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Bifurcation analysis for a ratio-dependent predator-prey system with multiple delays

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Abstract

In this paper, we consider a ratio-dependent predator-prey system with multiple delays where the dynamics are logistic with the carrying capacity proportional to prey population. By choosing the sum τ of two delays as the bifurcation parameter, the stability of the positive equilibrium and the existence of Hopf bifurcation are investigated. Furthermore, the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem for functional differential equations. Finally, some numerical simulations are carried out for illustrating the theoretical results.

Keywords: Ratio-dependent, delay, Hopf bifurcation, center manifold, periodic solutions.
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1. Introduction

The study of the dynamics of a predator-prey system is one of the dominant subjects in ecology and mathematical ecology due to its universal existence and importance. As to our knowledge, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. Thus, time delays of one type or another have been incorporated into mathematical models of population dynamics due

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to maturation time, capturing time or other reasons. In the last decades, many authors have explored the dynamics of systems with time delay and many interesting results have been obtained [4, 5, 11–18, 20, 23, 24].

Recently, Zhou et al. [25] considered the following system

$$\begin{cases} \frac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t)N(t), \\ \frac{dP(t)}{dt} = P(t)(r_2 - \theta \frac{P(t)}{N(t)}), \end{cases} \tag{1.1}$$

where r_1, r_2, ε and θ are positive constants. $N(t)$ and $P(t)$ can be interpreted as the densities of prey and predator populations at time t, respectively. Zhou et al. first studied the stability conditions of the equilibrium point for the system and then by introducing so called Allee effect [3, 6, 7] in different forms, they investigated the impact of this effect on the dynamics of this predator-prey system.

In system (1.1), there is no delay. In [1], Celik incorporated the delay into the system, then system (1.1) becomes

$$\begin{cases} \frac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t)N(t), \\ \frac{dP(t)}{dt} = P(t)(r_2 - \theta \frac{P(t-\tau)}{N(t)}), \end{cases} \tag{1.2}$$

where $\tau \geq 0$ denotes the delay time for the predator density. In system (1.2) predator density is logistic with time delay and the carrying capacity proportional to prey density.

In [2], Celik analyzed the following ratio-dependent delayed predator-prey system

$$\begin{cases} \frac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t)N(t), \\ \frac{dP(t)}{dt} = P(t)(r_2 - \theta \frac{P(t)}{N(t-\tau)}), \end{cases} \tag{1.3}$$

where $\tau \geq 0$ denotes the delay time for the prey density.

Inspired by system (1.2) and system (1.3), we easily obtained that besides the hunting delay for predator to prey, it is reasonable to incorporate delays into the prey density in the denominator of the ratio in the dynamics of predator and the predator density in the numerator. Based on this fact, we analyse the following more complicated ratio-dependent predator-prey system with multiple delays

$$\begin{cases} \frac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t-\tau_1)N(t), \\ \frac{dP(t)}{dt} = P(t)(r_2 - \theta \frac{P(t-\tau)}{N(t-\tau_2)}), \end{cases} \tag{1.4}$$

where $\tau_1 \geq 0$ denotes the hunting delay of predator to prey, $\tau_2 \geq 0$ denotes the delay time for the prey density, and $\tau \geq 0$ is the delay time for the predator density. We address the question how the time delay in predator density of ratio in the dynamics of predator affects the dynamical properties. Another interesting question is that what the effect is if the time delay not only be incorporated into prey density in the denominator of the ratio in the dynamics of predator but also be incorporated into the predator density in the numerator? Shall these delays destabilize the equilibrium and change the dynamical behavior of the system (1.4)? So the aim of this paper is to study the dynamical behavior of system (1.4), for which we investigate the stability of delayed Leslie-Gower predator-prey system (1.4) and analyse how the time delay τ effects the dynamics of this system. We would like to mention that the bifurcation in a predator-prey system with a single or multiple delays had been investigated by many researchers [9, 10, 19, 21, 22]. However, to the best of our knowledge, for this case, few results for system (1.4) have been obtained. Therefore, the research of this paper is worth considering.

This paper is organized as follows. In Section 2, we obtain the existence of the positive equilibrium. By analyzing the characteristic equation of the linearized system at positive equilibrium, we discuss the stability of the positive and the existence of the Hopf bifurcations occurring at the positive equilibrium. In Section 3, the formula determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions on the center manifold are obtained by using the normal form theory and the center manifold theorem due to Hassard et al. [8]. In order to verify our theoretical prediction, some numerical simulations are also included in Section 4.

