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HOW ALGEBRAIC IS ALGEBRA ?

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ABSTRACT. The 2-category VAR of finitary varieties is not varietal over CAT . We introduce the concept of an algebraically exact category and prove that the 2-category ALG of all algebraically exact categories is an equational hull of VAR w.r.t. all operations with rank. Every algebraically exact category $\mathcal{K}$ is complete, exact, and has filtered colimits which (a) commute with finite limits and (b) distribute over products; besides (c) regular epimorphisms in $\mathcal{K}$ are product-stable. It is not known whether (a) – (c) characterize algebraic exactness. An equational hull of VAR w.r.t. all operations is also discussed.

1. Introduction

1.1. IS ALGEBRA ALGEBRAIC?. The purpose of our paper is to study the non-full embedding

$$U : \mathrm{VAR} \to \mathrm{CAT}$$

of the 2-category of all finitary varieties into the 2-quasicategory of all categories. The morphisms (1-cells) of the former are indicated by the duality between varieties and algebraic theories introduced in [ALR_1]: they are the algebraically exact functors, i.e., finitary right adjoints preserving regular epimorphisms. And 2-cells are the natural transformations, of course. Is U monadic in some reasonable 2-categorical sense? (This is the precise meaning of the question in the subtitle.) The answer is negative, and we shortly explain it below.

Well, since algebra is not algebraic, what is an "algebraic hull" of algebra? To understand the question, let us use analogy with classical universal algebra: given a class $\mathcal{C}$ of algebras, considered as a concrete category w.r.t. the usual forgetful functor $U : \mathcal{C} \to \mathbf{Set}$, we can form the equational theory of $\mathcal{C}$ whose k-ary operations are the natural transformations $U^k \to U$, and whose clone is obtained by composition of n-tuples of k-ary operations (i.e., natural transformations from U^k to U^n). Here the arity k is a (usually finite) set. Now, for VAR, the arities k will be (usually small) categories, and a k-ary operation will be a natural transformation from U^k to U – no!, this is too strict. We will have to work with pseudonatural transformations from U^k to U. Example: for any small category k an operation "limits of type k" is a pseudonatural transformation from U^k to U, since (a) every variety has limits of type k and (b) every algebraically exact functor preserves them (non-strictly, of course). Analogously, for every small filtered category k we have

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an operation on VAR "filtered colimits of type k". And one important operation (of arity $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \bullet$) is "reflexive coequalizers". In fact, we will see that all ranked operations are obtained by combining the three types just mentioned.

But let us come back to the negative answer to the question in the subtitle: U : VAR $\to$ CAT is not 2-monadic nor pseudomonadic. The argument (quite analogous to that for the 2-category of locally finitely presentable categories in place of VAR, see [ALR$_2$]), is that if we take the "posetal shadow" of U, we get the forgetful functor from the category **ALat** of algebraic lattices (= those posets which are equivalent to a variety) to the category **Pos** of all posets. Here morphisms of **ALat** are maps preserving meets and directed joins, and morphisms of **Pos** are order-preserving maps, of course. The forgetful functor **ALat** $\to$ **Pos** is not varietal; in fact, the varietal hull of **ALat** is formed by the category **CLat** of continuous lattices of D. Scott. Now take a continuous lattice L which is not algebraic. Express L as the split coequalizer of free continuous (= free algebraic) lattices in the standard way. This gives a split coequalizer in CAT which our forgetful functor U : VAR $\to$ CAT fails to create. Thus, U is not pseudomonadic, see [LV].

1.2. An equational hull of VAR. The aim of our paper is to describe an equational hull of the category VAR of varieties over CAT. Here we refer to equations between k-ary operations, where k is a category and a k-ary operation ω assigns to every variety $\mathcal{V}$ an operation functor $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ in such a way that morphisms of VAR preserve ω; more precisely: that ω becomes a pseudonatural transformation from U^k : VAR $\to$ CAT into U. An equational hull is, roughly speaking, the smallest equationally defined quasicategory $\mathcal{K}$ containing VAR; or, a largest extension $\mathcal{K}$ of VAR such that every operation on VAR extends to $\mathcal{K}$ and all equations valid in VAR are also valid in $\mathcal{K}$. Formally, $\mathcal{K}$ is described as the quasicategory of algebras of a pseudomonad $\mathbb{D}^*$ on CAT which we describe in the last section of our paper.

The disadvantage of the pseudomonad $\mathbb{D}^*$ is that, so far, we have not been able to describe the category of its algebras. The same is, unfortunately, true concerning the small-core $\mathbb{D}^*_{\text{small}}$ of that pseudomonad which we obtain by restricting ourselves to all k-ary operations on VAR where the arity k is a small category. We therefore make a further restriction: to operations with rank. That is, we discuss k-ary operations ω, k small, for which there exists a cardinal λ such that each of the operation functors $\omega_{\mathcal{V}}$ preserves λ-filtered colimits. (Examples: "limits of type k" is an operation with rank.) We obtain a rank-core pseudomonad $\mathbb{D}^*_{\text{rank}}$ of $\mathbb{D}^*$ whose quasicategory of algebras is an equational hull of VAR w.r.t. all operations with rank. We also present another construction of $\mathbb{D}^*_{\text{rank}}$ which does not start with $\mathbb{D}^*$. And we are able to identify $\mathbb{D}^*_{\text{rank}}$-algebras as a subquasicategory of CAT consisting of so-called algebraically exact categories. That is, we introduce the 2-quasicategory

$$\text{ALG}$$

of algebraically exact categories, algebraically exact functors, and natural transformations, and we prove that ALG is an equational hull of VAR w.r.t. all operations with rank.

1.3. What are algebraically exact categories?. For the theory of varieties there is a concept of central importance generalizing filtered colimits:

sifted colimits.

A small category $\mathcal{D}$ is called *sifted* if $\mathcal{D}$-colimits commute in **Set** with finite products. And colimits of such types $\mathcal{D}$ are then called *sifted colimits*. Besides filtered colimits, also reflexive coequalizers are sifted. And morphisms of VAR are precisely the functors preserving limits and sifted colimits (see [ALR_1]); in general, functors (in CAT) with the last property will be called algebraically exact.

We have introduced in [AR_1] a free completion under sifted colimits, denoted by *Sind* (to remind of Grothendieck's free completion *Ind* under filtered colimits). This completion forms a pseudomonad *Sind* on CAT . Also a free limit completion forms a pseudomonad on CAT which we denote by *Lim* . And we prove below that *Sind* distributes over *Lim* (in the sense of Beck's distributive laws); thus, we obtain a composite pseudomonad

$$\mathit{Sind} \circ \mathit{Lim}$$

on CAT . The main result of our paper is that the quasicategory of all algebras of this pseudomonad is an equational hull of VAR w.r.t. ranked operations. It is easy to identify these algebras with categories $\mathcal{K}$ which

(a) have limits and sifted colimits

and

(b) sifted colimits distribute over limits in the sense that the functor

$$\mathit{Colim} : \mathit{Sind}\, \mathcal{K} \to \mathcal{K}$$

(of computation of sifted colimits in $\mathcal{K}$) preserves limits.

Categories satisfying (a) and (b) are called *algebraically exact.* All varieties are, of course, algebraically exact, but also all essential localizations of varieties are. Recall here the classical result of Roos [R] characterizing all essential localizations of varieties of modules: they are complete abelian categories $\mathcal{K}$ with a generator such that

(1) filtered colimits commute with finite limits;

(2) filtered colimits *distribute over products*, i.e., given filtered diagrams $D_i : \mathcal{D}_i \to \mathcal{K}$ $(i \in I)$, then for the filtered diagram $D : \Pi\mathcal{D} \to \mathcal{K}$, defined by

$$D(x_i) = \Pi D x_i$$

we have a canonical isomorphism

$$\operatorname{colim} D \cong \prod_{i\in I} (\operatorname{colim} D_i)$$

and

(3) regular epimorphisms are product-stable.

