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JORDAN HOMOMORPHISMS IN PROPER JCQ^*-TRIPLES

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday

ABSTRACT. In this paper, we investigate Jordan homomorphisms in proper JCQ^*-triples associated with the generalized 3-variable Jesnsen functional equation

$$rf(\frac{x+y+z}{r}) = f(x) + f(y) + f(z),$$

with $r \in (0, 3) \setminus \{1\}$. We moreover prove the Hyers-Ulam-Rassias stability of Jordan homomorphisms in proper JCQ^*-triples.

1. INTRODUCTION AND PRELIMINARIES

Let A be a linear space and A_0 is a $*$-algebra contained in A as a subspace. We say that A is a quasi $*$-algebra over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are defined and linear;

(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra (A, A_0) is said to be a locally convex quasi $*$-algebra if in A a locally convex topology τ is defined such that

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(i) the involution is continuous and the multiplications are separately continuous;

(ii) A_0 is dense in $A[\tau]$.

We should notify that a locally convex quasi $*$-algebra $(A[\tau], A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer to [2].

In a series of papers [7, 15, 17, 18], many authors have considered a special class of quasi $*$-algebras, called *proper* CQ^**-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a right Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper* CQ^**-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(ii) $(ab)^* = b^*a^*$ for all $a, b \in A_0$;

(iii) $\|y\|_0 = \sup_{a\in A, \|a\|\le 1} \|ay\|$ for all $y \in A_0$.

Several mathematician have contributed works on these subjects (see [1], [4]–[10], [12]–[16], [19], [20], [26], [28], [36]–[38], [60], [61], [63], [64]).

A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$". Such a problem was formulated by Ulam [65] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [32]. It gave rise to the *stability theory* for functional equations.

In 1978, Th.M. Rassias [51] formulated and proved the following theorem, which implies Hyers' Theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a function such that for each fixed $x \in E$ the mapping $t \longmapsto f(tx)$ is continuous on $\mathbb{R}$. If there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, then there exists a unique linear function $T : E \to F$ such that

$$\|f(x) - T(x)\| \le \frac{\epsilon\|x\|^p}{(1 - 2^{p-1})}$$

for all $x \in E$. In 1991, Gajda [29] answered the question for $p > 1$, which was rased by Th.M. Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations. It was shown by Gajda [29], as well as by Th.M. Rassias and Šemrl [57] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [29], as well as of Th.M. Rassias and Šemrl [57] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [30], who among others studied the Hyers-Ulam-Rassias stability of functional equations.

Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians (see [3], [21]–[24], [33]–[35], [42], [55]) and references therein.

J.M. Rassias [47] following the spirit of the innovative approach of Th.M. Rassias [51] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with

$p + q \neq 1$ (see also [48] for a number of other new results). Several mathematician have contributed works on these subjects (see [39]–[43], [49], [50], [53]–[56], [59]).

Let $\mathcal{C}$ be a C^*-algebra. Then $\mathcal{C}$ with the Jordan product $x \circ y := \frac{xy+yx}{2}$, is called a *Jordan C^*-algebra* (see [40], [41]). A C^*-algebra A, endowed with the Jordan triple product

$$\{z, x, w\} = \frac{1}{2}(zx^*w + wx^*z)$$

for all $z, x, w \in A$, is called a *JC^*-triple* (see [27]). Note that

$$\{z, x, w\} = (z \circ x^*) \circ w + (w \circ x^*) \circ z - (z \circ w) \circ x^*.$$

A proper CQ^*-algebra (A, A_0), endowed with the Jordan triple product

$$\{z, x, w\} = \frac{1}{2}(zx^*w + wx^*z)$$

for all $x \in A$ and all $z, w \in A_0$, is called a *proper JCQ^*-triple*, and denoted by $(A, A_0, \{\cdot, \cdot, \cdot\})$.

Let $(A, A_0\{\cdot, \cdot, \cdot\})$ and $(B, B_0\{\cdot, \cdot, \cdot\})$ be proper JCQ^*-triples. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper JCQ^*-triple Jordan homomorphism* if $H(z) \in B_0$ and

$$H(\{z, x, z\}) = \{H(z), H(x), H(z)\}$$

for all $z \in A_0$ and all $x \in A$. If, in addition, the mapping $H : A \to B$ and the mapping $H|_{A_0} : A_0 \to B_0$ are bijective, then the mapping $H : A \to B$ is called a *proper JCQ^*-triple Jordan isomorphism*.

