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Symmetric identities for degenerate generalized Bernoulli polynomials

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Abstract

In this paper, we give some interesting identities of symmetry for degenerate generalized Bernoulli polynomials attached to χ which are derived from the properties of symmetry of certain p-adic invariant integrals on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see } [9, 11]). \tag{1.1}$$

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Thus, by (1.1), we get

$$\int_{\mathbb{Z}_p} f(x+n)\, d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \sum_{l=0}^{n-1} f'(l), \quad (n \in \mathbb{N}). \tag{1.2}$$

In particular, for $n = 1$, we have

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = f'(0), \quad (\text{see [11]}). \tag{1.3}$$

As is well known, the Bernoulli polynomials are given by the generating function

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers (see [1]–[19]).

Carlitz introduced degenerate Bernoulli polynomials defined by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda) \frac{t^n}{n!}, \quad (\text{see [4]}). \tag{1.5}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0 \mid \lambda)$ are called degenerate Bernoulli numbers.

Recently, T. Kim introduced degenerate Bernoulli polynomials which are different from Carlitz degenerate Bernoulli polynomials as follows:

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \tag{1.6}$$

where $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$ (see [11]).

Let $d \in \mathbb{N}$ with $(d,p) = 1$. Then, we set

$$X = \varprojlim_N \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, \quad X^* = \bigcup_{\substack{0<a<dp \\ p\nmid a}} (a + dp\mathbb{Z}_p),$$

and

$$a + dp^N \mathbb{Z}_p = \left\{ x \mid x \equiv a \pmod{dp^N} \right\},$$

where $a \in \mathbb{Z}$ with $0 \le a \le dp^N - 1$.

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials are defined by the generating function

$$\left(\frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a) e^{at} \right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} \quad (\text{see [1]–[19]}). \tag{1.7}$$

We define the degenerate generalized Bernoulli polynomials attached to χ by the generating function

$$\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}} - 1} \sum_{a=0}^{d-1} \chi(a) (1+\lambda t)^{\frac{a}{\lambda}} \right) (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda,\chi}(x) \frac{t^n}{n!}. \tag{1.8}$$

When $x = 0$, $\beta_{n,\lambda,\chi} = \beta_{n,\lambda,\chi}(0)$ are called degenerate generalized Bernoulli numbers attached to χ. Note that $\lim_{\lambda \to 0} \beta_{n,\lambda,\chi}(x) = B_{n,\chi}(x)$, $(n \ge 0)$.

The purpose of this paper is to give some identities of symmetry for degenerate generalized Bernoulli polynomials attached to χ which are derived from certain symmetric identities of p-adic invariant integrals on $\mathbb{Z}_p$.

2. Symmetric identities of degenerate generalized Bernoulli polynomials

Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. It is known that

$$\int_X f(x)\, d\mu_0(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x), \quad (\text{see [9]}), \tag{2.1}$$

where $f(x)$ is uniformly differentiable function on $\mathbb{Z}_p$. From (1.2), we note that

$$\int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{a}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\lambda,\chi}\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.2), we get

$$\int_X \chi(y)(1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_0(y) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{a+x}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\lambda,\chi}(x)\frac{t^n}{n!}, \tag{2.3}$$

where χ is a Dirichlet character with conductor $d \in \mathbb{N}$. Now, we observe that

$$\int_X \chi(x)(1+\lambda t)^{\frac{x+nd}{\lambda}} d\mu_0(x) - \int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) = \log(1+\lambda t)^{\frac{1}{\lambda}}\sum_{a=0}^{nd-1}\chi(a)(1+\lambda t)^{\frac{a}{\lambda}}. \tag{2.4}$$

