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RANDOM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS: A FIXED POINT APPROACH

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday

Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following quadratic functional equations

$$cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)$$
$$= (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right),$$
$$Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right)$$

in random Banach spaces.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [42] concerning the stability of group homomorphisms. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [31] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [31] has provided a lot of influence in the development

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of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability of the quadratic functional equation was proved by Skof [41] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 8, 11, 15, 17], [32]–[38]).

2. Preliminaries

We define the notion of a random normed space, which goes back to Sherstnev (see, e.g., [12, 40]).

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in. Throughout this paper, let Δ^+ is the space of distribution functions, that is,

$$\begin{aligned}\Delta^+ := \{F : \mathbb{R}\cup\{-\infty,\infty\} \to [0,1] \ : \ &F \ is \ left-continuous, \\ &non-decreasing \ on \ \mathbb{R}, \ F(0)=0 \ and \ F(+\infty)=1\}\end{aligned}$$

and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([39]) A function $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a $t-$norm) if T satisfies the following conditions:

(TN_1) T is commutative and associative;
(TN_2) T is continuous;
(TN_3) $T(a,1) = a$ for all $a \in [0,1]$;
(TN_4) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous $t-$norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Łukasiewicz t-norm). Recall (see [12, 13]) that if T is a $t-$norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1, & \text{if } n=1, \\ T(T_{i=1}^{n-1} x_i, x_n), & \text{if } n \geq 2. \end{cases}$$

$T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$.

Definition 2.2. ([40]) A *random normed space* (briefly, RN-space) is a triple (X, Λ, T), where X is a vector space, T is a continuous $t-$norm, and Λ is a mapping from X into D^+ such that the following conditions hold:

(RN_1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN_2) $\Lambda_{\alpha x}(t) = \Lambda_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;

(RN_3) $\Lambda_{x+y}(t+s) \geq T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Every normed space $(X, \| \,.\, \|)$ defines a random normed space (X, Λ, T_M), where $\Lambda_u(t) = \frac{t}{t+\|u\|}$ for all $t > 0$ and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 2.3. Let (X, Λ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\Lambda_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\Lambda_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN-space (X, Λ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be a *random Banach space.*

Theorem 2.4. ([39]) *If* (X, Λ, T) *is an* RN*-space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$*, then* $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$ *almost everywhere.*

Starting with the paper [23], the stability of some functional equations in the framework of fuzzy normed spaces or random normed spaces has been investigated in [18]–[26].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 2.5. [3, 9] *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$*. Then* *for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5], [27]–[29], [30]).

This paper is organized as follows: In Section 3, we prove the generalized Hyers-Ulam stability of the quadratic functional equation

$$\begin{aligned} cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right) \qquad (2.1)\\ = (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right) \end{aligned}$$

in random Banach spaces by using the fixed point method. In Section 4, we prove the generalized Hyers-Ulam stability of the quadratic functional equation

$$Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\leq i<j\leq n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right) \qquad (2.2)$$

in random Banach spaces.

Throughout this paper, assume that X is a vector spaces and (Y, μ, T) is a complete RN-space.

3. Generalized Hyers-Ulam stability of the quadratic functional equation (2.1) in RN-spaces

For a given mapping $f : X \to Y$, consider the mapping $Pf : X^n \to Y$, defined by

$$\begin{aligned} Pf(x_1, x_2, \cdots, x_n) = cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)\\ -(n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $Pf(x_1, \cdots, x_n) = 0$ in complete RN-spaces.

