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SHORT COMMUNICATIONS

On Projective Functions with Bad Measurability Properties

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Under Martin's Axiom, we show the consistency of the existence of a projective function acting from **R** into itself, which has an extremely bad measurability property with respect to a wide class of measures on **R**.

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One of Ulam's problems posed by him in the Sottish Book [6] is formulated as follows:

Does there exist a measure on the real line **R**, extending the classical Lebesgue measure λ on **R** and containing in its domain the family of all projective subsets of **R**?

Nowadays it is clear that this problem cannot be resolved within the framework of the standard **ZFC** set theory. Indeed, assuming the Axiom of Projective Determinacy (see, for instance, [1]), one immediately gets that the algebra of all projective subsets of **R** is entirely contained in the domain of λ.

On the other hand, in [5] the following assumption (H) was considered:

There exists a well-ordering of **R** which is isomorphic to the standard well-ordering of ω_1 and, simultaneously, is a projective subset of the Euclidean plane $\mathbf{R}^2$.

Notice that the assumption (H) trivially holds in Gödel's Constructible Universe (see [1]).

It was shown in [5] that if (H) is fulfilled, then no extension of λ can make all projective sets in **R** be measurable.

Let us say a few words about the method presented in [5] for solving Ulam's above-mentioned problem. This method essentially relies on the two facts which are consequences of the assumption (H):

(1) the cardinality of **R** is equal to ω_1 (i.e., the Continuum Hypothesis is valid);

(2) every projective set $Z \subset \mathbf{R}^2$ all vertical sections of which are at most count-

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able admits a representation in the form

$$Z = \cup\{f_i : i \in I\},$$

where $\{f_i : i \in I\}$ is a countable family of some functions whose graphs are projective subsets of $\mathbf{R}^2$.

Actually, (1) and (2) imply that the method of [5] cannot be directly generalized to the case when the cardinality continuum $\mathbf{c}$ is strictly greater than ω_1.

In [2] a much stronger result was obtained. In order to formulate it, let us introduce several notions.

A measure μ defined on some σ-algebra of subsets of $\mathbf{R}$ is called diffused (continuous) if each singleton in $\mathbf{R}$ belongs to $\mathrm{dom}(\mu)$ and μ vanishes at all of them.

Let E be an uncountable ground (base) set and let $\mathcal{M}$ be a class of measures on E (in general, those measures are defined on different σ-algebras of subsets of E).

We shall say that a function $f : E \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}$ if f is nonmeasurable with respect to every measure from $\mathcal{M}$.

Let us denote by $\mathcal{M}(E)$ the class of all nonzero σ-finite diffused measures on E.

Remark 1: Obviously, the class $\mathcal{M}(\mathbf{R})$ is substantially wider than the class of all those measures on $\mathbf{R}$ which extend λ. In this context, it makes sense to recall that there exist many measures on $\mathbf{R}$ which extend λ and, simultaneously, are invariant under the group of all translations of $\mathbf{R}$ (see, e.g., [3], [9]). Moreover, as was demonstrated by Kakutani and Oxtoby in 1950, there exist nonseparable translation invariant measures on $\mathbf{R}$ which are extensions of λ.

In [2] it was proved that, under the same hypothesis (H), there exists a function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$ and whose graph is a projective subset of $\mathbf{R}^2$.

It follows from this result that there exists a countable family $\{P_i : i \in I\}$ of projective sets in $\mathbf{R}$ such that no measure μ from $\mathcal{M}(\mathbf{R})$ satisfies the relation

$$\{P_i : i \in I\} \subset \mathrm{dom}(\mu).$$

Moreover, all sets P_i $(i \in I)$ belong to the projective class $\Sigma^1_k(\mathbf{R})$, where k only depends on the index of the projective class of a well-ordering of $\mathbf{R}$.

Remark 2: It should be noted that if $g : \mathbf{R} \to \mathbf{R}$ is any function absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$, the graph $\mathrm{Gr}(g)$ of g is an absolute null subset of $\mathbf{R}^2$. The latter means that there exists no nonzero σ-finite diffused Borel measure on $\mathrm{Gr}(g)$.