By (2.4), we get

$$\left(\frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)\frac{1}{t}\sum_{m=0}^{\infty}\left(\beta_{m,\lambda,\chi}(nd)-\beta_{m,\lambda,\chi}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{a=0}^{nd-1}\chi(a)(a\mid\lambda)_m\frac{t^m}{m!}, \tag{2.5}$$

where $(a \mid \lambda)_m = a(a-\lambda)\cdots(a-\lambda(m-1))$ and $(a\mid\lambda)_0 = 1$. It is not difficult to show that

$$\begin{aligned}
&\frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\frac{1}{t}\sum_{m=0}^{\infty}\left(\beta_{m,\lambda,\chi}(nd)-\beta_{m,\lambda,\chi}\right)\frac{t^m}{m!}\\
&=\left(\sum_{l=0}^{\infty}b_l\frac{\lambda^l t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\frac{\beta_{k+1,\lambda,\chi}(nd)-\beta_{k+1,\lambda,\chi}}{k+1}\frac{t^k}{k!}\right)\\
&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}\frac{\beta_{k+1,\lambda,\chi}(nd)-\beta_{k+1,\lambda,\chi}}{k+1}b_{m-k}\lambda^{m-k}\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\sum_{k=0}^{m}\binom{m}{k}\left(\frac{\beta_{k+1,\lambda,\chi}(nd)-\beta_{k+1,\lambda,\chi}}{k+1}\right)b_{m-k}\lambda^{m-k} = \sum_{a=0}^{nd-1}\chi(a)(a\mid\lambda)_m, \tag{2.7}$$

where b_m is the m-th Bernoulli number of the second kind. Let us define $S_k(n,\lambda\mid\chi)$ as follows:

$$\sum_{l=0}^{n}\chi(l)(l\mid\lambda)_k = S_k(n,\lambda\mid\chi), \quad (n\in\mathbb{N}). \tag{2.8}$$

We observe that

$$\frac{1}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(\int_X \chi(x)(1+\lambda t)^{\frac{nd+x}{\lambda}} d\mu_0(x) - \int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x)\right) \tag{2.9}$$

$$= \frac{nd\int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\, d\mu_0(x)}{\int_X (1+\lambda t)^{\frac{ndx}{\lambda}}\, d\mu_0(x)} = \frac{(1+\lambda t)^{\frac{nd}{\lambda}}-1}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\left(\sum_{i=0}^{d-1}\chi(i)(1+\lambda t)^{\frac{i}{\lambda}}\right).$$

From (2.9), we have

$$\frac{1}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(\int_X \chi(x)(1+\lambda t)^{\frac{nd+x}{\lambda}}\, d\mu_0(x) - \int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\, d\mu_0(x)\right) = \sum_{a=0}^{nd-1}\chi(a)(1+\lambda t)^{\frac{a}{\lambda}} = \sum_{k=0}^{\infty}\left(\sum_{a=0}^{nd-1}\chi(a)(a\mid\lambda)_k\right)\frac{t^k}{k!}. \tag{2.10}$$

Thus, by (2.8) and (2.10), we get

$$\frac{1}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(\int_X \chi(x)(1+\lambda t)^{\frac{nd+x}{\lambda}}\, d\mu_0(x) - \int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\, d\mu_0(x)\right) = \sum_{k=0}^{\infty}\left(S_k(nd-1,\lambda\mid\chi)\right)\frac{t^k}{k!}. \tag{2.11}$$

From (2.7) and (2.11), we have

$$\sum_{k=0}^{m}\binom{m}{k}\left(\frac{\beta_{k+1,\lambda,\chi}(nd)-\beta_{k+1,\lambda,\chi}}{k+1}\right)b_{m-k}\lambda^{m-k} = S_m(nd-1,\lambda\mid\chi), \tag{2.12}$$

where $m \geq 0$. Let $w_1, w_2, d \in \mathbb{N}$. Then, we consider the following integral equation:

$$\begin{aligned}&\frac{d\int_X\int_X \chi(x_1)\chi(x_2)(1+\lambda t)^{\frac{w_1x_1+w_2x_2}{\lambda}}\, d\mu_0(x_1)\, d\mu_0(x_2)}{\int_X (1+\lambda t)^{\frac{dw_1w_2x}{\lambda}}\, d\mu_0(x)}\\ &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}\left((1+\lambda t)^{\frac{dw_1w_2}{\lambda}}-1\right)}{\left((1+\lambda t)^{\frac{w_1d}{\lambda}}-1\right)\left((1+\lambda t)^{\frac{w_2d}{\lambda}}-1\right)}\left(\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{w_1a}{\lambda}}\right)\\ &\quad\times\left(\sum_{b=0}^{d-1}\chi(b)(1+\lambda t)^{\frac{w_2b}{\lambda}}\right).\end{aligned} \tag{2.13}$$