2. Stability of positive equilibrium and existence of local Hopf bifurcations

For convenience, we introduce new variables $x(t) = N(t)$, $y(t) = P(t-\tau_1)$, and assume that $\tau = \tau_1+\tau_2$, so that system (1.4) can be written as the following system with a single delay:

$$\begin{cases} \frac{dx(t)}{dt} = r_1 x(t) - \varepsilon x(t)y(t), \\ \frac{dy(t)}{dt} = y(t)(r_2 - \theta\frac{y(t-\tau)}{x(t-\tau)}). \end{cases} \tag{2.1}$$

The positive equilibrium of the system (2.1) is $E^* := (x_0^*, y_0^*)$, where $x_0^* = \frac{r_1\theta}{r_2\varepsilon}$ and $y_0^* = \frac{r_1}{\varepsilon}$. To study the stability of the positive equilibrium E^*, we first use the linear transformation $\tilde{u}_1(t) = x(t) - x_0^*, \tilde{u}_2(t) = y(t) - y_0^*$, where $\tilde{u}_1(t) \ll 1$ and $\tilde{u}_2(t) \ll 1$, for which system (2.1) yields

$$\begin{cases} \frac{d\tilde{u}_1(t)}{dt} = (x_0^* + \tilde{u}_1(t))(r_1 - \varepsilon(y_0^* + \tilde{u}_2(t))), \\ \frac{d\tilde{u}_2(t)}{dt} = (y_0^* + \tilde{u}_2(t))(r_2 - \theta\frac{y_0^*+\tilde{u}_2(t-\tau)}{x_0^*+\tilde{u}_1(t-\tau)}). \end{cases} \tag{2.2}$$

By linearizing the system (2.2) around $(0,0)$ and using relations $r_1 - \varepsilon y_0^* = 0$ and $r_2 - \theta\frac{y_0^*}{x_0^*} = 0$, lead the following system

$$\begin{cases} \frac{d\tilde{u}_1(t)}{dt} = -\varepsilon x_0^* \tilde{u}_2(t), \\ \frac{d\tilde{u}_2(t)}{dt} = \theta(\frac{y_0^*}{x_0^*})^2\tilde{u}_1(t-\tau) - \theta(\frac{y_0^*}{x_0^*})\tilde{u}_2(t-\tau). \end{cases} \tag{2.3}$$

The characteristic equation of system (2.3) is

$$\lambda^2 + \theta(\frac{y_0^*}{x_0^*})e^{-\lambda\tau}\lambda + \theta\varepsilon\frac{(y_0^*)^2}{x_0^*}e^{-\lambda\tau} = 0. \tag{2.4}$$

Let $a = \theta(\frac{y_0^*}{x_0^*}) > 0$ and $b = \theta\varepsilon\frac{(y_0^*)^2}{x_0^*} > 0$, the Eq. (2.4) becomes

$$\lambda^2 + ae^{-\lambda\tau}\lambda + be^{-\lambda\tau} = 0. \tag{2.5}$$

Lemma 2.1. *The two roots $\lambda_{1,2} = \frac{-a\pm\sqrt{a^2-4b}}{2}$ of system (2.5) with $\tau = 0$ is asymptotically stable.*

Next, we examine when the characteristic equation has pairs of purely imaginary roots. For $\tau > 0$, if $i\omega(\omega > 0)$ is a root of (2.5), then ω should satisfy the following equations

$$\begin{aligned} &-\omega^2 + a\omega sin(\omega\tau) + bcos(\omega\tau) = 0, \\ &a\omega cos(\omega\tau) - bsin(\omega\tau) = 0. \end{aligned} \tag{2.6}$$

Thus

$$sin(\omega\tau) = \frac{a\omega^3}{a^2\omega^2 + b^2}, \qquad cos(\omega\tau) = \frac{b\omega^2}{a^2\omega^2 + b^2}. \tag{2.7}$$

Squaring and adding both the equations of (2.6), we have

$$a^2\omega^6 + (b^2 - a^4)\omega^4 - 2a^2b^2\omega^2 - b^4 = 0. \tag{2.8}$$

Let $z = \omega^2,\ a_1 = a^2,\ b_1 = b^2 - a^4,\ c_1 = -2a^2b^2,\ d_1 = -b^4$, Eq. (2.8 becomes

$$a_1z^3 + b_1z^2 + c_1z + d_1 = 0. \tag{2.9}$$

Since $d_1 < 0$, it is easy to see that the Eq. (2.9) has one positive root z_0, then Eq. (2.8) has a positive real root $\omega_0 = \sqrt{z_0}$. From Eq. (2.7), we have

$$tan(\omega\tau) = \frac{a}{b}\omega. \tag{2.10}$$

Define

$$\tau_k = \frac{1}{\omega_0}(\arctan(\frac{a\omega_0}{b}) + k\pi),\ k = 0, 1, 2, \cdots. \tag{2.11}$$

Therefore, Eq. (2.5) has a pair of imaginary roots $\pm i\omega_0$.