All algebraically exact categories are proved below to fulfil (1) – (3).

Open problem Is every complete category with sifted colimits satisfying (1) – (3) algebraically exact?

1.4. A comparison to the case LFP. In [ALR$_2$] we have studied the analogous question for the non-full embedding $U : \mathrm{LFP} \to \mathrm{CAT}$. Here LFP is the Gabriel-Ulmer 2-category of all locally finitely presentable categories, all finitary right adjoints, and all natural transformations. The functor U is not (pseudo-)monadic. An equational hull of LFP has been described in [ALR$_2$]. Let us shortly recall the result here. Free completions under filtered colimits form a pseudomonad *Ind* on CAT which distributes over *Lim* and thus yields a composite pseudomonad $Ind \circ Lim$. Algebras of that pseudomonad form an equational hull of LFP w.r.t. ranked operations. These algebras are called *precontinuous categories* and they are precisely the categories $\mathcal{K}$ such that

(i) $\mathcal{K}$ has limits and filtered colimits

and

(ii) filtered colimits distribute over limits in the sense that the functor $Colim : Ind\mathcal{K} \to \mathcal{K}$ preserves limits.

Recall that $\mathcal{K}$ is continuous (in the sense of [JJ]) if *Colim* has a left adjoint. An important tool for this result was a description of a "free LFP category" on a given small category k : it is

$$\mathrm{Lex}\,(\mathbf{Set}^{k}),$$

the category of all accessible functors from $\mathbf{Set}^{k}$ to $\mathbf{Set}$ preserving finite limits. We have shown in [ALR$_2$] that ranked k-ary operations on LFP are essentially encoded by objects of $\mathrm{Lex}\,(\mathbf{Set}^{k})$. This has raised the following question: are there, for k small, functors preserving finite limits from $\mathbf{Set}^{k}$ to $\mathbf{Set}$ which are not accessible? More precisely, is the following statement

(∗) every functor $\mathbf{Set}^{k} \to \mathbf{Set}$, k small, preserving finite limits is accessible

consistent with set theory? This has been answered affirmatively in [AK]. Thus, instead of "ranked operations" we may say "operations of small arity" — well, in some set theory at least.

Now below we make a similar identification by constructing a "free variety" on a small category k : it is

$$\mathbf{Mod}\,(\mathbf{Set}^{k}),$$

the category of all accessible functors from $\mathbf{Set}^{k}$ to $\mathbf{Set}$ preserving finite products. Here the analogous problem arises which has not been solved so far:

Open problem Is the following statement

(∗∗) every functor $\mathbf{Set}^{k} \to \mathbf{Set}$, k small, preserving finite products is accessible consistent with set theory?

1.5. Set-theoretical foundations. In the present paper we are not concerned with set theory beyond the open problem above, but here we want to make clear what (very meager) requirements are needed below. A chosen universe of small sets is assumed satisfying the Axiom of Choice (AC). In a higher universe *categories* (not assumed locally small in general) live. And when "categories" outside of this universe are considered, they are called *quasicategories*; example : CAT. Analogously with 2-categories and 2-quasicategories. But where a 2-quasicategory is biequivalent to a 2-category (such as LFP, due to Gabriel-Ulmer duality, or VAR, due to the duality of [ALR_1]), we call them 2-categories. And we assume that every category is an ∞-*filtered* union of its small subcategories (i.e., a large union which is λ-filtered for every cardinal λ). All limits and colimits are meant to be small (unless explicitly stated large).

It is an open problem to what extent our results apply without the assumption of AC; or even over an arbitrary base topos.

2. Operations on VAR

2.1. In the present section we introduce operations on the 2-category VAR of varieties, and give some examples of operations and equations among them. In the subsequent sections we will show that the operations introduced here generate all ranked operations on VAR and lead to a description of the equational hull of VAR w.r.t. ranked operations.

Recall from [Law_1] the concept of a k-ary operation on a concrete category, i.e. category $\mathcal{K}$ equipped by a faithful functor $U : \mathcal{K} \to \mathbf{Set}$: a k-ary operation ω is a natural transformation $\omega : U^k \to U$, i.e., to every object K of $\mathcal{K}$ an operation-map ω_K of arity k on the underlying set UK is assigned in such a way that all mappings Uf, where $f : K \to L$ is a morphism of $\mathcal{K}$, preserve the operation maps in the expected sense: $Uf \cdot \omega_K = \omega_L \cdot (Uf)^k$. Observe that k can, in general, by any set, but we often want to restrict ourselves to k finite.

By analogy, we might at first want to introduce k-ary operations ω on VAR as natural transformations from U^k to U, where

$$U : \mathrm{VAR} \to \mathrm{CAT}$$

is the (non-full) inclusion functor; here k is an arbitrary category. Such ω would assign to every variety $\mathcal{V}$ an "operation" functor $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ in such a way that every algebraically exact functor $F : \mathcal{V} \to \mathcal{W}$ preserves the operation functor in the strict sense, i.e., $F \cdot \omega_{\mathcal{V}} = \omega_{\mathcal{W}} \cdot F^k$. However, this is too strict. We have, for example, an important k-ary operation "limits of type k" for any small category k : the operation functor $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ assigns to every diagram of type k in $\mathcal{V}$ a limit in $\mathcal{V}$. Since algebraically exact functors $F : \mathcal{V} \to \mathcal{W}$ preserve limits, we have a natural isomorphism $F \cdot \omega_{\mathcal{V}} \cong \omega_{\mathcal{W}} \cdot F^k$, but not an equality. We thus have to work with pseudonatural transformations:

2.2. Definition. *For every category k the pseudonatural transformations from U^k to U are called* k-ary operations *on* VAR.

REMARK. Explicitly, a k-ary operation ω assigns

(i) to every variety $\mathcal{V}$ an "operation" functor

$$\omega_{\mathcal{V}} : \mathcal{V}^k \longrightarrow \mathcal{V}$$

and

(ii) to every algebraically exact functor $F : \mathcal{V} \to \mathcal{W}$ a natural isomorphism $\widehat{\omega}_F$:

$$\begin{array}{ccc} \mathcal{V}^k & \xrightarrow{\ \omega_{\mathcal{V}}\ } & \mathcal{V} \\ {\scriptstyle F^k}\downarrow & \overset{\widehat{\omega}_F}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathcal{W}^k & \xrightarrow[\ \omega_{\mathcal{W}}\ ]{} & \mathcal{W} \end{array}$$

such that the expected coherence conditions are satisfied: (a) $\widehat{\omega}_{\mathrm{id}} = \mathrm{id}$ and (b) if $F = F_2 \cdot F_1$ then $\widehat{\omega}_F = (F_2 * \widehat{\omega}_{F_1}) \cdot (\widehat{\omega}_{F_2} * F_1^k)$.