By (2.9), we get

$$\frac{dw_1\int_X \chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\, d\mu_0(x)}{\int_X (1+\lambda t)^{\frac{dw_1x}{\lambda}}\, d\mu_0(x)} = \sum_{k=0}^{\infty}S_k(dw_1-1,\lambda\mid\chi)\frac{t^k}{k!}. \tag{2.14}$$

Let

$$T_\chi(w_1,w_2\mid\lambda) = \frac{d\int_X\int_X \chi(x_1)\chi(x_2)(1+\lambda t)^{\frac{w_1x_1+w_2x_2+w_1w_2x}{\lambda}}\, d\mu_0(x_1)\, d\mu_0(x_2)}{\int_X (1+\lambda t)^{\frac{dw_1w_2x}{\lambda}}\, d\mu_0(x)}. \tag{2.15}$$

From (2.15), we note that $T_\chi(w_1,w_2\mid\lambda)$ is symmetric in w_1 and w_2, and

$$\begin{aligned}&T_\chi(w_1,w_2\mid\lambda)\\ &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}\left((1+\lambda t)^{\frac{dw_1w_2}{\lambda}}-1\right)(1+\lambda t)^{\frac{w_1w_2x}{\lambda}}}{\left((1+\lambda t)^{\frac{w_1d}{\lambda}}-1\right)\left((1+\lambda t)^{\frac{w_2d}{\lambda}}-1\right)}\\ &\quad\times\left(\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{w_1a}{\lambda}}\right)\left(\sum_{b=0}^{d-1}\chi(b)(1+\lambda t)^{\frac{w_2b}{\lambda}}\right).\end{aligned} \tag{2.16}$$

Thus, by (2.15) and (2.16), we get

$$\begin{aligned} T_{\chi}(w_1, w_2 \mid \lambda) &= \left(\frac{1}{w_1}\int_X \chi(x_1)(1+\lambda t)^{\frac{w_1(x_1+w_2x)}{\lambda}} d\mu_0(x_1)\right) \\ &\quad\times\left(\frac{dw_1\int_X \chi(x_2)(1+\lambda t)^{\frac{w_2x_2}{\lambda}} d\mu_0(x_2)}{\int_X (1+\lambda t)^{\frac{dw_1w_2x}{\lambda}} d\mu_0(x)}\right) \\ &= \left(\frac{1}{w_1}\sum_{i=0}^{\infty}\beta_{i,\frac{\lambda}{w_1},\chi}(w_2x)\frac{w_1^i t^i}{i!}\right)\left(\sum_{k=0}^{\infty} S_k\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)\frac{w_2^k t^k}{k!}\right) \\ &= \frac{1}{w_1}\sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\frac{\beta_{i,\frac{\lambda}{w_1},\chi}(w_2x)S_{l-i}\left(dw_1-1,\frac{\lambda}{w_2}\mid\chi\right)w_1^i w_2^{l-i} l!}{i!(l-i)!}\right)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}\beta_{i,\frac{\lambda}{w_1},\chi}(w_2x)S_{l-i}\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)w_1^{i-1}w_2^{l-i}\right)\frac{t^l}{l!}. \end{aligned} \tag{2.17}$$

On the other hand,

$$T_{\chi}(w_1, w_2 \mid \lambda) = \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}\beta_{i,\frac{\lambda}{w_2},\chi}(w_1x)S_{l-i}\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right)w_2^{i-1}w_1^{l-i}\right)\frac{t^l}{l!}. \tag{2.18}$$