Let $\lambda_k(\tau) = \alpha_k(\tau) + i\omega_k(\tau)$ be the root of Eq. (2.5) near $\tau = \tau_k$ satisfying $\alpha_k(\tau_k) = 0, \quad \omega_k(\tau_k) = \omega_0,\ (k = 0, 1, 2, \cdots)$. Then we have the following transversality condition.

Lemma 2.2. $\frac{dRe\lambda_k(\tau)}{d\tau} > 0$

Proof. Differentiating both sides of Eq. (2.5) with respect to τ, we get

$$2\lambda\frac{d\lambda}{d\tau} + ae^{-\lambda\tau}\frac{d\lambda}{d\tau} + a\lambda e^{-\lambda\tau}(-\tau\frac{d\lambda}{d\tau} - \lambda) + be^{-\lambda\tau}(-\tau\frac{d\lambda}{d\tau} - \lambda) = 0,$$

that is

$$\begin{aligned}(\frac{d\lambda}{d\tau})^{-1} &= \frac{2\lambda + ae^{-\lambda\tau} - a\lambda\tau e^{-\lambda\tau} - b\tau e^{-\lambda\tau}}{a\lambda^2 e^{-\lambda\tau} + b\lambda e^{-\lambda\tau}} \\ &= \frac{2\lambda + ae^{-\lambda\tau}}{a\lambda^2 e^{-\lambda\tau} + b\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda + ae^{-\lambda\tau}}{-\lambda^3} - \frac{\tau}{\lambda}.\end{aligned}$$

Thus

$$\begin{aligned}Re(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=i\omega_0} &= Re(\frac{2(i\omega_0) + a(cos(\omega_0\tau_k) - isin(\omega_0\tau_k))}{-(i\omega_0)^3} - \frac{\tau_k}{i\omega_0}) \\ &= \frac{a^2\omega_0^2 + 2b^2}{\omega_0^2(a^2\omega_0^2 + b^2)} > 0.\end{aligned}$$

This completes the proof of Lemma 2.2. □

Summarizing the above results, we have the following theorem on stability and Hopf bifurcation of system (2.1).

Theorem 2.3. For system (2.1),
(i) If $\tau \in [0, \tau_0)$, then the equilibrium $E^* : (x_0^*, y_0^*)$ of system (2.1) is asymptotically stable.
(ii) If $\tau > \tau_0$, then the equilibrium $E^* : (x_0^*, y_0^*)$ of system (2.1) is unstable.
(iii) $\tau_k, (k = 0, 1, 2, \cdots)$ are Hopf bifurcation values for system (2.1).

3. Direction of Hopf bifurcations and stability of bifurcating periodic solutions

In the previous section, we have already obtained that, under certain conditions, the system (2.1) can undergo Hopf bifurcation at the positive equilibrium $E^* : (x_0^*, y_0^*)$ when τ takes some critical values $\tau = \tau_k, (k = 0, 1, 2, \cdots)$. In this section, by employing the normal form theory and the center manifold Theorem introduced by Hassard et al. [8], we shall present the formula determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions of (2.1).

Let $u_i(t) = \tilde{u}_i(\tau t)$, $\tau = \tau_k + \mu$, $\mu \in R$, then $\mu = 0$ is the Hopf bifurcation value for system (2.1), dropping the bars for simplification of notations, system (2.1) becomes the following functional differential equation in $C = C([-1, 0], R^2)$,

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t))^T \in R^2$, and $L_\mu : C \to R^2, f : R \times C \to R^2$ are given by

$$L_\mu(\phi) = (\tau_k + \mu)\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_k + \mu)\begin{pmatrix} 0 & 0 \\ G & S \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \tag{3.2}$$

and

$$f(\mu, \phi) = (\tau_k + \mu)\begin{pmatrix} a_{11}\phi_1(0)\phi_2(0) \\ a_{21}\phi_1^2(-1) + a_{22}\phi_1(-1)\phi_2(0) + a_{23}\phi_1(-1)\phi_2(-1) + a_{24}\phi_2(0)\phi_2(-1) \end{pmatrix}, \tag{3.3}$$

where

$$N = -\varepsilon x_0^*, \qquad G = \theta(\frac{y_0^*}{x_0^*})^2, \qquad S = -\theta(\frac{y_0^*}{x_0^*}), \qquad a_{11} = -\varepsilon,$$

$$a_{21} = -\theta\frac{(y_0^*)^2}{(x_0^*)^3}, \qquad a_{22} = \frac{\theta y_0^*}{(x_0^*)^2}, \qquad a_{23} = \frac{\theta y_0^*}{(x_0^*)^2}, \qquad a_{24} = \frac{-\theta}{x_0^*}.$$