In this paper, we investigate Jordan homomorphisms in proper JCQ^*-triples for the 3-variable Jensen functional equation. Moreover, we prove the Hyers-Ulam-Rassias stability of Jordan homomorphisms in proper JCQ^*-triples.

From now on, assume that $(A, A_0, \{\cdot, \cdot, \cdot\})$ is a proper JCQ^*-triple with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that $(B, B_0, \{\cdot, \cdot, \cdot\})$ is a proper JCQ^*-triple with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

2. Jordan isomorphisms in proper JCQ^*-triples

We start our work with the following theorem, which investigate Jordan isomorphisms in proper JCQ^*-triples.

Theorem 2.1. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping such that*

$$\left\| rf\left(\frac{\mu x + \mu y + \mu z}{r}\right) - \mu f(x) - \mu f(y) - \mu f(z) \right\|_B \leq \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}}, \quad (2.1)$$

$$\left\| rf\left(\frac{w_0 + w_1 + w_2}{r}\right) - f(w_0) - f(w_1) - f(w_2) \right\|_B \leq \theta \cdot \|w_0\|_{A_0}^{\frac{r}{3}} \cdot \|w_1\|_{A_0}^{\frac{r}{3}} \cdot \|w_2\|_{A_0}^{\frac{r}{3}}, \quad (2.2)$$

$$f(\{w, x, w\}) = \{f(w), f(x), f(w)\} \quad (2.3)$$

for all $\mu \in \mathbb{T}^1$, all $w, w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. If $\lim_{n\to\infty} \frac{r^n}{3^n} f\left(\frac{3^n e}{r^n}\right) = e'$ and $f|_{A_0} : A_0 \to B_0$ is bijective, then the mapping $f : A \to B$ is a proper JCQ^*-triple Jordan isomorphism.

Proof. Let us assume $\mu = 1$ and $x = y = z$ in (2.1). Then we get

$$\left\| rf\left(\frac{3x}{r}\right) - 3f(x)\right\|_B \le \theta \|x\|_A^r \tag{2.4}$$

for all $x \in A$. So

$$\left\| f(x) - \frac{r}{3}f\left(\frac{3x}{r}\right)\right\|_B \le \frac{\theta}{3}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\| \frac{r^l}{3^l}f\left(\frac{3^l x}{r^l}\right) - \frac{r^m}{3^m}f\left(\frac{3^m x}{r^m}\right)\right\|_B &\le \sum_{j=l}^{m-1}\left\| \frac{r^j}{3^j}f\left(\frac{3^j x}{r^j}\right) - \frac{r^{j+1}}{3^{j+1}}f\left(\frac{3^{j+1}x}{r^{j+1}}\right)\right\|_B \\ &\le \frac{\theta}{3}\sum_{j=l}^{m-1}\frac{r^j 3^{rj}}{3^j r^{rj}}\|x\|_A^r\end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\left\{\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)\right\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)$$

for all $x \in A$. Since $f(\{w, x, w\}) = \{f(w), f(x), f(w)\}$ for all $w \in A_0$ and all $x \in A$,

$$\begin{aligned}H(\{w,x,w\}) &= \lim_{n\to\infty}\frac{r^{3n}}{3^{3n}}\left\{f\left(\frac{3^n w}{r^n}\right), f\left(\frac{3^n x}{r^n}\right), f\left(\frac{3^n w}{r^n}\right)\right\} \\ &= \lim_{n\to\infty}\left\{\frac{r^n}{3^n}f\left(\frac{3^n w}{r^n}\right), \frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right), \frac{r^n}{3^n}f\left(\frac{3^n w}{r^n}\right)\right\} \\ &= \{H(w), H(x), H(w)\}\end{aligned}$$

for all $w \in A_0$ and all $x \in A$.

It follows from (2.2) that $H(w) := \lim_{n\to\infty}\frac{r^n}{3^n}f\left(\frac{3^n w}{r^n}\right) \in B_0$ for all $w \in A_0$. By (2.2), we can show that H is additive, and by (2.1), we can show that H is $\mathbb{C}$-linear.