2.3. EXAMPLES. of operations on VAR.

(a) **Limit operations.** For every small category k we have a k-ary operation

$$k\text{-}\lim \tag{1}$$

of formation of limits of type k in varieties: to every variety $\mathcal{V}$ this assigns a functor

$$\omega_{\mathcal{V}} : \mathcal{V}^k \longrightarrow \mathcal{V}$$

choosing, for every diagram $D : k \to \mathcal{V}$, a limit $\omega_{\mathcal{V}}(D)$ in $\mathcal{V}$. To every algebraically exact functor $F : \mathcal{V} \to \mathcal{W}$ (which, being a right adjoint, preserves limits) it assigns the natural isomorphism

$$\widehat{\omega}_F : \omega_{\mathcal{W}} \cdot F^k \to F \cdot \omega_{\mathcal{V}}$$

defined as expected: given a diagram $D : k \to \mathcal{V}$, we have two limits of FD in $\mathcal{W}$, viz, $\omega_{\mathcal{W}}(FD)$ and $F(\omega_{\mathcal{W}} D)$, and the connecting isomorphism of these two limits is the D-component of $\widehat{\omega}_F$

In case k is discrete, we use the notation

$$\prod_k \tag{2}$$

instead of k-lim (for the operation of k-ary products).

(b) **Filtered-Colimit Operations.** For every small filtered category f we have an f-ary operation

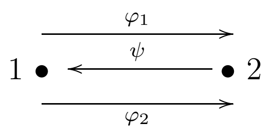
$$f\text{-}\underset{\rightarrow}{\mathrm{colim}} \tag{3}$$

of formation of filtered colimits of type f in varieties: to every variety $\mathcal{V}$ it assigns a functor $\omega_{\mathcal{V}}$ choosing colimits of diagrams of type f in $\mathcal{V}$. Since algebraically exact functors

$F : \mathcal{V} \to \mathcal{W}$ preserve filtered colimits, $\widehat{\omega_F}$ is defined as above: the D-component is the connecting isomorphism of the colimits

$$\omega_{\mathcal{W}}(FD) \text{ and } F(\omega_{\mathcal{V}}D) \text{ of the diagram } FD \text{ in } \mathcal{W}\,.$$

(c) **The Reflexive-Coequalizer Operation.** This is one operation of arity r, the finite category given by the following graph



and the commutativity conditions

$$\varphi_1\psi = \mathrm{id} = \varphi_2\psi\,.$$

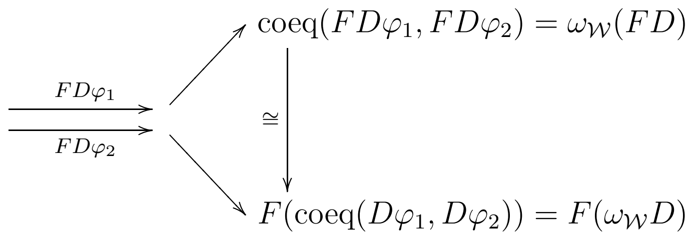
We denote by

$$r\text{-coeq} \tag{4}$$

the operation ω whose operation functors are

$$\omega_{\mathcal{V}} : \mathcal{V}^r \to \mathcal{V}\,,\ D \longmapsto \ \mathrm{coeq}(D\varphi_1, D\varphi_2) \qquad (\text{for } D : r \to \mathcal{V})\,.$$

Then for every algebraically exact functor $F : \mathcal{V} \to \mathcal{W}$ (which preserves reflexive coequalizers, see [ALR$_2$]) we have the natural isomorphism $\widehat{\omega}_F : \omega_{\mathcal{W}} \cdot F^k \to F \cdot \omega_{\mathcal{V}}$ whose D-component is the following isomorphism:



2.4. Remark. As mentioned above, in set-based (universal) algebra we can work with k-ary operations for any set k, but in a number of situations we want to restrict ourselves to k finite.

Analogously, we can (and will, in Section 6 below) work with k-ary operations for any category k. However, as already mentioned in the introduction, we have not been able to describe the corresponding category of algebras over CAT. We therefore restrict ourselves here to operations which (a) have small arities k and (b) the operation functors $\omega_{\mathcal{V}}$: $\mathcal{V}^k \to \mathcal{V}$ are accessible in a coherent sense, i.e., the accessibility rank is the same for all varieties $\mathcal{V}$. Than we say that ω has a rank:

DEFINITION. *An operation* ω *on* VAR *is called* ranked *provided that its arity* k *is small and there exists an infinite regular cardinal* λ *such that* $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ *preserves* λ*-filtered colimits for every variety* $\mathcal{V}$*. (We call* λ *a rank of* ω*.)*

EXAMPLES. (a) The operation k-lim is ranked: any infinite regular cardinal λ such that k has less than λ morphism is a rank of k-lim. In fact, k-limits commute with λ-filtered colimits in **Set**. Therefore, they commute in each variety $\mathcal{V}$ (in fact, in every locally finitely presentable category), i.e., the k-limit functor $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ preserve λ-filtered colimits.

(b) The operations k-$\underrightarrow{\text{colim}}$ and r-coeq have rank ω since colimits commute with colimits.

REMARK. As stated in the Introduction (1.4) we do not know at this moment whether it is consistent with set theory to assume that every operation of small arity on VAR is ranked. Let us remark that, however, this statement is not true absolutely: in any theory such that **Set** contains a proper class of measurable cardinals J. Reiterman has constructed in [Re] a functor $F : \mathbf{Set} \to \mathbf{Set}$ which is not accessible but preserves finite limits. This functor defines a non-ranked unary operation ω on VAR as follows:

Given a variety $\mathcal{V} = \mathbf{Mod}\,\mathcal{T}$ (where $\mathcal{T}$ is an algebraic theory for $\mathcal{V}$) let $\omega_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ be the functor which precomposes every model $M : \mathcal{T} \to \mathbf{Set}$ with F, i.e., $\omega_{\mathcal{V}} : M \longmapsto MF$. Since F preserves finite products, MF is a model, and it is easy to see that $\omega_{\mathcal{V}}$ is indeed an operation of arity 1 on VAR. It is not ranked because $\omega_{\mathbf{Set}}$ is not accessible.

2.5. COMPOSED OPERATIONS. Besides operations as described above we want to work with composed operations, e.g., a limit of filtered colimits. Recall that in the classical case mentioned above the discovery of [Law$_1$] was that one can work with natural transformations

$$U^k \to U^n$$

(which, since $U^n = U \times U \times \cdots \times U$, are just n-tuples of k-ary operations). This makes the task of composing operations very easy: they were just composites of natural transformations between powers of U.

Analogously in our case of $U : \text{VAR} \to \text{CAT}$: we work with *$n$-tuples of k-ary operations* (where n and k are categories), which are pseudonatural transformations

$$\omega : U^k \to U^n$$

2.6. EXAMPLES. (a) **Morphisms of operations.** Let $n = \to$ (the chain of length 1). A pseudonatural transformation

$$\omega : U^k \to U^{\to}$$

consists of two k-ary operations

$$\omega_1, \omega_2 : U^k \to U$$

obtained by composing ω with the obvious natural transformation U^d, $U^c : U^{\to} \to U$. (Here $d, c : 1 \to (\to)$ are the domain and codomain functors, respectively.) Plus a modification

$$\omega_1 \Rightarrow \omega_2$$

which is formed by the values of $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}^{\to}$ at the unique nonidentity arrow of $\to$. Thus, the pseudonatural transformation ω is what we will call a *morphism of k-ary operations* from ω_1 to ω_2. Notation: $\omega : \omega_1 \Rightarrow \omega_2$. The constant functor $t : (\to) \to 1$ gives *identity morphisms* $U^k \to U \xrightarrow{U^t} U^{\to}$.