By the Riesz representation Theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$, such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \qquad for \quad \phi \in C. \tag{3.4}$$

In fact,we can choose

$$\eta(\theta,\mu) = (\tau_k+\mu)\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\delta(\theta) + (\tau_k+\mu)\begin{pmatrix} 0 & 0 \\ G & S \end{pmatrix}\delta(\theta+1), \tag{3.5}$$

where δ is the Dirac delta function. For $\phi \in C([-1,0],R^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (3.1) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.6}$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$. For $\psi \in C^1([0,1],(R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} \psi(-t)d\eta(t,0), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.7}$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of A^*. We first need to compute the eigenvectors of $A(0)$ and A^* corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$ respectively. Suppose $q(\theta) = (1,q_1)^T e^{i\omega_0\tau_k\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_k$, then $A(0)q(0) = i\omega_0\tau_k q(0)$. From the definition of $A(0)$ and (3.2), (3.4), (3.5) we have

$$\tau_k\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}q(0) + \tau_k\begin{pmatrix} 0 & 0 \\ G & S \end{pmatrix}q(-1) = i\omega_0\tau_k q(0).$$

For $q(-1) = q(0)e^{-i\omega_0\tau_k}$, then we obtain

$$q_1 = \frac{i\omega_0}{N}.$$

Similarly, we can obtain the eigenvector $q^*(s) = D(1,q_1^*)e^{i\omega_0\tau_k s}$ of A^* corresponding to $-i\omega_0\tau_k$, where

$$q_1^* = \frac{iN}{\omega_0}.$$

In order to assure $\langle q^*(s), q(\theta)\rangle = 1$, we need to determine the value of D. By (3.7), we have

$$\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \bar{q}^*(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi \\
&= \bar{D}(1,\bar{q}_1^*)(1,q_1)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1,\bar{q}_1^*)e^{-i\omega_0\tau_k(\xi-\theta)}d\eta(\theta)(1,q_1)^T e^{i\omega_0\tau_k\xi}d\xi \\
&= \bar{D}\{1 + q_1\bar{q}_1^* - (1,\bar{q}_1^*)\int_{-1}^{0}\theta e^{i\omega_0\tau_k\theta}d\eta(\theta)(1,q_1)^T\} \\
&= \bar{D}\{1 + q_1\bar{q}_1^* + \tau_k(G\bar{q}_1^* + Sq_1\bar{q}_1^*)e^{-i\omega_0\tau_k}\}.
\end{aligned}$$

Therefore, we can choose D as

$$D = \frac{1}{1 + \bar{q}_1 q_1^* + \tau_k (G q_1^* + S \bar{q}_1 q_1^*) e^{i\omega_0 \tau_k}}.$$

Next we will compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (3.6) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t \rangle, \qquad W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{3.8}$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.9}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We only consider real solutions. For solutions $u_t \in C_0$ of (3.6). Since $\mu = 0$, we have

$$\dot{z}(t) = i\omega_0\tau_k z + \bar{q}^*(0) f(0, W(z,\bar{z},0) + 2Re\{zq(\theta)\}) \triangleq i\omega_0\tau_k z + \bar{q}^*(0) f_0(z,\bar{z}).$$

We rewrite this equation as

$$\dot{z}(t) = i\omega_0\tau_k z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = \bar{q}^*(0) f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.10}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} u_t(\theta) &= W(t,\theta) + 2Re\{z(t)q(\theta)\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1, q_1)^T e^{i\omega_0\tau_k\theta} z + (1, \bar{q}_1)^T e^{-i\omega_0\tau_k\theta}\bar{z} + \cdots. \end{aligned} \tag{3.11}$$

It follows together with (3.3) that

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0) f_0(z,\bar{z}) = \bar{q}^*(0) f(0, u_t) \\ &= \tau_k \bar{D}(1, \bar{q}_1^*) \begin{pmatrix} a_{11} u_{1t}(0) u_{2t}(0) \\ a_{21} u_{1t}^2(-1) + a_{22} u_{1t}(-1) u_{2t}(0) + a_{23} u_{1t}(-1) u_{2t}(-1) + a_{24} u_{2t}(0) u_{2t}(-1) \end{pmatrix} \\ &= \tau_k \bar{D}[a_{11} u_{1t}(0) u_{2t}(0) + a_{21}\bar{q}_1^* u_{1t}^2(-1) + a_{22}\bar{q}_1^* u_{1t}(-1) u_{2t}(0) \\ &\quad + a_{23}\bar{q}_1^* u_{1t}(-1) u_{2t}(-1) + a_{24}\bar{q}_1^* u_{2t}(0) u_{2t}(-1)] \\ &= \tau_k \bar{D} a_{11}\{W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z + \bar{z} + \cdots\} \\ &\quad \{W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + q_1 z + \bar{q}_1\bar{z} + \cdots\} \\ &\quad + \tau_k \bar{D} a_{21}\bar{q}_1^*\{W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_k} z + e^{i\omega_0\tau_k}\bar{z} + \cdots\} \\ &\quad \{W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_k} z + e^{i\omega_0\tau_k}\bar{z} + \cdots\} \\ &\quad + \tau_k \bar{D} a_{22}\bar{q}_1^*\{W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_k} z + e^{i\omega_0\tau_k}\bar{z} + \cdots\} \\ &\quad \{W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + q_1 z + \bar{q}_1\bar{z} + \cdots\} \end{aligned}$$

