a+b-\sum_{i=1}^n \lambda_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^n \lambda_i f(x_i). \tag{4.5}$$

Using the secant line passing through the graph points $A(a, f(a))$ and $B(b, f(b))$, the Jensen inequality can be extended to the right side, and so enlarged can be written in the symmetric form. Using the Jensen inequality, the Jensen-Mercer inequality can be refined by inserting the intermediate term.

Theorem 4.2. *Let $a \neq b$ be a pair of points in $\mathbb{R}^k$, let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in \operatorname{conv}\{a, b\}$, and let $\sum_{i=1}^{n} \lambda_i x_i = \alpha a + \beta b$ be its unique convex combination of segment endpoints a and b.*

Then every convex function $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$ satisfies the double inequalities

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b) \tag{4.6}$$

and

$$f\left(a + b - \sum_{i=1}^{n} \lambda_i x_i\right) \leq (1-\alpha) f(a) + (1-\beta) f(b) \leq f(a) + f(b) - \sum_{i=1}^{n} \lambda_i f(x_i). \tag{4.7}$$

The Hermite-Hadamard inequality can be carried out from the inequality in formula (4.6) by using the integral method. That procedure is suitable to perform on the invex line segments, where we will do it.

Lemma 4.3. *Let $a \neq b$ be a pair of points in $\mathbb{R}^k$.*

Then every convex function $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\|b-a\|} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{4.8}$$

Using the segment equation $x = a + t(b - a)$ through the real parameter $t \in [0, 1]$, the middle term of (4.8) can be expressed by

$$\int_0^1 f\big(a + t(b-a)\big) dt. \tag{4.9}$$

To refine the Hermite-Hadamard inequality, we first determine three midpoints with respect to segment endpoints and any segment point. Then we combine the application of the Jensen and Hermite-Hadamard inequality.

Theorem 4.4. *Let $a \neq b$ be a pair of points in $\mathbb{R}^k$, let $c \in \operatorname{conv}\{a, b\}$ be a segment point, and let $c = (1-\alpha)a + (1-\beta)b$ be its unique convex combination of segment endpoints a and b.*

Then every convex function $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$ satisfies the series of inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \alpha f\left(\frac{a+c}{2}\right) + \beta f\left(\frac{b+c}{2}\right) \\ &\leq \frac{1}{\|b-a\|} \int_a^b f(x)\, dx \\ &\leq \frac{\alpha f(a) + \beta f(b) + f(c)}{2} \leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{4.10}$$

Proof. If $c \in \{a, b\}$, then the inequality in equation (4.10) is actually reduced to the Hermite-Hadamard inequality in equation (4.8).

Suppose that $c \notin \{a, b\}$. Using the assumption $c = (1-\alpha)a + (1-\beta)b$, we get the convex combination equality

$$\frac{a+b}{2} = \alpha \frac{a+c}{2} + \beta \frac{b+c}{2}. \tag{4.11}$$

Applying the convexity of the function f to the combination at the right side of equation (4.11), and the left-hand side of the Hermite-Hadamard inequality to midpoints $(a+c)/2$ and $(b+c)/2$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \alpha f\left(\frac{a+c}{2}\right)+\beta f\left(\frac{b+c}{2}\right) \\ &\le \frac{1}{\|b-a\|}\int_a^c f(x)\,dx+\frac{1}{\|b-a\|}\int_c^b f(x)\,dx \\ &= \frac{1}{\|b-a\|}\int_a^b f(x)\,dx \end{aligned} \tag{4.12}$$

proving the first double inequality of formula (4.10).

Now we will use the convex combination equality

$$\frac{\alpha}{2}a+\frac{\beta}{2}b+\frac{1}{2}c=\frac{1}{2}a+\frac{1}{2}b \tag{4.13}$$

in terms of Theorem 4.2. Applying the right-hand side of the Hermite-Hadamard inequality to segments $\operatorname{conv}\{a,c\}$ and $\operatorname{conv}\{c,b\}$, and the right-hand side of formula (4.6) to combinations of equation (4.13), we obtain

$$\begin{aligned} \frac{1}{\|b-a\|}\int_a^b f(x)\,dx &= \frac{1}{\|b-a\|}\int_a^c f(x)\,dx+\frac{1}{\|b-a\|}\int_c^b f(x)\,dx \\ &\le \frac{\alpha}{2}f(a)+\frac{\beta}{2}f(b)+\frac{1}{2}f(c) \\ &\le \frac{1}{2}f(a)+\frac{1}{2}f(b) \end{aligned} \tag{4.14}$$

proving the last double inequality of formula (4.10). □

The coefficients α and β used in the previous theorem are as follows

$$\alpha=\frac{\|c-a\|}{\|b-a\|},\ \beta=\frac{\|b-c\|}{\|b-a\|}. \tag{4.15}$$

5. Application to preinvex functions

The following lemma explores invex line segments.

Lemma 5.1. *Let $a,b\in\mathbb{R}^k$ be a pair of points, and let the segment* $\operatorname{conv}\{a,b\}$ *be invex respecting a mapping v.*

Then $v(y,x)$ is collinear with $b-a$ for every pair of segment points x and y.