Theorem 3.1. *Let $v := 2 - n - c > 1$.* Let $\rho : X^n \to D^+$ be a mapping ($\rho(x_1, \cdots, x_n)$ is denoted by $\rho_{x_1,\cdots,x_n}$) such that, for some $0 < \alpha < v^2$,

$$\rho_{vx_1,\cdots,vx_n}(\alpha t) \geq \rho_{x_1,\cdots,x_n}(t) \tag{3.1}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\mu_{Pf(x_1,\cdots,x_n)}(t) \geq \rho_{x_1,\cdots,x_n}(t) \tag{3.2}$$

for all $x_1, \cdots x_n \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \rho_{0,x,0,\cdots,0}\left((v^2 - \alpha)t\right)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_2 = x$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (3.2), we get

$$\mu_{f((2-c-n)x)-(2-c-n)^2 f(x)}(t) \geq \rho_{0,x,0,\cdots,0}(t) \tag{3.3}$$

for all $x \in X$ and all $t > 0$. Replacing $2 - c - n$ by v in (3.3), we get

$$\mu_{f(vx)-v^2 f(x)}(t) \geq \rho_{0,x,0,\cdots,0}(t) \tag{3.4}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$\mu_{\frac{f(vx)}{v^2}-f(x)}(t) \geq \rho_{0,x,0,\cdots,0}(v^2 t) \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Let S be the set of all even mappings $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h, k) = \inf \left\{ u \in \mathbb{R}^+ : \mu_{h(x)-k(x)}(ut) \geq \rho_{0,x,0,\cdots,0}(t), \ \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is a generalized complete metric space (see [19, Lemma 2.1]).

Now we define the mapping $J : S \to S$

$$Jh(x) := \frac{h(vx)}{v^2}$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f, g) < \varepsilon$. Therefore

$$\begin{aligned} \mu_{Jg(x)-Jf(x)}\left(\frac{\alpha u}{v^2} t\right) &= \mu_{\frac{g(vx)}{v^2}-\frac{f(vx)}{v^2}}\left(\frac{\alpha u}{v^2} t\right) = \mu_{g(vx)-f(vx)}(\alpha u t) \\ &\geq \rho_{0,vx,0,\cdots,0}(\alpha t) \geq \rho_{0,x,0,\cdots,0}(t), \end{aligned}$$

that is, if $d(f, g) < \varepsilon$ we have $d(Jf, Jg) < \frac{\alpha}{v^2}\varepsilon$. Hence

$$d(Jf, Jg) \leq \frac{\alpha}{v^2} d(f, g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $\alpha/v^2 (< 1)$.

It follows from (3.5) that

$$\mu_{Jf(x)-f(x)}\left(\frac{1}{v^2} t\right) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$, which means that $d(Jf, f) \leq \frac{1}{v^2}$.

By Theorem 2.5, there exists a unique mapping $Q : X \to Y$ such that Q is a fixed point of J, i.e., $Q(2x) = 4Q(x)$ for all $x \in X$.

Also, $d(J^m g, Q) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} \frac{f(v^m x)}{v^{2m}} = Q(x)$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \mu_{\frac{Pf(v^m x_1,\cdots,v^m x_n)}{v^{2m}}}(t) \geq \rho_{v^m x_1,\cdots,v^m x_n}\left(v^{2m}t\right) &= \rho_{v^m x_1,\cdots,v^m x_n}\left(\alpha^m\left(\frac{v^2}{\alpha}\right)^m t\right) \\ &\geq \rho_{x_1,\cdots,x_n}\left(\left(\frac{v^2}{\alpha}\right)^m t\right) \end{aligned} \tag{3.6}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Letting $m \to \infty$ in (3.6), we find that $\mu_{PQ(x_1,\cdots,x_n)}(t) = 1$ for all $t > 0$, which implies $PQ(x_1, \cdots, x_n) = 0$. Therefore, the mapping $Q : X \to Y$ is quadratic.