The method of [2] essentially relies on properties of so-called Luzin subsets of $\mathbf{R}$. An extensive information on Luzin sets can be found in many sources (see, e.g., [4], [7], [8]).

In this note, assuming Martin's Axiom (**MA**) instead of the Continuum Hypothesis (**CH**), we would like to present a certain analogue of the just formulated result.

Recall that a set $X \subset \mathbf{R}$ is a generalized Luzin set if $\mathrm{card}(X) = \mathbf{c}$ and, for every first category subset Y of $\mathbf{R}$, the inequality $\mathrm{card}(Y \cap X) < \mathbf{c}$ is valid.

Under Martin's Axiom, there exist generalized Luzin sets in $\mathbf{R}$ and all of them are absolute null. It follows from this fact that, under the same axiom, there exist injective functions acting from $\mathbf{R}$ into $\mathbf{R}$ which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Denote by $\mathcal{I}(\mathbf{R})$ the σ-ideal of all those subsets of $\mathbf{R}$ whose cardinalities are strictly less than $\mathbf{c}$.

Also, denote by $\mathcal{M}_0(\mathbf{R})$ the class of all those nonzero σ-finite measures μ on $\mathbf{R}$ which satisfy the relation

$$(\forall Z \in \mathcal{I}(\mathbf{R}))(\mu_*(Z) = 0),$$

where $\mu_*(Z)$ stands, as usual, for the inner μ-measure of Z.

Clearly, under Martin's Axiom, $\mathcal{M}_0(\mathbf{R})$ properly contains the class of all those measures on $\mathbf{R}$ which extend λ.

Remark 3 : At present, models of **ZFC** theory are known in which there exists a measure ν on $\mathbf{R}$ extending λ and such that $\nu(\mathbf{R} \setminus Z) = 0$ for some set $Z \in \mathcal{I}(\mathbf{R})$. Moreover, the existence of such a measure ν can be deduced from the assumption that the cardinal $\mathbf{c}$ is real-valued measurable.

Remark 4 : Let μ be an arbitrary measure from the class $\mathcal{M}_0(\mathbf{R})$. It is not hard to show that there exists a measure μ' on $\mathbf{R}$ such that:

(a) μ' extends μ;

(b) $\mathcal{I}(\mathbf{R}) \subset \mathrm{dom}(\mu')$ and $\mu'(Z) = 0$ for any set $Z \in \mathcal{I}(\mathbf{R})$.

Lemma 1 : *Suppose that there exists a well-ordering of $\mathbf{R}$ which is a projective subset of $\mathbf{R}^2$.*

Then, for every projective set $Z \subset \mathbf{R}^2$, there is a projective uniformization of Z, i.e., there exists a function

$$h : \mathrm{pr}_1(Z) \to \mathbf{R}$$

whose graph $\mathrm{Gr}(h)$ is contained in Z and is also a projective subset of $\mathbf{R}^2$.

This lemma is well known and its proof is not difficult (cf. [5]).

Lemma 2 : *Assume Martin's Axiom and let $f : \mathbf{R} \to \mathbf{R}$ be a function having the following two properties:*

(1) the range $\mathrm{ran}(f)$ of f is a generalized Luzin set in $\mathbf{R}$;

(2) for each point $t \in \mathbf{R}$, the cardinality of the set $f^{-1}(t)$ is strictly less than the cardinality of the continuum.

Then f is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$.

The proof of this lemma is completely analogous to the proof of Theorem 1 from Chapter 5 in [3].

Now, we are able to formulate and prove the main result of this note.

Theorem 1 : *Assume Martin's Axiom and suppose also that there is a projective well-ordering $\preceq$ of $\mathbf{R}$ isomorphic to the well-ordering of the least ordinal number of cardinality $\mathbf{c}$.*

Then there exists a function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$ and whose graph is a projective subset of $\mathbf{R}^2$.