$$+ \tau_k \bar{D} a_{23} \bar{q}_1^* \{W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + e^{-i\omega_0\tau_k}z + e^{i\omega_0\tau_k}\bar{z} + \cdots\}$$
$$\{W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + q_1 e^{-i\omega_0\tau_k}z + \bar{q}_1 e^{i\omega_0\tau_k}\bar{z} + \cdots\}$$
$$+ \tau_k \bar{D} a_{24} \bar{q}_1^* \{W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + q_1 z + \bar{q}_1\bar{z} + \cdots\}$$
$$\{W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + q_1 e^{-i\omega_0\tau_k}z + \bar{q}_1 e^{i\omega_0\tau_k}\bar{z} + \cdots\}$$

Comparing the coefficients with (3.10), we have

$$\begin{aligned}
g_{20} &= 2\tau_k \bar{D} a_{11} q_1 + 2\tau_k \bar{D} a_{21} \bar{q}_1^* e^{-2i\omega_0\tau_k} + 2\tau_k \bar{D} a_{22} \bar{q}_1^* q_1 e^{-i\omega_0\tau_k} \\
&\quad + 2\tau_k \bar{D} a_{23} \bar{q}_1^* q_1 e^{-2i\omega_0\tau_k} + 2\tau_k \bar{D} a_{24} \bar{q}_1^* q_1^2 e^{-i\omega_0\tau_k}, \\
g_{11} &= 2\tau_k \bar{D} a_{11} Re\{q_1\} + 2\tau_k \bar{D} a_{21} \bar{q}_1^* + 2\tau_k \bar{D} a_{22} \bar{q}_1^* Re\{q_1 e^{i\omega_0\tau_k}\} \\
&\quad + 2\tau_k \bar{D} a_{23} \bar{q}_1^* Re\{q_1\} + 2\tau_k \bar{D} a_{24} \bar{q}_1^* Re\{q_1 \bar{q}_1 e^{i\omega_0\tau_k}\}, \\
g_{02} &= 2\tau_k \bar{D} a_{11} \bar{q}_1 + 2\tau_k \bar{D} a_{21} \bar{q}_1^* e^{2i\omega_0\tau_k} + 2\tau_k \bar{D} a_{22} \bar{q}_1^* \bar{q}_1 e^{i\omega_0\tau_k} \\
&\quad + 2\tau_k \bar{D} a_{23} \bar{q}_1^* q_1 e^{2i\omega_0\tau_k} + 2\tau_k \bar{D} a_{24} \bar{q}_1^* (\bar{q}_1)^2 e^{i\omega_0\tau_k}, \\
g_{21} &= \tau_k \bar{D} a_{11} \bar{q}_1 W_{20}^{(1)}(0) + 2\tau_k \bar{D} a_{11} q_1 W_{11}^{(1)}(0) + (2\tau_k \bar{D} a_{11} + 2\tau_k \bar{D} a_{22} \bar{q}_1^* e^{-i\omega_0\tau_k} \\
&\quad + 2\tau_k \bar{D} a_{24} \bar{q}_1^* q_1 e^{-i\omega_0\tau_k}) W_{11}^{(2)}(0) + (\tau_k \bar{D} a_{11} + \tau_k \bar{D} a_{22} \bar{q}_1^* e^{i\omega_0\tau_k} \\
&\quad + \tau_k \bar{D} a_{24} \bar{q}_1^* \bar{q}_1 e^{i\omega_0\tau_k}) W_{20}^{(2)}(0) + (2\tau_k \bar{D} a_{21} \bar{q}_1^* e^{i\omega_0\tau_k} + \tau_k \bar{D} a_{22} \bar{q}_1^* \bar{q}_1 \\
&\quad + \tau_k \bar{D} a_{23} \bar{q}_1^* \bar{q}_1 e^{i\omega_0\tau_k}) W_{20}^{(1)}(-1) + (4\tau_k \bar{D} a_{21} \bar{q}_1^* e^{-i\omega_0\tau_k} + 2\tau_k \bar{D} a_{22} \bar{q}_1^* q_1 \\
&\quad + 2\tau_k \bar{D} a_{23} \bar{q}_1^* q_1 e^{-i\omega_0\tau_k}) W_{11}^{(1)}(-1) + (2\tau_k \bar{D} a_{23} \bar{q}_1^* e^{-i\omega_0\tau_k} + 2\tau_k \bar{D} a_{24} \bar{q}_1^* q_1) W_{11}^{(2)}(-1) \\
&\quad + (\tau_k \bar{D} a_{23} \bar{q}_1^* e^{i\omega_0\tau_k} + \tau_k \bar{D} a_{24} \bar{q}_1^* \bar{q}_1) W_{20}^{(2)}(-1).
\end{aligned}$$

To determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. By (3.6) and (3.8), we have

$$\begin{aligned}
\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\
&= \begin{cases} A(0)W - 2Re\{\bar{q}^*(0) f_0 q(\theta)\}, & \theta \in [-1, 0) \\ A(0)W - 2Re\{\bar{q}^*(0) f_0 q(0)\} + f_0, & \theta = 0 \end{cases} \\
&\triangleq A(0)W + H(z, \bar{z}, \theta),
\end{aligned} \tag{3.12}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.13}$$

Substituting the corresponding series into (3.12) and comparing the coefficients, we have

$$(A(0) - 2i\omega_0\tau_k I)W_{20}(\theta) = -H_{20}(\theta), \qquad A(0)W_{11}(\theta) = -H_{11}(\theta). \tag{3.14}$$

From (3.12), we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{3.15}$$

Comparing the coefficients with (3.13) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{3.16}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.17}$$

From the definition of A and (3.14) and (3.16), we have

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

For $q(\theta) = (1, q_1)^T e^{i\omega_0\tau_k\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_1e^{2i\omega_0\tau_k\theta}, \tag{3.18}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector.
Similarly, from (3.14) and (3.17), we know

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_2, \tag{3.19}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector.

In what follows, we shall seek the values of E_1 and E_2. From the definition of $A(0)$ and (3.14), we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0), \tag{3.20}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.21}$$

where $\eta(\theta) = \eta(\theta, 0)$. By (3.12), we know when $\theta = 0$,

$$\begin{aligned} H(z, \bar{z}, 0) &= -2Re\{\bar{q}^*(0)f_0q(0)\} + f_0 \\ &= -\bar{q}^*(0)f_0q(0) - q^*(0)\bar{f}_0\bar{q}(0) + f_0 \\ &= -g(z, \bar{z})q(0) - \bar{g}(z, \bar{z})\bar{q}(0) + f_0. \end{aligned}$$

That is

$$\begin{aligned} &H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \\ &= -q(0)(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) - \bar{q}(0)(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots) + f_0. \end{aligned} \tag{3.22}$$

By (3.3), we have

$$f_0 = \tau_k \begin{pmatrix} a_{11}u_{1t}(0)u_{2t}(0) \\ a_{21}u_{1t}^2(-1) + a_{22}u_{1t}(-1)u_{2t}(0) + a_{23}u_{1t}(-1)u_{2t}(-1) + a_{24}u_{2t}(0)u_{2t}(-1) \end{pmatrix}.$$

By (3.11), we obtain

$$\begin{aligned} f_0 =&\tau_k \begin{pmatrix} a_{11}q_1 \\ (a_{21} + a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} \end{pmatrix} z^2 \\ &+ \tau_k \begin{pmatrix} a_{11}q_1 + a_{11}\bar{q}_1 \\ 2a_{21} + a_{23}(q_1 + \bar{q}_1) + (a_{22}q_1 + a_{24}q_1\bar{q}_1)e^{i\omega_0\tau_k} + (a_{22}\bar{q}_1 + a_{24}q_1\bar{q}_1)e^{-i\omega_0\tau_k} \end{pmatrix} z\bar{z} \\ &+ \cdots. \end{aligned} \tag{3.23}$$

By (3.22) and (3.23), we have

$$\begin{aligned} H_{20}(0) =& - g_{20}q(0) - \bar{g}_{02}\bar{q}(0) \\ &+ 2\tau_k \begin{pmatrix} a_{11}q_1 \\ (a_{21} + a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} \end{pmatrix}, \end{aligned} \tag{3.24}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_k \begin{pmatrix} a_{11}q_1 + a_{11}\bar{q}_1 \\ 2a_{21} + a_{23}(q_1+\bar{q}_1) + (a_{22}q_1 + a_{24}q_1\bar{q}_1)e^{i\omega_0\tau_k} + (a_{22}\bar{q}_1 + a_{24}q_1\bar{q}_1)e^{-i\omega_0\tau_k} \end{pmatrix}. \tag{3.25}$$