Since Q is the unique fixed point of J in the set $\Omega = \{g \in S : d(f, g) < \infty\}$, Q is the unique mapping such that

$$\mu_{f(x)-Q(x)}(ut) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$. Using the fixed point alternative, we obtain that

$$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{v^2(1-L)} = \frac{1}{v^2\left(1-\frac{\alpha}{v^2}\right)},$$

which implies the inequality

$$\mu_{f(x)-Q(x)}\left(\frac{1}{v^2-\alpha}t\right) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{f(x)-Q(x)}(t) \geq \rho_{0,x,0,\cdots,0}\left((v^2-\alpha)t\right)$$

for all $x \in X$ and all $t > 0$. □

Theorem 3.2. Let $\rho : X^n \to D^+$ be a mapping ($\rho(x_1, \cdots, x_n)$ is denoted by $\rho_{x_1,\cdots,x_n}$) such that, for some $\alpha > v^2$,

$$\rho_{\frac{x_1}{v},\cdots,\frac{x_n}{v}}(t) \geq \rho_{x_1,\cdots,x_n}(\alpha t) \tag{3.7}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Suppose that an even mapping $f : X \to Y$ satisfying $f(0) = 0$ and (3.2). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \rho_{0,x,0,\cdots,0}\left((\alpha-v^2)t\right)$$

for *all* $x \in X$ and *all* $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the mapping $J : S \to S$ defined by

$$Jh(x) := v^2 h\left(\frac{x}{v}\right)$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f,g) < \varepsilon$. Then

$$\begin{aligned}\mu_{Jg(x)-Jf(x)}\left(\frac{v^2 u}{\alpha}t\right) &= \mu_{v^2 g(\frac{x}{v})-v^2 f(\frac{x}{v})}\left(\frac{v^2 u}{\alpha}t\right) = \mu_{g(\frac{x}{v})-f(\frac{x}{v})}\left(\frac{u}{\alpha}t\right) \\ &\geq \rho_{0,\frac{x}{v},0,\cdots,0}\left(\frac{t}{\alpha}\right) \geq \rho_{0,x,0,\cdots,0}(t), \end{aligned} \tag{3.8}$$

that is, if $d(f,g) < \varepsilon$ we have $d(Jf, Jg) < \frac{v^2}{\alpha}\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{v^2}{\alpha} d(f,g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $\frac{v^2}{\alpha}(< 1)$.

By Theorem 2.5, there exists a unique mapping $Q : X \to Y$ such that Q is a fixed point of J, i.e., $Q\left(\frac{x}{v}\right) = \frac{1}{v^2}Q(x)$ for all $x \in X$.

Also, $d(J^m g, Q) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} v^2 m f\left(\frac{x}{v^m}\right) = Q(x)$$

for all $x \in X$.

It follows from (3.7) that

$$\mu_{Jf(x)-f(x)}\left(\frac{1}{\alpha}t\right) \geq \rho_{0,\frac{x}{v},0,\cdots,0}(\frac{t}{\alpha}) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$, which implies that $d(Jf, f) \leq \frac{1}{\alpha}$.

Since Q is the unique fixed point of J in the set $\Omega = \{g \in S : d(f,g) < \infty\}$, Q is the unique mapping such that

$$\mu_{f(x)-Q(x)}(ut) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$. Using the fixed point alternative, we obtain that

$$d(f,Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{\alpha(1-L)} = \frac{1}{\alpha\left(1 - \frac{v^2}{\alpha}\right)},$$

which implies the inequality

$$\mu_{f(x)-Q(x)}\left(\frac{1}{(\alpha - v^2)}t\right) \geq \rho_{0,x,0,\cdots,0}(t)$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{f(x)-Q(x)}(t) \geq \rho_{0,x,0,\cdots,0}\left((\alpha - v^2)t\right)$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

4. Generalized Hyers-Ulam stability of the quadratic functional equation (2.2) in RN-spaces

For a given mapping $Q : X \to Y$, we define

$$DQ(x_1, \cdots, x_n) := Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1 \le i < j \le n} d_i d_j Q\left(x_i - x_j\right) - \sum_{i=1}^{n} d_i \left(\sum_{i=1}^{n} d_i Q\left(x_i\right)\right)$$

for all $x_1, \cdots, x_n \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $DQ(x_1, \cdots, x_n) = 0$ in random Banach spaces