(b) **Composing morphisms.** Let $n = \to\to$ (the chain of length 2). A pseudonatural transformation

$$\omega : U^k \longrightarrow U^{\to\to}$$

consists of three k-ary operations $\omega_1, \omega_2, \omega_3 : U^k \to U$ (obtained by composing ω with the three canonical projections $U^{\to\to} \to U$) together with three modifications $u : \omega_1 \Rightarrow \omega_2$, $v : \omega_2 \Rightarrow \omega_3$ and $w = vu : \omega_1 \Rightarrow \omega_3$ corresponding to the three non-identity morphisms of $\to\to$. Thus, the pseudonatural transformation ω is what we will call a *composite w of morphisms u and v* of k-ary operations.

(c) **Kernel pair** is an example of an r-tuple of operations of arity $\to$, where r denotes the category of 2.3 (c). More precisely, we have a pseudonatural transformation

$$\ker\ : U^{\to} \to U^r$$

whose components $\ker_{\mathcal{V}} : \mathcal{V}^{\to} \to \mathcal{V}^r$ assign to every morphism of $\mathcal{V}$ a kernel pair. Given an algebraically exact functor $F : \mathcal{V} \to \mathcal{W}$ then F preserves (limits thus) kernel pairs which yields a unique invertible 2-cell $\widehat{\ker_F}$: $F^r \cdot \ker_v \to \ker_w \cdot F^{\to}$.

(d) **Trivial operations.** As in any category of "two-dimensional" algebras, we have a trivial n-tuple of k-ary operations associated with an arbitrary functor $H : n \to k$: to every variety $\mathcal{V}$ assign the functor $\mathcal{V}^H : \mathcal{V}^k \to \mathcal{V}^n$ of precomposition with H.

2.7. Equations. between k-ary operations on VAR. We want to investigate an equational hull of VAR. What do we understand as equation $\omega \approx \omega'$ (for operations $\omega, \omega' : U^k \to U$)? Certainly not the strict equality: $\omega_{\mathcal{V}} = \omega'_{\mathcal{V}}$ for all varieties $\mathcal{V}$. We of course mean the non-strict variant of natural isomorphisms $\omega_{\mathcal{V}} \xrightarrow{\cong} \omega'_{\mathcal{W}}$ satisfying the appropriate coherence conditions. Our concept of morphism and composition above makes this quite formal as follows:

2.8. Definition. *By an* equation *between k-ary operations on* VAR *is meant an invertible morphism $\omega_1 \Rightarrow \omega_2$.*

2.9. Examples of equations. (1) Filtered colimits commute with finite limits. This is a property that all varieties have. The commutation of f-$\underrightarrow{\text{colim}}$ (f small, filtered) with k-lim (k finite) can be expressed by an equation as follows: we can form an operation of arity $f \times k$ by composing the k-tuple

$$(f\text{-}\underrightarrow{\text{colim}})^k : (\mathcal{V}^f)^k \to \mathcal{V}^k$$

with k-lim. And another operation of the same arity $f \times k$ by composing the f-tuple

$$(k\text{-}\lim)^f : (\mathcal{V}^k)^f \to \mathcal{V}^f$$

with f-lim. The fact that f-colimits commute with k-limits is expressed by the following equation

$$(k\text{-}\lim) \cdot (f\text{-}\underset{\rightarrow}{\mathrm{colim}})^k \approx (f\text{-}\underset{\rightarrow}{\mathrm{colim}}) \cdot (k\text{-}\lim)^f \qquad (f \text{ filtered, } k \text{ finite}) \qquad \text{(E1)}$$

since the required naturality conditions are easy to verify.

(2) Filtered colimits distribute over (infinite) products. This, less famous, property of all varieties (in fact, of all locally finitely presentable categories) has been observed already in [AGV]: given a set $D_i : f_i \to \mathcal{K}$ $(i \in k)$ of filtered diagrams in $\mathcal{K}$, denote by $D : \prod_{i\in k} f_i \to \mathcal{K}$ the diagram obtained by forming products in $\mathcal{K}$:

$$(d_i)_{i\in k} \longmapsto Dd_i \, .$$

then the canonical morphism $\mathrm{colim}\, D \to \prod_{i\in k} \mathrm{colim}\, D_i$ is invertible.

To express this as an equation, denote by f the product of the categories f_i (for $i \in k$) and by g their coproduct; observe that f is filtered (due to the Axiom of Choice).

The passage from $(D_i)_{i\in k}$ to $\prod \mathrm{colim}\, D_i$ is the composite of the product operation $\prod_k : \mathcal{V}^k \to \mathcal{V}$ (k considered as a discrete category) with the product of the operations f_i-colim:

$$\mathcal{V}^g \cong \prod_{i\in k} \mathcal{V}^{f_i} \xrightarrow{\prod(f_i\text{-}\underset{\rightarrow}{\mathrm{colim}})} \mathcal{V}^k \xrightarrow{\prod_k} \mathcal{V}$$

And the passage to $\mathrm{colim}\, D$ is expressed by using the trivial operations $o_i : \mathcal{V}^{f_i} \to \mathcal{V}^f$ corresponding to the i-th projection of f, and composing them with $f\text{-}\underset{\rightarrow}{\mathrm{colim}} : \mathcal{V}^f \to \mathcal{V}$:

$$\mathcal{V}^g \cong \prod_{i\in k} \mathcal{V}^{f_i} \xrightarrow{(o_i)_{i\in k}} \mathcal{V}^f \xrightarrow{f\text{-}\underset{\rightarrow}{\mathrm{colim}}} \mathcal{V}$$

This gives an equation expressing distributivity of filtered colimits over products as follows

$$\prod_k \cdot \prod_{i\in k} (f_i\text{-}\underset{\rightarrow}{\mathrm{colim}}) \approx (\prod_{i\in k} f_i\text{-}\underset{\rightarrow}{\mathrm{colim}}) \cdot (o_i)_{i\in k} \qquad (k \text{ a set, } f_i \text{ filtered for } i \in k)\,. \qquad \text{(E2)}$$

Again, the naturality conditions for the above canonical morphisms are easy to verify.

(3) Regular epimorphisms are product-stable. That is, given regular epimorphisms $e_i : A_i \to B_i$ $(i \in k,\ k \text{ a set})$ in a variety, the product morphism

$$\prod_{i\in k} e_i : \prod_{i\in k} A_i \to \prod_{i\in k} B_i$$

is a regular epimorphism. The reason why this holds in every variety is that products commute with coequalizers of equivalence relations: form a kernel pair $\mathrm{E}_i \rightrightarrows A_i$ of e_i, then a product $\Pi\,\mathrm{E}_i \rightrightarrows \Pi A_i$ yields a kernel pair of $\prod_{i\in k} e_i$ whose coequalizer is $\prod_{i\in k} e_i$.

To express this as an equation, combine the above operations $\prod_k : U^k \to U$, coeq : $U^r \to U$ and ker : $U^\to \to U^r$ as follows: we first multiply e_i, then form kernel pair and finally a coequalizer:

$$(U^k)^\to \xrightarrow{(\prod_k)^\to} U^\to \xrightarrow{\text{ker}} U^r \xrightarrow{\text{coeq}} U\,.$$

This is an operation of arity $(\to)\times k$ on VAR. Or we first form kernel pairs, then multiply and then form a coequalizer:

$$(U^\to)^k \xrightarrow{\text{ker}^k} (U^r)^k \cong (U^k)^r \xrightarrow{(\prod_k)^r} U^r \xrightarrow{\text{coeq}} U\,.$$

Thus, product-stability of regular epimorphisms is expressed by the following equation

$$\text{coeq}\cdot\text{ker}\cdot(\prod_k)^\to \approx \text{coeq}\cdot(\prod_k)^r\cdot\text{ker}^k \quad (k \text{ a set})\,. \tag{E3}$$

(4) There is a collection (E4) of equations which stem from the general fact that limits commute with limits, and colimits commute with colimits.