By comparing the coefficients on both sides of (2.17) and (2.18), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, d \in \mathbb{N}$, let χ be a Dirichlet character with conductor d. Then we have*

$$\begin{aligned} &\sum_{i=0}^{l}\binom{l}{i}\beta_{i,\frac{\lambda}{w_1},\chi}(w_2x)S_{l-i}\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)w_1^{i-1}w_2^{l-i} \\ &\quad= \sum_{i=0}^{l}\binom{l}{i}\beta_{i,\frac{\lambda}{w_2},\chi}(w_1x)S_{l-i}\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right)w_2^{i-1}w_1^{l-i}. \end{aligned}$$

In particular, taking $x = 0$, we have

$$\sum_{i=0}^{l}\beta_{i,\frac{\lambda}{w_1},\chi}S_{l-i}\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)w_1^{i-1}w_2^{l-i}\binom{l}{i} = \sum_{i=0}^{l}\beta_{i,\frac{\lambda}{w_2},\chi}S_{l-i}\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right)w_2^{i-1}w_1^{l-i}\binom{l}{i}.$$

From (2.9), (2.15) and (2.17), we have

$$\begin{aligned} T_{\chi}(w_1, w_2 \mid \lambda) &= \left(\frac{(1+\lambda t)^{\frac{w_1w_2}{\lambda}x}}{w_1}\int_X \chi(x_1)(1+\lambda t)^{\frac{w_1x_1}{\lambda}} d\mu_0(x_1)\right) \\ &\quad\times\left(\frac{dw_1\int_X \chi(x_2)(1+\lambda t)^{\frac{w_2x_2}{\lambda}} d\mu_0(x_2)}{\int_X (1+\lambda t)^{\frac{dw_1w_2x}{\lambda}} d\mu_0(x)}\right) \\ &= \left(\frac{(1+\lambda t)^{\frac{w_1w_2x}{\lambda}}}{w_1}\int_X \chi(x_1)(1+\lambda t)^{\frac{w_1x_1}{\lambda}} d\mu_0(x_1)\right) \\ &\quad\times\left(\frac{(1+\lambda t)^{\frac{dw_1w_2}{\lambda}}-1}{(1+\lambda t)^{\frac{w_2d}{\lambda}}-1}\right)\left(\sum_{i=0}^{d-1}\chi(i)(1+\lambda t)^{\frac{w_2i}{\lambda}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2 x}{\lambda}}}{w_1} \int_X \chi(x_1)(1+\lambda t)^{\frac{w_1 x_1}{\lambda}} d\mu_0(x_1) \right) \\
&\quad \times \left(\sum_{l=0}^{w_1-1} \sum_{i=0}^{d-1} (1+\lambda t)^{\frac{w_2(i+ld)}{\lambda}} \chi(i+ld) \right) \\
&= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2 x}{\lambda}}}{w_1} \int_X \chi(x_1)(1+\lambda t)^{\frac{w_1 x_1}{\lambda}} d\mu_0(x_1) \right) \left(\sum_{i=0}^{dw_1-1} (1+\lambda t)^{\frac{w_2 i}{\lambda}} \chi(i) \right) \qquad (2.19) \\
&= \frac{1}{w_1} \sum_{i=0}^{dw_1-1} \chi(i) \int_X \chi(x_1)(1+\lambda t)^{\frac{w_1}{\lambda}\left(x_1+w_2x+\frac{w_2}{w_1}i\right)} d\mu_0(x_1) \\
&= \frac{1}{w_1} \sum_{i=0}^{dw_1-1} \chi(i) \sum_{k=0}^{\infty} \beta_{k,\frac{\lambda}{w_1},\chi}\left(w_2 x + \frac{w_2}{w_1} i\right) \frac{w_1^k t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{dw_1-1} \chi(i) \beta_{k,\frac{\lambda}{w_1},\chi}\left(w_2 x + \frac{w_2}{w_1} i\right) w_1^{k-1} \right) \frac{t^k}{k!}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
T_\chi(w_1, w_2 \mid \lambda) &= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2 x}{\lambda}}}{w_2} \int_X \chi(x_2)(1+\lambda t)^{\frac{w_2 x_2}{\lambda}} d\mu_0(x_2) \right) \\
&\quad \times \left(\frac{dw_2 \int_X \chi(x_1)(1+\lambda t)^{\frac{w_1 x_1}{\lambda}} d\mu_0(x_1)}{\int_X (1+\lambda t)^{\frac{dw_1 w_2 x}{\lambda}} d\mu_0(x)} \right) \\
&= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2}{\lambda} x}}{w_2} \int_X \chi(x_2)(1+\lambda t)^{\frac{w_2 x_2}{\lambda}} d\mu_0(x_2) \right) \\
&\quad \times \left(\frac{(1+\lambda t)^{\frac{dw_1 w_2}{\lambda}} - 1}{(1+\lambda t)^{\frac{w_1 d}{\lambda}} - 1} \right) \left(\sum_{i=0}^{d-1} \chi(i)(1+\lambda t)^{\frac{w_1 i}{\lambda}} \right) \qquad (2.20) \\
&= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2 x}{\lambda}}}{w_2} \int_X \chi(x_2)(1+\lambda t)^{\frac{w_2 x_2}{\lambda}} d\mu_0(x_2) \right) \\
&\quad \times \left(\sum_{l=0}^{w_2-1} \sum_{i=0}^{d-1} (1+\lambda t)^{\frac{w_1(i+ld)}{\lambda}} \chi(i+ld) \right) \\
&= \left(\frac{(1+\lambda t)^{\frac{w_1 w_2 x}{\lambda}}}{w_2} \int_X \chi(x_2)(1+\lambda t)^{\frac{w_2 x_2}{\lambda}} d\mu_0(x_2) \right) \left(\sum_{i=0}^{dw_2-1} (1+\lambda t)^{\frac{w_1 i}{\lambda}} \chi(i) \right) \\
&= \frac{1}{w_2} \sum_{i=0}^{dw_2-1} \chi(i) \int_X \chi(x_2)(1+\lambda t)^{\frac{w_2}{\lambda}\left(x_2+w_1x+\frac{w_1}{w_2}i\right)} d\mu_0(x_2) \\
&= \frac{1}{w_2} \sum_{i=0}^{dw_2-1} \chi(i) \sum_{k=0}^{\infty} \beta_{k,\frac{\lambda}{w_2},\chi}\left(w_1 x + \frac{w_1}{w_2} i\right) \frac{w_2^k t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{dw_2-1} \chi(i) \beta_{k,\frac{\lambda}{w_2},\chi}\left(w_1 x + \frac{w_1}{w_2} i\right) w_2^{k-1} \right) \frac{t^k}{k!}.
\end{aligned}
$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.2. *For $w_1, w_2, d \in \mathbb{N}$, let χ be a Dirichlet character with conductor d. Then we have*