Theorem 4.1. *Let $d := \sum_{i=1}^{n} d_i$. Let $\rho : X^n \to D^+$ be a mapping ($\rho(x_1, \cdots, x_n)$ is denoted by $\rho_{x_1, \cdots, x_n}$) such that, for some $0 < \alpha < d^2$,*

$$\rho_{dx_1, \cdots, dx_n}(\alpha t) \ge \rho_{x_1, \cdots, x_n}(t) \tag{4.1}$$

for all $x_1, \cdots, x_d \in X$ and all $t > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\mu_{DQ(x_1, \cdots, x_n)}(t) \ge \rho_{x_1, \cdots, x_n}(t) \tag{4.2}$$

for all $x_1, \cdots x_{2l} \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $R : X \to Y$ such that

$$\mu_{Q(x) - R(x)}(t) \ge \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}\left((d^2 - \alpha)t\right)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_1 = \cdots = x_n = x$ in (4.2), we get

$$\mu_{Q(dx) - d^2 Q(x)}(t) \ge \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}(t) \tag{4.3}$$

for all $x \in X$ and all $t > 0$. It follows from (4.3) that

$$\mu_{\frac{Q(dx)}{d^2} - Q(x)}(t) \ge \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}(d^2 t) \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

Let S be the set of all even mappings $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h, k) = \inf\left\{u \in \mathbb{R}^+ : \mu_{h(x) - k(x)}(ut) \ge \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}(t),\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is a generalized complete metric space (see [19, Lemma 2.1]).

Now we consider the mapping $J : S \to S$ defined by

$$Jh(x) := \frac{h\left(dx\right)}{d^2}$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f,g) < \varepsilon$. Then

$$\begin{aligned}\mu_{Jg(x)-Jf(x)}\left(\frac{\alpha u}{d^2}t\right) &= \mu_{\frac{g(dx)}{d^2}-\frac{f(dx)}{d^2}}\left(\frac{\alpha u}{d^2}t\right) = \mu_{g(dx)-f(dx)}(\alpha ut) \\ &\geq \rho_{\underbrace{dx,\cdots,dx}_{n\text{ times}}}(\alpha t) \geq \rho_{\underbrace{x,\cdots,x}_{n\text{ times}}}(t),\end{aligned}$$

that is, if $d(f,g) < \varepsilon$ we have $d(Jf, Jg) < \frac{\alpha}{d^2}\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{\alpha}{d^2} d(f,g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $\frac{\alpha}{d^2}$.

It follows from (4.4) that

$$\mu_{JQ(x)-Q(x)}\left(\frac{1}{d^2}t\right) \geq \rho_{\underbrace{x,\cdots,x}_{n\text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$, which implies that $d(JQ, Q) \leq \frac{1}{d^2}$.

By Theorem 2.5, there exists a unique mapping $R : X \to Y$ such that R is a fixed point of J, i.e., $R(dx) = d^2R(x)$ for all $x \in X$.

Also, $d(J^m g, Q) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} \frac{Q(d^m x)}{d^{2m}} = R(x)$$

for all $x \in X$.

It follows from (4.1) and (4.2) that

$$\mu_{\frac{DQ(d^m x_1,\cdots,d^m x_n)}{d^{2m}}}(t) \geq \rho_{d^m x_1,\cdots,d^m x_n}\left(d^{2m}t\right) = \rho_{x_1,\cdots,x_n}\left(\left(\frac{d^2}{\alpha}\right)^m t\right) \quad (4.5)$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Letting $m \to \infty$ in (4.5), we find that $\mu_{DR(x_1,\cdots,x_n)}(t) = 1$ for all $t > 0$, which implies $DR(x_1, \cdots, x_n) = 0$. Since Q is even, R is even. So the mapping $R : X \to Y$ is quadratic.