Remark. We will later see our reasons for believing it possible that

(1) there are no other ranked operations on VAR than those obtained by composing the three types above: k-lim, f-$\underrightarrow{\text{colim}}$ and coeq,

and

(2) there are no other equations between ranked operations on VAR than those which follow from (E1) – (E4).

We will see that the process of composition of the three types of operations has a particularly simple structure due to the fact, proved below, that both filtered colimits and reflexive coequalizers distribute over limits in varieties.

3. Limits and Sifted colimits

In the present section we consider two pseudomonads on CAT: *Lim*, the pseudomonad of free limit-completions of categories, and *Sind*, the pseudomonad of free completions under sifted colimits. The first one is well-known and the second one is closely related to Grothendieck's free completion *Ind* under filtered colimits. The main result is that

Sind distributes over *Lim*,

which implies, as proved in [M₃], that the composite *Sind* ∘ *Lim* obtains a structure of a pseudomonad on CAT. In subsequent sections we show that algebras of that pseudomonad form an equational hull of VAR with respect to ranked operations.

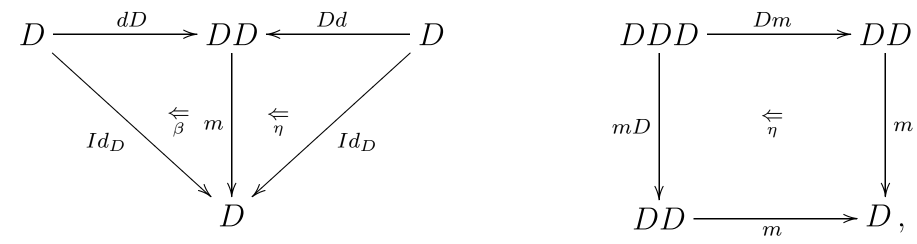
3.1. Pseudomonads and Algebras. (1) Recall the concept of a *pseudomonad* on a 2-category $\mathcal{K}$, as introduced by B. Day and R. Street [DS]; we use the notation of F. Marmolejo [M_2]. A pseudomonad $\mathbb{D} = (D, d, m, \eta, \beta, \mu)$ consists of a bifunctor

$$D : \mathcal{K} \to \mathcal{K}$$

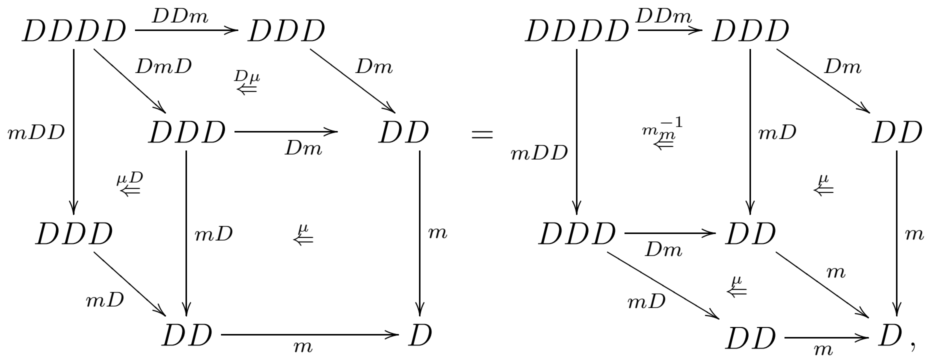
with 1-cells

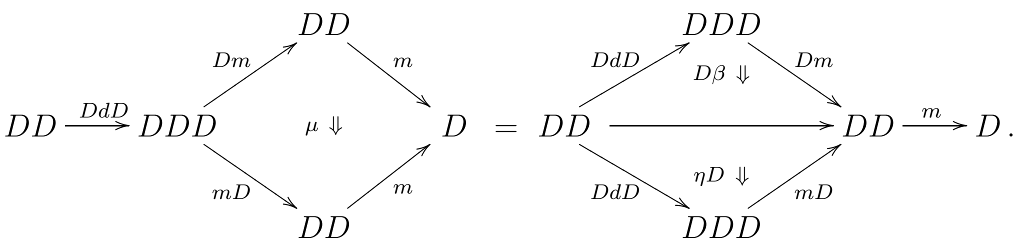
$$d : \mathrm{id}_{\mathcal{K}} \to D \quad \text{and} \quad m : D \cdot D \to D$$

together with invertible 2-cells



such that the following two equalities are satisfied:





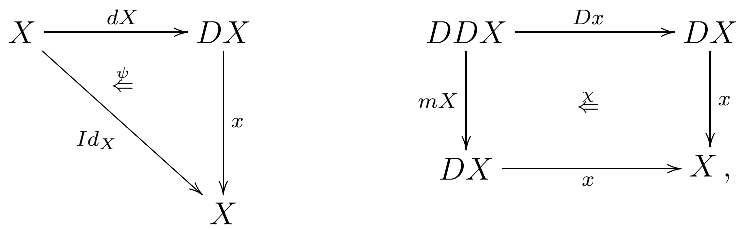
(2) Recall further the concept of an *algebra* for a pseudomonad $\mathbb{D}$: it is a quadruple

$$(X, x, \psi, \chi)$$

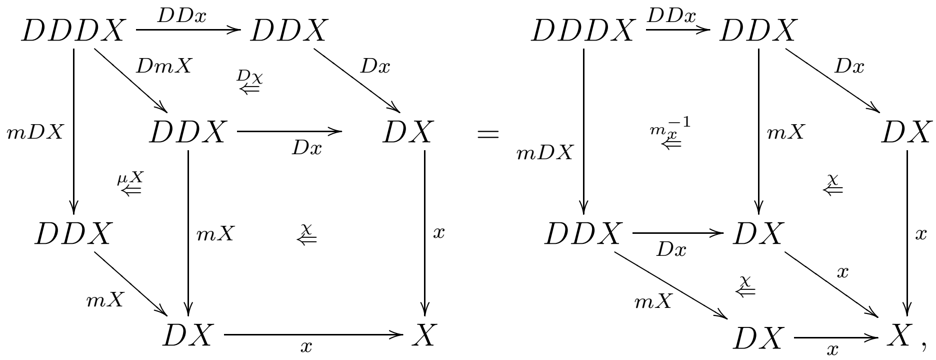
consisting of

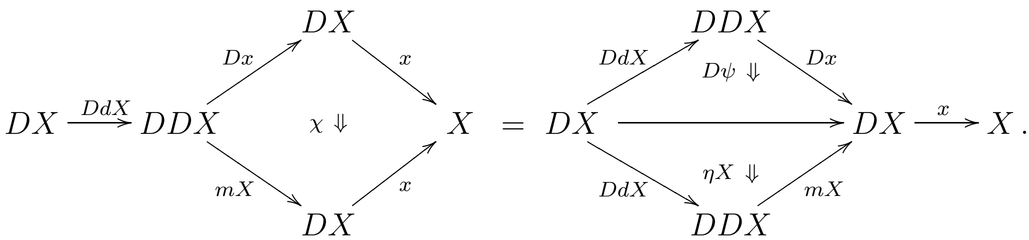
an object (0-cell) X,

a morphism (1-cell) $x : DX \to X$
and
invertible 2-cells



This data must satisfy the following two equations





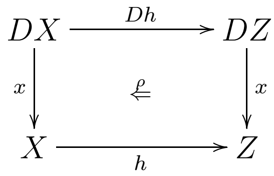
We denote by

$$\mathcal{K}^{\mathbb{D}}$$

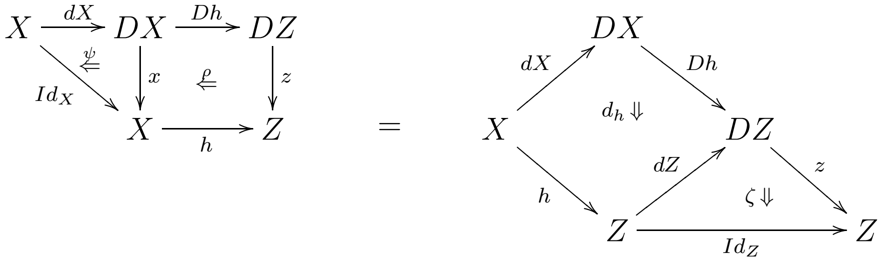
the 2-category of $\mathbb{D}$-algebras. Its 1-cells, called *homomorphisms*, from (X, x, ψ, χ) to (Z, z, ξ, θ) are pairs (h, ϱ) where