On the other hand, by the assumption,

$$\begin{aligned}H(x) &= H(ex) = \lim_{n\to\infty}\frac{r^{2n}}{3^{2n}}f\left(\frac{3^{2n}ex}{r^{2n}}\right) = \lim_{n\to\infty}\frac{r^{2n}}{3^{2n}}f\left(\left\{\frac{3^n e}{r^n}, \frac{3^n e}{r^n}, x\right\}\right) \\ &= \lim_{n\to\infty}\left\{\frac{r^n}{3^n}f\left(\frac{3^n e}{r^n}\right), \frac{r^n}{3^n}f\left(\frac{3^n e}{r^n}\right), f(x)\right\} = \{e', e', f(x)\} \\ &= f(x)\end{aligned}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a proper JCQ^*-triple Jordan isomorphism. □

Theorem 2.2. *Let* $1 < r < 3$ *and* θ *be nonnegative real numbers, and let* $f : A \to B$ *be a bijective mapping satisfying* (2.1), (2.2) *and* (2.3). *If* $\lim_{n\to\infty}\frac{3^n}{r^n}f\left(\frac{r^n e}{3^n}\right) = e'$

and $f|_{A_0} : A_0 \to B_0$ *is bijective, then the mapping* $f : A \to B$ *is a proper* JCQ^**-triple Jordan isomorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.1, there is a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying $H(w) \in B_0$ for all $w \in A_0$. The mapping $H : A \to B$ is given by

$$H(x) := \lim_{n\to\infty} \frac{3^n}{r^n} f\left(\frac{r^n x}{3^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Jordan homomorphisms in proper JCQ^*-triples

We investigate Jordan homomorphisms in proper JCQ^*-triples.

Theorem 3.1. *Let* $r \in (0,3) \setminus \{1\}$ *and* θ *be nonnegative real numbers and* $f : A \to B$ *a mapping satisfying* $f(w) \in B_0$ *for all* $w \in A_0$ *such that*

$$\|\mu f(x) + f(y) + f(z)\|_B + \|f(\{w,a,w\}) - \{f(w), f(a), f(w)\}\|_B \leq \left\|rf\left(\tfrac{\mu x+y+z}{r}\right)\right\|_B + \theta(2\|w\|_A^{3r} + \|a\|_A^{3r}) \quad (3.1)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$*, all* $w \in A_0$ *and all* $x, y, z \in A$*. Then the mapping* $f : A \to B$ *is a proper* JCQ^**-triple Jordan homomorphism.*

Proof. Letting $\mu = 1, x = y = z = a = w = 0$ in (3.1), we obtain

$$\|3f(0)\|_B + \|f(0) - \{f(0), f(0), f(0)\}\|_B \leq \|rf(0)\|_B.$$

It follows that

$$\|3f(0)\|_B \leq \|rf(0)\|_B,$$

and $f(0) = 0$.

Letting $a = w = 0$ in (3.1), we obtain

$$\|\mu f(x) + f(y) + f(z)\|_B \leq \left\|rf\left(\frac{\mu x + y + z}{r}\right)\right\|_B \quad (3.2)$$

for all $x, y, z \in A$.

Letting $\mu = 1, x = -y, z = 0$ in (3.2), we obtain

$$\|f(x) + f(-x) + f(0)\|_B \leq \|rf(0)\|_B = 0.$$

Hence, $f(-x) = -f(x)$ for all $x \in A$.

Letting $\mu = 1, z = -x - y$ in (3.2), by the oddness of f, we get

$$\|f(x) + f(y) - f(x+y)\|_B = \|f(x) + f(y) + f(-x-y)\|_B \leq \|rf(0)\|_B = 0$$

for all $x, y \in A$. It follows that $f(x) + f(y) = f(x+y)$ for all $x, y \in A$. So $f : A \to B$ is Cauchy additive.

Letting $z = 0$ and $y = -\mu x$ in (3.2), we get $f(\mu x) = \mu f(x)$ for all $x \in A$. By the same reasoning as in the proof of Theorem 2.1 of [40], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. It follows from (3.1) that

$$\|f(\{w,x,w\}) - \{f(w),f(x),f(w)\}\|_B \leq \theta(2\|w\|_A^{3r} + \|x\|_A^{3r}) \quad (3.3)$$

for all $w \in A_0$ and all $x \in A$. By (3.3), we get

$$\begin{aligned}\|f(\{w,x,w\}) &- \{f(w),f(x),f(w)\}\|_B \\ &= \lim_{n\to\infty} \frac{1}{8^n}\|f(8^n\{w,x,w\}) - \{f(2^n w), f(2^n x), f(2^n w)\}\|_B \\ &\leq \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta(2\|w\|_A^{3r} + \|x\|_A^{3r}) = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(\{w,x,w\}) = \{f(w),f(x),f(w)\}$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(\{w,x,w\}) = \{f(w),f(x),f(w)\}$$

for all $w \in A_0$ and all $x \in A$.