Since $i\omega_0\tau_k$ is the eigenvalue of $A(0)$ and $q(0)$ is the corresponding eigenvector, we obtain

$$(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta}d\eta(\theta))q(0) = 0, \qquad (-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta}d\eta(\theta))\bar{q}(0) = 0.$$

Substituting (3.18) and (3.24) into (3.20), we obtain

$$(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta}d\eta(\theta))E_1 = 2\tau_k \begin{pmatrix} a_{11}q_1 \\ (a_{21}+a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} \end{pmatrix}.$$

That is

$$\begin{aligned} &\begin{pmatrix} 2i\omega_0 & -N \\ -Ge^{-2i\omega_0\tau_k} & 2i\omega_0 - Se^{-2i\omega_0\tau_k} \end{pmatrix} E_1 \\ &= 2\begin{pmatrix} a_{11}q_1 \\ (a_{21}+a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} E_1^{(1)} &= \frac{2}{M_1}\begin{vmatrix} a_{11}q_1 & -N \\ (a_{21}+a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} & 2i\omega_0 - Se^{-2i\omega_0\tau_k} \end{vmatrix}, \\ E_1^{(2)} &= \frac{2}{M_1}\begin{vmatrix} 2i\omega_0 & a_{11}q_1 \\ -Ge^{-2i\omega_0\tau_k} & (a_{21}+a_{23}q_1)e^{-2i\omega_0\tau_k} + (a_{22}q_1 + a_{24}q_1^2)e^{-i\omega_0\tau_k} \end{vmatrix}, \end{aligned}$$

where

$$M_1 = \begin{vmatrix} 2i\omega_0 - M & -N \\ -Ge^{-2i\omega_0\tau_k} & 2i\omega_0 - Se^{-2i\omega_0\tau_k} \end{vmatrix}.$$

Similarly, substituting (3.19) and (3.25) into (3.21), we get

$$\begin{aligned} E_2^{(1)} &= \frac{1}{M_2}\begin{vmatrix} a_{11}q_1 + a_{11}\bar{q}_1 & -N \\ 2a_{21} + a_{23}(q_1+\bar{q}_1) + (a_{22}q_1 + a_{24}q_1\bar{q}_1)e^{i\omega_0\tau_k} + (a_{22}\bar{q}_1 + a_{24}q_1\bar{q}_1)e^{-i\omega_0\tau_k} & -S \end{vmatrix}, \\ E_2^{(2)} &= \frac{1}{M_2}\begin{vmatrix} 0 & a_{11}q_1 + a_{11}\bar{q}_1 \\ -G & 2a_{21} + a_{23}(q_1+\bar{q}_1) + (a_{22}q_1 + a_{24}q_1\bar{q}_1)e^{i\omega_0\tau_k} + (a_{22}\bar{q}_1 + a_{24}q_1\bar{q}_1)e^{-i\omega_0\tau_k} \end{vmatrix}, \end{aligned}$$

where

$$M_2 = \begin{vmatrix} 0 & -N \\ -G & -S \end{vmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.18) and (3.19). Furthermore, we can compute g_{21} by (3.12). Thus we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\frac{d\lambda(\tau_k)}{d\tau}\}}, \\ \beta_2 &= 2Re\{C_1(0)\}, \\ T_2 &= -\frac{Im\{C_1(0)\} + \mu_2 Im\{\frac{d\lambda(\tau_k)}{d\tau}\}}{\omega_0\tau_k}, \ (k = 0, 1, 2, \cdots). \end{aligned}$$

Theorem 3.1.

(i) μ_2 *determines the direction of the Hopf bifurcation: if* $\mu_2 > 0$ $(\mu_2 < 0)$*, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for* $\tau > \tau_k$ $(\tau < \tau_k)$*;*

(ii) β_2 *determines the stability of the bifurcating periodic solutions: if* $\beta_2 < 0$ $(\beta_2 > 0)$*, then the bifurcating periodic solutions are stable (unstable);*

(iii) T_2 *determines the period of the bifurcating periodic solutions: if* $T_2 > 0$ $(T_2 < 0)$*, then the period increases (decreases).*

4. Numerical simulations

In this section, we give some numerical simulations supporting our theoretical predictions. As an example, we consider the following system

$$\begin{cases} \frac{dN(t)}{dt} = 0.9N(t) - 0.09P(t-\tau_1)N(t), \\ \frac{dP(t)}{dt} = P(t)(0.3 - 0.15\frac{P(t-\tau_1-\tau_2)}{N(t-\tau_2)}). \end{cases} \tag{4.1}$$

By computing, we may obtain that the unique positive equilibrium $E^* : (5.000000000,\ 10.00000000)$, $a = 0.3000000000$, $b = 0.2700000000$, $a_1 = 0.09000000000$, $b_1 = 0.06480000000$, $c_1 = -0.01312200000$, $d_1 = -0.005314410000$, $\omega_0 = 0.5645567411$, $\tau_0 = \tau_1 + \tau_2 = 0.9923554636$. The computer simulations see Fig.1, Fig.2 show that $E^* : (5.000000000,\ 10.00000000)$ is asymptotically stable when $\tau = 0.9 < \tau_0 = 0.9923554636$.