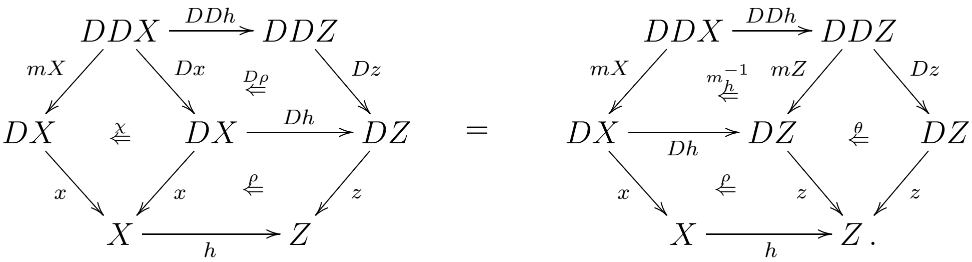
$$h : X \to Z$$

is a 1-cell of $\mathcal{K}$ and



is an invertible 2-cell, such that the following two equations are satisfied.





The 2-cells of $\mathcal{K}^{\mathbb{D}}$ are defined as follows:

Given homomorphisms (h, ρ), $(h', \rho') : (X, x, \psi, \chi) \longrightarrow (Z, z, \zeta, \theta)$, a 2-cell

$$\xi : (h, \rho) \longrightarrow (h', \rho')$$

is a 2-cell $\xi : h \longrightarrow h'$ of $\mathcal{K}$ such that $(\xi \circ x) \cdot \rho = \rho' \cdot (z \circ D\xi)$. Vertical composition is the obvious one.

Horizontal composition: for $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$ and $(k, \pi) : (\zeta, \theta) \longrightarrow (\tau, \sigma)$ we define $(k, \pi) \circ (h, \rho) = (k \circ h, (k \circ \rho) \cdot (\pi \circ Dh))$.

3.2. Example: Free limit completion *Lim*. This is a well-known pseudomonad on CAT of freely completing categories under limits. In fact, A. Kock has described it as a strict 2-monad in [K]. Recall that for small categories we have

$$Lim\ \mathcal{K} = (\mathbf{Set}^{\mathcal{K}})^{\mathrm{op}}.$$

The pseudomonad *Lim* is a co-KZ-doctrine (see [K], [M$_1$]), i.e.

$$\mu_{\mathcal{K}}^{Lim} \vdash Lim\ \eta_{\mathcal{K}}^{Lim} \tag{5}$$

for every category $\mathcal{K}$.

If $\mathcal{K}$ is locally small then $Lim\ \mathcal{K}$ is well-known to be equivalent to the closure of representable functors in $(\mathbf{Set}^{\mathcal{K}})^{\mathrm{op}}$ under limits.

3.3. Sifted colimits play a very analogous role in the theory of varieties to the role of filtered colimits in the theory of locally finitely presentable categories, see [ALR_1].

The following has been introduced by C. Lair [L] under the name tamisante:

DEFINITION. *A small category $\mathcal{D}$ is called* sifted *if $\mathcal{D}$-colimits commute in* **Set** *with finite products.*

Colimits of diagrams with sifted domains are called *sifted colimits*.

3.4. EXAMPLE. (1) Filtered categories are sifted.

(2) The category r of 2.3 (c) is sifted – that is, reflexive coequalizers are sifted colimits.

(3) Every small category with finite coproducts is sifted, see [AR_1].

3.5. LEMMA. (see [ALR_1]) *A functor between varieties is algebraically exact iff it preserves limits and sifted colimits.*

REMARK. For categories other than varieties we extend the terminology and call a functor *algebraically exact* if it preserves limits and sifted colimits. Observe that due to 3.4 (2) every algebraically exact functor is exact.

3.6. EXAMPLE. of an operation on VAR: for every small sifted category s we can form the operation

$$s\text{-colim}$$

of colimits of type s, analogously to 2.3 (b) above (and generalizing 2.3 (c), of course).

3.7. EXAMPLE: FREE COMPLETION SIND UNDER SIFTED COLIMITS. This is a pseudomonad on CAT quite analogous to Grothendieck's free completion $Ind\mathcal{K}$ under filtered colimits. The latter can be identified, for locally small categories $\mathcal{K}$, with the category of all flat presheaves in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ (i.e., a closure of representable functors under filtered colimits in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$), with the universal map $\eta_{\mathcal{K}}^{Ind} : \mathcal{K} \to Ind\mathcal{K}$ being a codomain restriction of the Yoneda embedding $Y_{\mathcal{K}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$. And whenever $\mathcal{K}$ is small and has finite colimits, then

$$Ind\mathcal{K} = \mathrm{Lex}\,\mathcal{K}^{\mathrm{op}}$$

is the category of all presheaves preserving finite limits. In general $Ind\mathcal{K}$ can be, for any category $\mathcal{K}$, described as a suitable category of all filtered diagrams in $\mathcal{K}$, see [JJ].

Quite analogously, we can introduce a free completion

$$\eta_{\mathcal{K}}^{Sind} : \mathcal{K} \to Sind\ \mathcal{K}$$

of any category $\mathcal{K}$ under sifted colimits.

3.8. LEMMA. (1) *For every locally small category $\mathcal{K}$, Sind $\mathcal{K}$ can be identified with the category of all presheaves in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ which are sifted colimits of representables (where $\eta_{\mathcal{K}}^{Sind}$ is a codomain restriction of $Y_{\mathcal{K}}$).*

(2) *If $\mathcal{K}^{\mathrm{op}}$ is locally presentable, then Sind $\mathcal{K}$ can be identified with the category of all accessible presheaves in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ preserving finite products.*

PROOF. The proof of (1) for $\mathcal{K}$ small has been performed in [AR$_1$], the generalization to $\mathcal{K}$ locally small is straightforward.