Proof. Take a pair of points $x,y\in\operatorname{conv}\{a,b\}$, suppose that the $x=a+t_1(b-a)$, and take a coefficient $t\in(0,1]$. Since the point

$$x+tv(y,x)=a+t_1(b-a)+tv(y,x) \tag{5.1}$$

belongs to $\operatorname{conv}\{a,b\}$ by assumption, then it follows that

$$a+t_1(b-a)+tv(y,x)=a+t_2(b-a), \tag{5.2}$$

and consequently

$$v(y,x)=\frac{t_2-t_1}{t}(b-a), \tag{5.3}$$

which proves the required collinearity. □

If the conditions of the above lemma are satisfied, then $v(y, x)$ is collinear with $y - x$ for every pair of segment points x and y.

Corollary 5.2. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping v, let $a, b \in K$ be a pair of points such that the generated segment* $\operatorname{conv}\{a, a + v(b, a)\}$ *is invex respecting v.*

Then $v(y, x)$ is collinear with $v(b, a)$ for every pair of segment points x and y.

If K is invex respecting v, and if $a, b \in K$, then the generated segment $\operatorname{conv}\{a, a + v(b, a)\}$ is not necessarily invex respecting v. The requirement that the generated segments of the invex set be invex provides the condition introduced in [8].

Definition 5.3. Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a vector function $v : K \times K \to \mathbb{R}^k$. It is said that the function v satisfies condition C if the equalities

$$v\left(x, x + tv(y, x)\right) = -tv(y, x), \tag{5.4}$$

$$v\left(y, x + tv(y, x)\right) = (1 - t)v(y, x) \tag{5.5}$$

hold for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

A consequence of condition C is the equality

$$v\left(x + t_2 v(y, x), x + t_1 v(y, x)\right) = (t_2 - t_1)v(y, x) \tag{5.6}$$

which holds for all points $x, y \in K$ and coefficients $t_1, t_2 \in [0, 1]$.

Assuming the presence of condition C, the following theorem shows where the preinvexity coincides with convexity.

Theorem 5.4. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping v that satisfies condition C, and let $f : K \to \mathbb{R}$ be a preinvex function respecting v.*

Then the function f is convex on the generated segment $\operatorname{conv}\{a, a + v(b, a)\}$ *for every pair of points $a, b \in K$.*

Proof. Let $a, b \in K$ be a pair of set points, let $x, y \in \operatorname{conv}\{a, a + v(b, a)\}$ be a pair of segment points, and let $t \in [0, 1]$ be a coefficient. We will verify the equality of combinations $(1 - t)x + ty$ and $x + tv(y, x)$. Using the representations

$$x = a + t_1 v(b, a),\ \ y = a + t_2 v(b, a)$$

via formula (5.6), we get

$$\begin{aligned} (1 - t)x + ty &= (1 - t)\left(a + t_1 v(b, a)\right) + t\left(a + t_2 v(b, a)\right) \\ &= a + t_1 v(b, a) + t(t_2 - t_1)v(b, a) \\ &= a + t_1 v(b, a) + tv\left(a + t_2 v(b, a), a + t_1 v(b, a)\right) \\ &= x + tv(y, x). \end{aligned} \tag{5.7}$$

Taking into account the above equality, and applying the preinvexity of f to the invex combination $x + tv(y, x)$, we obtain the inequality

$$f\left((1 - t)x + ty\right) = f\left(x + tv(y, x)\right) \le (1 - t)f(x) + tf(y) \tag{5.8}$$

which proves the convexity of f on the segment $\operatorname{conv}\{a, a + v(b, a)\}$. $\square$

Formula (5.7) specifies the mapping v, it follows that $v(y, x) = y - x$ for all points x and y of the invex generated segment $\operatorname{conv}\{a, a + v(b, a)\}$.

The type of convexity given in Theorem 5.4 enables us to apply the convex function inequalities to preinvex functions. First and foremost, it refers to fundamental inequalities for convex functions on the line segment which are prepared in the previous section.

Versions of the Jensen and Jensen-Mercer inequality for preinvex functions are the first that follow.

Corollary 5.5. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping v that satisfies condition C, and let $f : K \to \mathbb{R}$ be a preinvex function respecting v. Let $\lambda_1, \ldots, \lambda_n \in [0, 1]$ be coefficients such that $\sum_{i=1}^n \lambda_i = 1$, let $t_1, \ldots, t_n \in [0, 1]$ be coefficients, and let $t = \sum_{i=1}^n \lambda_i t_i$.*

Then the inequalities

$$f(a + tv(b, a)) \le \sum_{i=1}^n \lambda_i f(a + t_i v(b, a)) \le (1 - t)f(a) + tf(a + v(b, a)) \tag{5.9}$$

and

$$\begin{aligned} f\left(a + \sum_{i=1}^n \lambda_i (1 - t_i) v(b, a)\right) &\le tf(a) + (1 - t)f(a + v(b, a)) \\ &\le f(a) + f(a + v(b, a)) - \sum_{i=1}^n \lambda_i f(a + t_i v(b, a)) \end{aligned} \tag{5.10}$$

hold for every pair of points $a, b \in K$.