Since R is the unique fixed point of J in the set $\Omega = \{g \in S : d(f,g) < \infty\}$, R is the unique mapping such that

$$\mu_{Q(x)-R(x)}(ut) \geq \rho_{\underbrace{x,\cdots,x}_{n\text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$. Using the fixed point alternative, we obtain that

$$d(Q,R) \leq \frac{1}{1-L}d(Q, JQ) \leq \frac{1}{d^2(1-L)} = \frac{1}{d^2\left(1-\frac{\alpha}{d^2}\right)} = \frac{1}{d^2-\alpha},$$

which implies the inequality

$$\mu_{Q(x)-R(x)}\left(\frac{1}{d^2-\alpha}t\right) \geq \rho_{\underbrace{x,\cdots,x}_{n\text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{Q(x)-R(x)}(t) \geq \rho_{\underbrace{x,\cdots,x}_{n\text{ times}}}\left((d^2-\alpha)t\right)$$

for all $x \in X$ and all $t > 0$. □

Theorem 4.2. *Let $\rho : X^n \to D^+$ be a mapping ($\rho(x_1, \cdots, x_n)$ is denoted by $\rho_{x_1,\cdots,x_n}$) such that, for some $\alpha > d^2$,*

$$\rho_{\frac{x_1}{d},\cdots,\frac{x_n}{d}}(t) \geq \rho_{x_1,\cdots,x_n}(\alpha t) \tag{4.6}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Suppose that an even mapping $Q : X \to Y$ satisfying $Q(0) = 0$ and (4.2). Then there exists a unique quadratic mapping $R : X \to Y$ such that

$$\mu_{R(x)-Q(x)}(t) \geq \rho_{\underbrace{x,\cdots,x}_{n\ times}}\left((\alpha - d^2)t\right)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1.

Now we consider the mapping $J : S \to S$ defined by

$$Jh(x) := d^2 h\left(\frac{x}{d}\right)$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f, g) < \varepsilon$. Then

$$\begin{aligned}\mu_{Jg(x)-Jf(x)}\left(\frac{d^2\varepsilon}{\alpha}t\right) &= \mu_{d^2g\left(\frac{x}{d}\right)-d^2f\left(\frac{x}{d}\right)}\left(\frac{d^2\varepsilon}{\alpha}t\right) = \mu_{g\left(\frac{x}{d}\right)-f\left(\frac{x}{d}\right)}\left(\frac{\varepsilon}{\alpha}t\right) \\ &\geq \rho_{\underbrace{\frac{x}{d},\cdots,\frac{x}{d}}_{n \text{ times}}}\left(\frac{t}{\alpha}\right) \geq \rho_{\underbrace{x,\cdots,x}_{n \text{ times}}}(t),\end{aligned}$$

that is, if $d(f, g) < \varepsilon$ we have $d(Jf, Jg) < \frac{d^2}{\alpha}\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{d^2}{\alpha} d(f, g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $\frac{d^2}{\alpha}$.

By Theorem 2.5, there exists a unique mapping $R : X \to Y$ such that R is a fixed point of J, i.e., $R\left(\frac{x}{d}\right) = \frac{1}{d^2}R(x)$ for all $x \in X$.

Also, $d(J^m g, R) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} d^{2m} Q\left(\frac{x}{d^m}\right) = R(x)$$

for all $x \in X$.

It follows from (4.3) that

$$\mu_{JQ(x)-Q(x)}\left(\frac{1}{\alpha}t\right) \geq \rho_{\underbrace{x,\cdots,x}_{n \text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$, which implies that $d(JQ, Q) \leq \frac{1}{\alpha}$.

Since R is the unique fixed point of J in the set $\Omega = \{g \in S : d(f, g) < \infty\}$, R is the unique mapping such that

$$\mu_{Q(x)-R(x)}(ut) \geq \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$. Using the fixed point alternative, we obtain that

$$d(Q, R) \leq \frac{1}{1-L} d(Q, JQ) \leq \frac{1}{\alpha(1-L)} = \frac{1}{\alpha\left(1 - \frac{d^2}{\alpha}\right)} = \frac{1}{\alpha - d^2},$$

which implies the inequality

$$\mu_{Q(x)-R(x)}\left(\frac{1}{\alpha - d^2} t\right) \geq \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}(t)$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{Q(x)-R(x)}(t) \geq \rho_{\underbrace{x, \cdots, x}_{n \text{ times}}}\left((\alpha - d^2)t\right)$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.1. □

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