For (2), observe that a sifted colimit of representables preserves finite products (since they commute in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ with sifted colimits) and is accessible. Conversely, given a λ-accessible functor $F : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ where $\mathcal{K}^{\mathrm{op}}$ is locally λ-presentable, then in the usual diagram $El(F)$ of elements of F (i.e., pairs (K, x) with $K \in \mathrm{obj}\,\mathcal{K}$ and $x \in FK$) all objects (K, x) with K λ-presentable obviously form a cofinal, essentially small subdiagram $El_\lambda(F)$. If F preserves finite products, then $El(F)$ has finite coproducts, and a closure of $El_\lambda(F)$ under finite coproducts yields an essentially small diagram of representables with colimit object F; that diagram is sifted by 3.4 (3). ■

3.9. LEMMA. ([AR$_1$]) *For every algebraic theory $\mathcal{K}$, Sind $\mathcal{K}^{\mathrm{op}}$ is a variety. Conversely, every variety is a free sifted-colimit completion of the dual of any of its algebraic theories.*

3.10. SIFTED COLIMITS DISTRIBUTE OVER LIMITS. A crucial fact that enables us to describe an equational hull of VAR w.r.t. ranked operations is that the 2-monad *Sind* distributes over the 2-monad *Lim*. Recall that for "ordinary" (1-dimensional) monads $\mathbb{T}$ and $\mathbb{S}$ over a category $\mathcal{K}$, a distributive law of J. Beck [B] is, equivalently, one of the following structures:

(i) a natural transformation from $S \circ T$ to $T \circ S$ compatible with the units and multiplications of $\mathbb{T}$ and $\mathbb{S}$,

or

(ii) a lifting $\widehat{\mathbb{T}}$ of the monad $\mathbb{T}$ to the category $\mathcal{K}^{\mathbb{S}}$ of algebras of $\mathbb{S}$.

Given (i) or (ii), the composite $T \circ S$ obtains a structure of a monad, say $\mathbb{T} \circ \mathbb{S}$, whose category of algebras is equivalent to the category of algebras of the lifted monad $\widehat{\mathbb{T}}$, i.e.,

$$\mathcal{K}^{\mathbb{T}\circ\mathbb{S}} \text{ is equivalent to } (\mathcal{K}^{\mathbb{S}})^{\widehat{\mathbb{T}}} \text{ over } \mathcal{K}.$$

We now want to apply all this to $\mathbb{T} = Sind$ and $\mathbb{S} = Lim$: we prove below that *Sind* lifts to the quasicategory CAT^{Lim} of algebras of *Lim* (= the quasicategory of all complete categories and functors preserving limits). And we would like to conclude that the composite $Sind \circ Lim$ is a pseudomonad over CAT whose algebras are precisely the algebras of *Sind* lifted to complete categories.

Distributive laws for pseudomonads have been studied by F. Marmolejo in [M$_2$] and [M$_3$]. We want to apply his results to a distributive law of *Sind* (a KZ-doctrine) over *Lim* (a co-KZ-doctrine). In that case there exists at most one distributive law, as proved in [M$_2$], so that we can formulate the result by saying that

$$Sind \text{ distributes over } Lim\ .$$

The form we prove this is by exhibiting a lifting of *Sind* from CAT to CAT^{Lim}:

3.11. Theorem. (*Sind* distributes over *Lim*)

(1) *For every complete category* $\mathcal{K}$ *the category* $\mathit{Sind}\ \mathcal{K}$ *is complete.*

(2) *For every functor* $F : \mathcal{K} \to \mathcal{L}$ *preserving limits, where* $\mathcal{K}$ *and* $\mathcal{L}$ *are complete, the functor* $\mathit{Sind}\ F : \mathit{Sind}\ \mathcal{K} \to \mathit{Sind}\ \mathcal{L}$ *also preserves limits.*

(3) *Both* $\eta_{\mathcal{K}}^{\mathit{Sind}}$ *and* $\mu_{\mathcal{K}}^{\mathit{Sind}}$ *preserve limits (for any category* $\mathcal{K}$*).*

Proof. I. We first prove the statements (1) and (2) for the case that $\mathcal{K} = \mathit{Lim}\ \mathcal{K}_0$ is a free completion of a category $\mathcal{K}_0$ under limits, and analogously $\mathcal{L} = \mathit{Lim}\ \mathcal{L}_0$, and $F = \mathit{Lim}\ F_0$ for some $F_0 : \mathcal{K}_0 \to \mathcal{L}_0$.

(I.A) Suppose that $\mathcal{K}_0$ and $\mathcal{L}_0$ are small. Then $\mathit{Lim}\ \mathcal{K}_0 \cong (\mathbf{Set}^{\mathcal{K}_0})^{\mathrm{op}}$ is locally presentable, therefore, $\mathit{Sind}\ \mathcal{K}$ is the category of all accessible functors in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ preserving finite products. Having a diagram $D : \mathcal{D} \to \mathit{Sind}\ \mathcal{K}$, there is a cardinal λ such that $\mathcal{D}$ has less than λ morphisms and, for each object $d \in \mathcal{D}$, the functor Dd is λ-accessible. Since limits over categories having less than λ morphisms commute in $\mathbf{Set}$ with λ-filtered colimits, $\lim \mathcal{D}$ is λ-accessible. We have proved that limits of accessible functors from $\mathcal{K}^{op}$ to $\mathbf{Set}$ are accessible. Consequently, $\mathit{Sind}\ \mathcal{K}$ is closed under limits in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$, see 3.7, thus, $\mathit{Sind}\ \mathcal{K}$ is complete.

For every functor $F_0 : \mathcal{K}_0 \to \mathcal{L}_0$ the functor $\mathit{Lim}\ F_0 : (\mathbf{Set}^{\mathcal{K}_0})^{\mathrm{op}} \to (\mathbf{Set}^{\mathcal{L}_0})^{\mathrm{op}}$ is right adjoint to $(\mathbf{Set}^{F_0})^{\mathrm{op}} : (\mathbf{Set}^{\mathcal{L}_0})^{\mathrm{op}} \to (\mathbf{Set}^{\mathcal{K}_0})^{\mathrm{op}}$. *Sind* is a pseudofunctor, thus, $\mathit{Sind}\,(\mathit{Lim}\ F_0)$ is a right adjoint to $\mathit{Sind}\,(\mathbf{Set}^{F_0})^{\mathrm{op}}$; consequently, $\mathit{Sind}\ F = \mathit{Sind}\,(\mathit{Lim}\ F_0)$ preserves limits.

(I.B) Let $\mathcal{K}_0$ and $\mathcal{L}_0$ be arbitrary categories. Express $\mathcal{K}_0$ as an ∞-filtered colimit of small categories $\mathcal{K}_{0i} \subseteq \mathcal{K}_0$ $(i \in I)$, see Introduction, 1.5. The connecting morphisms of that colimit are the inclusions $E_{ij} : \mathcal{K}_{0i} \hookrightarrow \mathcal{K}_{0j}$ for $i \leq j$. Both *Lim* and *Sind* obviously preserve ∞-filtered colimits (in fact, any free completion functor dealing with small diagrams does). Thus,

$$\mathit{Sind}\,(\mathit{Lim}\ \mathcal{K}_0) = \operatorname*{colim}_{i \in I} \mathit{Sind}\,(\mathit{Lim}\ \mathcal{K}_{0i})$$

is an ∞-filtered colimit of categories $\mathit{Sind}\,(\mathit{Lim}\ \mathcal{K}_{0i})$ which are complete, due to I.A, with the connecting functors $\mathit{Sind}\,(\mathit{Lim}\ E_{ij})$ preserving limits, also by I.A. It follows that $\mathit{Sind}\,(\mathit{Lim}\ \mathcal{K}_0)$ is complete.