Since $f(w) \in B_0$ for all $w \in A_0$, the mapping $f : A \to B$ is a proper JCQ^*-triple homomorphism, as desired. □

Theorem 3.2. *Let $r \in (0,3) \setminus \{1\}$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (3.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f(\{w,x,w\}) - \{f(w),f(x),f(w)\}\|_B \leq \theta \cdot \|w\|_A^{2r} \cdot \|x\|_A^r \quad (3.4)$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper JCQ^*-triple Jordan homomorphism.

Proof. By the same reasoning as in the proof of Theorem 3.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.4),

$$\begin{aligned}\|f(\{w,x,w\}) &- \{f(w),f(x),f(w)\}\|_B \\ &= \lim_{n\to\infty} \frac{1}{8^n}\|f(8^n\{w,x,w\}) - \{f(2^n w), f(2^n x), f(2^n w)\}\|_B \\ &\leq \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta \cdot \|w\|_A^{2r} \cdot \|x\|_A^r = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(\{w,x,w\}) = \{f(w),f(x),f(w)\}$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(\{w,x,w\}) = \{f(w),f(x),f(w)\}$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper JCQ^*-triple Jordan homomorphism. □

4. Stability of Jordan homomorphisms in proper JCQ^*-triples

In this section, we prove the Hyers-Ulam-Rassias stability of Jordan homomorphisms in proper JCQ^*-triples.

Theorem 4.1. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that $f(w) \in B_0$ for all $w \in A_0$ and*

$$\begin{aligned} &\left\| rf\left(\frac{\mu x + \mu y + \mu z}{r}\right) - \mu f(x) - \mu f(y) - \mu f(z)\right\|_B \\ &\quad + \left\| rf\left(\frac{w_0 + w_1 + w_2}{r}\right) - f(w_0) - f(w_1) - f(w_2)\right\|_B \\ &\quad \le \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}} + \theta \cdot \|w_0\|_{A_0}^{\frac{r}{3}} \cdot \|w_1\|_{A_0}^{\frac{r}{3}} \cdot \|w_2\|_{A_0}^{\frac{r}{3}}, \end{aligned} \tag{4.1}$$

$$\|f(\{w, x, w\}) - \{f(w), f(x), f(w)\}\|_B \le \theta \cdot \|w\|_A^{2r} \cdot \|x\|_A^r \tag{4.2}$$

for all $\mu \in \mathbb{T}^1$, all $w, w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper JCQ^*-triple Jordan homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{r^r\theta}{3 \cdot r^r - r \cdot 3^r}\|x\|_A^r \tag{4.3}$$

for all $x \in A$.

Proof. It follows from (4.1) that

$$\left\| rf\left(\frac{\mu x + \mu y + \mu z}{r}\right) - \mu f(x) - \mu f(y) - \mu f(z)\right\|_B \le \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}} \tag{4.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Let us assume $\mu = 1$ and $x = y = z$ in (4.4). Then we get

$$\left\| rf\left(\frac{3x}{r}\right) - 3f(x)\right\|_B \le \theta\|x\|_A^r$$

for all $x \in A$. So

$$\left\| f(x) - \frac{r}{3}f\left(\frac{3x}{r}\right)\right\|_B \le \frac{\theta}{3}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} \left\| \frac{r^l}{3^l}f\left(\frac{3^l x}{r^l}\right) - \frac{r^m}{3^m}f\left(\frac{3^m x}{r^m}\right)\right\|_B &\le \sum_{j=l}^{m-1}\left\| \frac{r^j}{3^j}f\left(\frac{3^j x}{r^j}\right) - \frac{r^{j+1}}{3^{j+1}}f\left(\frac{3^{j+1}x}{r^{j+1}}\right)\right\|_B \\ &\le \frac{\theta}{3}\sum_{j=l}^{m-1}\frac{r^j 3^{rj}}{3^j r^{rj}}\|x\|_A^r \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\left\{\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)\right\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{r^n}{3^n}f\left(\frac{3^n x}{r^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