Proof. Let a and b be a pair of points of K, and let $I = \operatorname{conv}\{a, a + v(b, a)\}$ be the generated segment with endpoints a and $a + v(b, a)$. The function f is convex on the segment I by Theorem 5.4.

Let us prove the inequality in formula (5.9). Since the points $a + t_i v(b, a)$ belong to the segment I, their convex combination

$$\begin{aligned} \sum_{i=1}^n \lambda_i (a + t_i v(b, a)) &= \sum_{i=1}^n \lambda_i a + \sum_{i=1}^n \lambda_i t_i v(b, a) = a + tv(b, a) \\ &= (1 - t)a + t(a + v(b, a)) \end{aligned} \tag{5.11}$$

also belongs to I. Respecting the above equalities, and applying formula (4.6) by using a as a, $a + v(b, a)$ as b, $a + t_i v(b, a)$ as x_i, and t as β, we obtain the inequality in formula (5.9).

The inequality in formula (5.10) can be proved similarly by using the extended form of the Jensen-Mercer inequality in formula (4.7). □

Since

$$(1 - t)f(a) + tf(a + v(b, a)) \le (1 - t)f(a) + tf(b), \tag{5.12}$$

the inequality in formula (5.9) can be extended to the right side. If $v(b, a) = 0$, the inequality in formula (5.9) is reduced to $f(a) \le f(a) \le f(a)$.

The left-hand side of the inequality in formula (5.9) representing the Jensen inequality for preinvex functions can be written in the form

$$f\left(\sum_{i=1}^n \lambda_i (a + t_i v(b, a))\right) \le \sum_{i=1}^n \lambda_i f(a + t_i v(b, a)). \tag{5.13}$$

Implementing the integral method through the reflection moment applied to formula (5.9), we obtain the Hermite-Hadamard inequality for preinvex functions as follows.

Corollary 5.6. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping v that satisfies condition C, and let $f : K \to \mathbb{R}$ be a preinvex function respecting v.*

Then the double inequality

$$f\left(a+\frac{v(b,a)}{2}\right) \leq \frac{1}{\|v(b,a)\|}\int_a^{a+v(b,a)} f(x)\,dx \leq \frac{f(a)+f\left(a+v(b,a)\right)}{2} \tag{5.14}$$

holds for every pair of points $a, b \in K$ such that $v(b,a) \neq 0$.

Proof. We utilize formula (5.9) with the following elements. Take a positive integer n, and select coefficients $\lambda_{ni} = 1/n$ and $t_{ni} = i/n$. Then the coefficient

$$t_n = \sum_{i=1}^{n} \lambda_{ni} t_{ni} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{n+1}{2n}, \tag{5.15}$$

and the middle term is

$$\sum_{i=1}^{n} \frac{1}{n} f\left(a+\frac{i}{n}v(b,a)\right) = \frac{1}{\|v(b,a)\|}\sum_{i=1}^{n} \frac{\|v(b,a)\|}{n} f\left(a+i\frac{v(b,a)}{n}\right). \tag{5.16}$$

Sending n to infinity, we have that the coefficient t_n approaches $1/2$, the segment point $x_{n1} = a + v(b,a)/n$ approaches a and the segment point $x_{nn} = a + v(b,a)$ approaches $a + v(b,a)$, and therefore the inequality in adjusted formula (5.9) approaches the Hermite-Hadamard inequality in formula (5.14). □

The middle term of the inequality in formula (5.14) can be replaced with

$$\int_0^1 f\left(a + tv(b,a)\right) dt. \tag{5.17}$$

The type of the Hermite-Hadamard inequality involving the Riemann-Liouville integrals and gamma function were considered in [4], wherein some results were achieved for positive preinvex functions on the open invex set $K \subseteq \mathbb{R}$. The results regarding the Hermite-Hadamard inequality for functions whose absolute values of derivatives are preinvex were obtained in [1].

A refinement of the inequality in formula (5.14) is based on formula (4.10).

Corollary 5.7. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a mapping v that satisfies condition C, and let $f : K \to \mathbb{R}$ be a preinvex function respecting v.*

Then the series of inequalities

$$\begin{aligned} f\left(a+\frac{1}{2}v(b,a)\right) &\leq tf\left(a+\frac{t}{2}v(b,a)\right) + (1-t)f\left(a+\frac{1+t}{2}v(b,a)\right) \\ &\leq \frac{1}{\|v(b,a)\|}\int_a^{a+v(b,a)} f(x)\,dx \\ &\leq \frac{tf(a)+(1-t)f\left(a+v(b,a)\right)+f\left(a+tv(b,a)\right)}{2} \leq \frac{f(a)+f\left(a+v(b,a)\right)}{2} \end{aligned} \tag{5.18}$$

holds for every pair of points $a, b \in K$ such that $v(b,a) \neq 0$, and every coefficient $t \in [0,1]$.

Proof. The inequality in formula (4.10) should be used with a as a, $a + v(b,a)$ as b, $a + tv(b,a)$ as c, and t as α. □

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