Proof: We follow the method of [2]. Obviously, it suffices to demonstrate the existence of function

$$\phi : [0,1] \to [0,1]$$

which is absolutely nonmeasurable with respect to the analogous class $\mathcal{M}_0([0,1])$ and whose graph is a projective subset of the square $[0,1]^2$.

Let $\preceq$ be a projective well-ordering $\preceq$ of $[0,1]$ isomorphic to the well-ordering of the least ordinal number of cardinality $\mathbf{c}$.

First, denote by E the compact metric space consisting of all nonempty closed subsets of $[0,1]$, and consider its subspace E' consisting of all nonempty nowhere dense closed subsets of $[0,1]$. Observe that this E' is of type G_δ in E, so E' can be treated as a Polish topological space (see [4]).

Further, identify the Baire canonical space ω^ω with the set $\mathbf{I}$ of all irrational numbers in $[0,1]$ and introduce a continuous surjection

$$\Phi : \mathbf{I} \to E'.$$

The existence of Φ is well known (see again [4]). Then, for each point $x \in [0,1]$, consider the set

$$Z(x) = \{y \in [0,1] : x \preceq y \ \& \ y \notin \cup\{\Phi(i) : i \in \mathbf{I} \ \& \ i \preceq x\}\}$$

and define the subset Z of the plane $\mathbf{R}^2$ by putting

$$Z = \cup\{\{x\} \times Z(x) : x \in [0,1]\}.$$

It is not difficult to check that Z is a projective subset of the square $[0,1]^2$ and $\mathrm{pr}_1(Z) = [0,1]$.

Taking into account the fact that Z admits a projective uniformization (see Lemma 1), we come to the existence of a function

$$\phi : [0,1] \to [0,1]$$

whose graph is a projective subset of $[0,1]^2$ and is entirely contained in Z.

Now, a straightforward verification shows that the range $\mathrm{ran}(\phi)$ of ϕ is a generalized Luzin set in $[0,1]$ which simultaneously is a projective subset of $[0,1]$. In addition to this,

$$\mathrm{card}(\phi^{-1}(t)) < \mathbf{c}$$

for every point $t \in [0,1]$. Keeping in mind Lemma 2, we can conclude that ϕ is absolutely nonmeasurable with respect to the class $\mathcal{M}_0([0,1])$.

This completes the proof of Theorem 1. □

As a consequence of the above theorem, we have the following statement.

Theorem 2: *Assume Martin's Axiom and suppose that there is a projective well-ordering $\preceq$ of $\mathbf{R}$ isomorphic to the well-ordering of the least ordinal number of cardinality $\mathbf{c}$.*

Then there exists a countable family $\{P_j : j \in J\}$ *of projective subsets of* $\mathbf{R}$ *such that no measure* $\mu \in \mathcal{M}_0(\mathbf{R})$ *satisfies the relation*

$$\{P_j : j \in J\} \subset \mathrm{dom}(\mu).$$

Proof: Let $f : \mathbf{R} \to \mathbf{R}$ be a function of Theorem 1, which is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$. Take some countable base $\{U_k : k < \omega\}$ of open sets in $\mathbf{R}$ and some countable base $\{V_k : k < \omega\}$ of open sets in the space $\mathrm{ran}(f)$. Denote

$$\{P_j : j \in J\} = \{U_k : k < \omega\} \ \cup \ \{f^{-1}(V_k) : k < \omega\}.$$

By virtue of the absolute nonmeasurability of f with respect to $\mathcal{M}_0(\mathbf{R})$, it is not hard to demonstrate that the family $\{P_j : j \in J\}$ is as required.

Moreover, if $\preceq$ belongs to the projective class $\Sigma^1_n(\mathbf{R}^2)$, then all sets P_j $(j \in J)$ belong to the projective class $\Sigma^1_{m(n)}(\mathbf{R})$, where the index $m(n)$ depends on n. $\square$

Remark 5: The following question naturally arises: under the assumptions of Theorem 1, does there exist a projective function $g : \mathbf{R} \to \mathbf{R}$ absolutely nonmeasurable with respect to the whole class $\mathcal{M}(\mathbf{R})$? At present, we do not know the answer to this question.

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