When τ passes through the critical value τ_0, $E^* = (5.000000000, 10.00000000)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from $E^* : (5.000000000,\ 10.00000000)$.

When $\tau = 1.1 > \tau_0 = 0.9923554636$, $E^* : (5.000000000,\ 10.00000000)$ is unstable, see Fig.3, Fig.4.

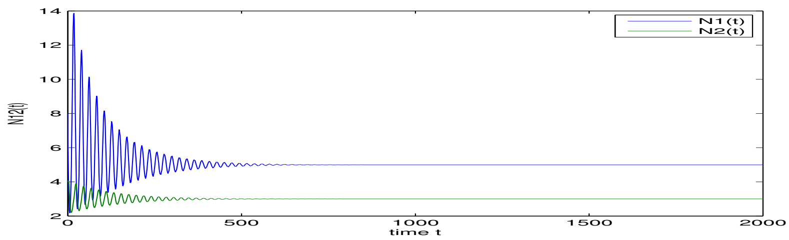

Figure 1: The trajectory graph of system (4.1) with $\tau = 0.9 < \tau_0 = 0.9923554636$. The positive equilibrium E^* : $(5.000000000, 10.00000000)$ is asymptotically stable with initial value $(x(0) = 8.0, y(0) = 4.0)$.

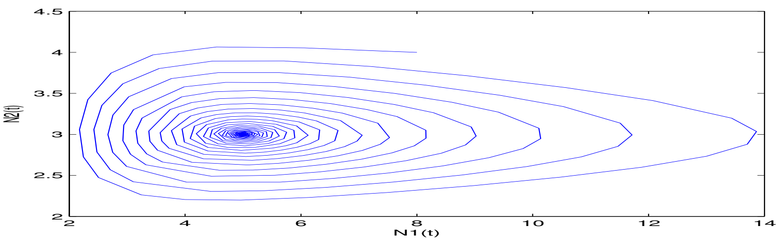

Figure 2: The phase graph of system (4.1) with $\tau = 0.9 < \tau_0 = 0.9923554636$. The positive equilibrium E^* : $(5.000000000, 10.00000000)$ is asymptotically stable with initial value $(x(0) = 8.0, y(0) = 4.0)$.

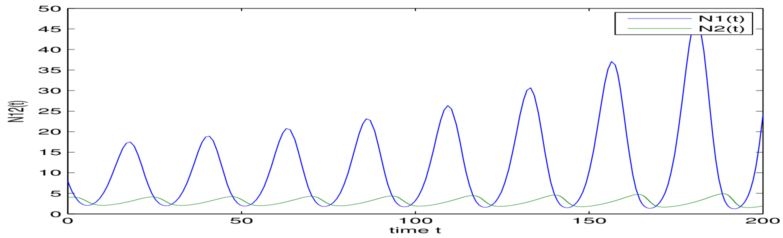

Figure 3: The trajectory graph of system (4.1) with $\tau = 1.1 > \tau_0 = 0.9923554636$. The positive equilibrium E^* : $(5.000000000, 10.00000000)$ is unstable with initial value $(x(0) = 8.0, y(0) = 4.0)$.

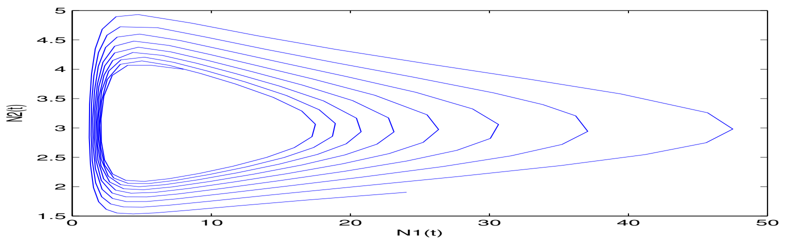

Figure 4: The phase graph of system (4.1) with $\tau = 1.1 > \tau_0 = 0.9923554636$. The positive equilibrium E^* : $(5.000000000, 10.00000000)$ is unstable with initial value $(x(0) = 8.0, y(0) = 4.0)$.

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