It follows from (4.4) that

$$\begin{aligned}
&\left\| rH\left(\frac{\mu(x+y+z)}{r}\right) - \mu H(x) - \mu H(y) - \mu H(z)\right\|_B \\
&= \lim_{n\to\infty} \frac{r^n}{3^n}\left\| 2f\left(\frac{3^n\mu(x+y+z)}{r^n}\right) - \mu f\left(\frac{3^n x}{r^n}\right) - \mu f\left(\frac{3^n y}{r^n}\right) - \mu f\left(\frac{3^n z}{r^n}\right)\right\|_B \\
&\le \lim_{n\to\infty} \frac{r^n 3^{nr}\theta}{3^n 2^{nr}} \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}} = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. So

$$rH\left(\frac{\mu x + \mu y + \mu z}{r}\right) = \mu H(x) + \mu H(y) + \mu H(z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By the same reasoning as in the proof of Theorem 2.1 of [40], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Now, let $T : A \to B$ be another 3-variable Jensen mapping satisfying (4.3). Then we have

$$\begin{aligned}
\|H(x) \quad - \quad T(x)\|_B &= \frac{r^n}{3^n}\left\| H\left(\frac{3^n x}{r^n}\right) - T\left(\frac{3^n x}{r^n}\right)\right\|_B \\
&\le \frac{r^n}{3^n}\left(\left\| H\left(\frac{3^n x}{r^n}\right) - f\left(\frac{3^n x}{r^n}\right)\right\|_B + \left\| T\left(\frac{3^n x}{r^n}\right) - f\left(\frac{3^n x}{r^n}\right)\right\|_B\right) \\
&\le \frac{r^{r+1} r^n 3^{nr}\theta}{3^n r^{nr}(3\cdot r^r - r\cdot 3^r)}\|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H.

On the other hand, by (4.1), we get

$$\left\| rf\left(\frac{w_0 + w_1 + w_2}{r}\right) - f(w_0) - f(w_1) - f(w_2)\right\|_B \le \theta \cdot \|w_0\|_{A_0}^{\frac{r}{3}} \cdot \|w_1\|_{A_0}^{\frac{r}{3}} \cdot \|w_2\|_{A_0}^{\frac{r}{3}} \quad (4.6)$$

for all $w, w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$.

It follows from (4.6) that $H(w) = \lim_{n\to\infty} \frac{r^n}{3^n} f\left(\frac{3^n w}{r^n}\right) \in B_0$ for all $w \in A_0$. So it follows from (4.2) that

$$\begin{aligned}
\|H(\{w, x, w\}) \quad &- \quad \{H(w), H(x), H(w)\}\|_B \\
&= \lim_{n\to\infty} \frac{r^{3n}}{3^{3n}}\left\| f\left(\frac{3^{3n}\{w, x, w\}}{r^{3n}}\right) - \left\{ f\left(\frac{3^n w}{r^n}\right), f\left(\frac{3^n x}{r^n}\right), f\left(\frac{3^n w}{r^n}\right)\right\}\right\|_B \\
&\le \lim_{n\to\infty} \frac{r^{3n} 3^{3nr}}{3^{3n} r^{3nr}}\theta \cdot \|w\|_A^{2r} \cdot \|x\|_A^r = 0
\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$H(\{w, x, w\}) = \{H(w), H(x), H(w)\}$$

for all $w \in A_0$ and all $x \in A$.

Thus the mapping $H : A \to B$ is a unique proper JCQ^*-triple Jordan homomorphism satisfying (4.3), and the proof is complete. □

Theorem 4.2. *Let $1 < r < 3$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (4.1) and (4.2) such that $f(w) \in B_0$ for all $w \in A_0$. Then there exists a unique proper JCQ^*-triple Jordan homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{r^r\theta}{r \cdot 3^r - 3 \cdot r^r}\|x\|_A^r \tag{4.7}$$

for all $x \in A$.

Proof. It follows from (4.1) that

$$\left\|f(x) - \frac{3}{r}f\left(\frac{rx}{3}\right)\right\|_B \le \frac{r^r\theta}{r \cdot 3^r}\|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned}\left\|\frac{3^l}{r^l}f\left(\frac{r^lx}{3^l}\right) - \frac{3^m}{r^m}f\left(\frac{r^mx}{3^m}\right)\right\|_B &\le \sum_{j=l}^{m-1}\left\|\frac{3^j}{r^j}f\left(\frac{r^jx}{3^j}\right) - \frac{3^{j+1}}{r^{j+1}}f\left(\frac{r^{j+1}x}{3^{j+1}}\right)\right\|_B \\ &\le \frac{r^r\theta}{r \cdot 3^r}\sum_{j=l}^{m-1}\frac{3^jr^{jr}}{r^j3^{jr}}\|x\|_A^r \end{aligned} \tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\left\{\frac{3^n}{r^n}f\left(\frac{r^nx}{3^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{\frac{3^n}{r^n}f\left(\frac{r^nx}{3^n}\right)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{3^n}{r^n}f\left(\frac{r^nx}{3^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

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