$$w_1^{k-1} \sum_{i=0}^{dw_1-1} \chi(i)\, \beta_{k,\frac{\lambda}{w_1},\chi}\left(w_2 x + \frac{w_2}{w_1} i\right) = w_2^{k-1} \sum_{i=0}^{dw_2-1} \chi(i)\, \beta_{k,\frac{\lambda}{w_2},\chi}\left(w_1 x + \frac{w_1}{w_2} i\right).$$

Remark 2.3. Let $x = 0$ in Theorem 2.2. Then we note that

$$w_1^{k-1} \sum_{i=0}^{dw_1-1} \chi(i)\, \beta_{k,\frac{\lambda}{w_1},\chi}\left(\frac{w_2}{w_1} i\right) = w_2^{k-1} \sum_{i=0}^{dw_2-1} \chi(i)\, \beta_{k,\frac{\lambda}{w_2},\chi}\left(\frac{w_1}{w_2} i\right).$$

Further, if we take $w_2 = 1$, then we get

$$w_1^{k-1} \sum_{i=0}^{dw_1-1} \chi(i)\, \beta_{k,\frac{\lambda}{w_1},\chi}\left(\frac{1}{w_1} i\right) = \sum_{i=0}^{d-1} \chi(i)\, \beta_{k,\lambda,\chi}(w_1 i).$$

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