Analogously do we derive that $\mathit{Sind}\,(\mathit{Lim}\ F_0)$ preserves limits: express $\mathcal{K}_0$ and $\mathcal{L}_0$ as ∞-filtered colimits of small subcategories

$$\mathcal{K}_{0i} \subseteq \mathcal{K}_0 \quad \text{and} \quad \mathcal{L}_{0i} \subseteq \mathcal{L}_0 \qquad (i \in I)$$

such that F_0 restricts to

$$F_{0i} : \mathcal{K}_{0i} \to \mathcal{L}_{0i} \qquad (i \in I)\,.$$

Then $F_0 = \operatorname*{colim}_{i \in I} F_{0i}$ implies $\mathit{Sind}\,(\mathit{Lim}\ F_0) = \operatorname*{colim}_{i \in I} \mathit{Sind}\,(\mathit{Lim}\ F_{0i})$. The latter is an ∞-filtered colimit of functors $\mathit{Sind}\,(\mathit{Lim}\ F_{0i})$ which, by I.A, preserve limits. And the connecting functors of that colimit also preserve limits. Consequently, $\mathit{Sind}\,(\mathit{Lim}\ F_0)$ preserves limits.

II. We prove (1) generally: let $\mathcal{K}$ be complete, and let

$$L_{\mathcal{K}} : \mathit{Lim}\ \mathcal{K} \to \mathcal{K}$$

be a functor of computation of limits in $\mathcal{K}$. Then

$$L_{\mathcal{K}}\eta_{K}^{\mathit{Lim}} \cong \mathrm{id} \qquad \text{and} \qquad \eta_{\mathcal{K}}^{\mathit{Lim}} \dashv L_{\mathcal{K}}\,.$$

Since Sind is a pseudofunctor, we conclude

$$\mathit{Sind}\ L_{\mathcal{K}}\,\mathit{Sind}\ \eta_{\mathcal{K}}^{\mathit{Lim}} \cong \mathrm{id} \quad \text{and} \quad \mathit{Sind}\ \eta_{\mathcal{K}}^{\mathit{Lim}} \dashv \mathit{Sind}\ L_{\mathcal{K}}\,.$$

Thus a limit of a diagram $D : \mathcal{D} \to \mathit{Sind}\ \mathcal{K}$ can be computed from a limit of $\mathit{Sind}\ \eta_{\mathcal{K}}^{\mathit{Lim}} \cdot D$ in the (complete, see I.) category $\mathit{Sind}\ \mathit{Lim}\ \mathcal{K}$ as follows:

$$\begin{aligned} &\mathit{Sind}\ L_{\mathcal{K}}(\lim[\mathit{Sind}\ \eta_{\mathcal{K}}^{\mathit{Lim}} \cdot D]) \\ &\cong \lim[\mathit{Sind}\ L_{\mathcal{K}} \cdot \mathit{Sind}\ \eta_{\mathcal{K}}^{\mathit{Lim}} \cdot D] \\ &\cong \lim D\,. \end{aligned} \tag{6}$$

Analogously we prove (2) generally: let $F : \mathcal{K} \to \mathcal{L}$ preserve limits, and let

$$L_{\mathcal{K}} : \mathit{Lim}\ \mathcal{K} \to \mathcal{K} \quad \text{and} \quad L_{\mathcal{L}} : \mathit{Lim}\ \mathcal{L} \to \mathcal{L}$$

be as above. Since F preserves limit, we have a natural isomorphism

$$\begin{array}{ccc} \mathit{Lim}\ \mathcal{K} & \xrightarrow{\mathit{Lim}\ F} & \mathit{Lim}\ \mathcal{L} \\ {\scriptstyle L_{\mathcal{K}}}\downarrow & \overset{\cong}{\Rightarrow} & \downarrow{\scriptstyle L_{\mathcal{L}}} \\ \mathcal{K} & \xrightarrow[F]{} & L \end{array}$$

This yields a natural isomorphism

$$\begin{array}{ccc} \mathit{Sind}\,(\mathit{Lim}\ \mathcal{K}) & \xrightarrow{\mathit{Sind}\,(\mathit{Lim}\ F)} & \mathit{Sind}\,(\mathit{Lim}\ \mathcal{L}) \\ {\scriptstyle \mathit{Sind}\ L_{\mathcal{K}}}\downarrow & \overset{\cong}{\Rightarrow} & \downarrow{\scriptstyle \mathit{Sind}\,(L_{\mathcal{L}})} \\ \mathit{Sind}\ \mathcal{K} & \xrightarrow[\mathit{Sind}\ F]{} & \mathit{Sind}\ L \end{array}$$

Since $\mathit{Sind}\ L_{\mathcal{L}}$ and $\mathit{Sind}\,(\mathit{Lim}\ F)$ preserve limits, by I., and limits in $\mathit{Sind}\ \mathcal{K}$ are computed as above, see (6), we conclude that $\mathit{Sind}\ F$ preserves limits:

$$\begin{aligned}
&\mathit{Sind}\ F(\lim D) \cong \mathit{Sind}\ F\big(\,\mathit{Sind}\ L_{\mathcal{K}}(\mathit{lim}[\mathit{Sind}\ \eta^{\mathit{Lim}}_{\mathcal{K}}\cdot D])\big)\\
&\cong \mathit{Sind}\ L_{\mathcal{L}}\cdot \mathit{Sind}\,(\mathit{Lim}\ F)\big(\lim[\mathit{Sind}\ \eta^{\mathit{Lim}}_{\mathcal{K}}\cdot D]\big)\\
&\cong \lim\big(\,\mathit{Sind}\ L_{\mathcal{L}}\cdot \mathit{Sind}\,(\mathit{Lim}\ F)\cdot \mathit{Sind}\ \eta^{\mathit{Lim}}_{\mathcal{K}}\cdot D\big)\\
&\cong \lim(\mathit{Sind}\ L_{\mathcal{L}}\cdot \mathit{Sind}\ \eta^{\mathit{Lim}}_{\mathcal{L}}\cdot \mathit{Sind}\ F\cdot D)\\
&\cong \lim(\mathit{Sind}\ F\cdot D)\,.
\end{aligned}$$

III. The proof of (3) is clear: if $\mathcal{K}$ is (locally) small then $\eta^{\mathit{Sind}}_{\mathcal{K}}$ preserves limits because it is a codomain restriction of the Yoneda embedding $Y_{\mathcal{K}}:\mathcal{K}\to \mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ and the category $\mathit{Sind}\ \mathcal{K}$ is closed under limits in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ (since it contains all representable functors). In a general case, we express $\mathcal{K}$ as an ∞-filtered colimit of small subcategories.

And $\mu^{\mathit{Sind}}_{\mathcal{K}}$ preserves limits because it is a right adjoint, see 3.2. ■

3.12. Corollary. *$\mathit{Sind}\circ \mathit{Lim}$ is a pseudomonad over* CAT *whose quasicategory of algebras is equivalent to the quasicategory of algebras of $\widehat{\mathit{Sind}}$. The latter is the lifting of Sind to the quasicategory of complete categories.*

In fact, given pseudomonads $\mathbb{S}$ and $\mathbb{T}$ on a 2-category $\mathcal{K}$, then a lifting $\widehat{\mathbb{S}}$ of $\mathbb{S}$ to the category $\mathcal{K}^{\mathbb{T}}$ of algebras for $\mathbb{T}$ is equivalent to providing a pseudomonad structure $\mathbb{S}\circ\mathbb{T}$ of the corresponding composite endofunctor of $\mathcal{K}$ – see [M_2]. And the category of algebras for $\mathbb{S}\circ\mathbb{T}$ is biequivalent to that of $\widehat{\mathbb{S}}$ as proved in [M_3].

4. Equational Hull of VAR

4.1. We are going to describe an equational hull of VAR w.r.t. all operations with rank. Before proceeding formally, let us informally say that the basic observation is that for every small category k we can almost form a "free variety" on k: it is the category

$$\mathbf{Mod}\,(\mathbf{Set}^{k})$$

of all accessible functors from $\mathbf{Set}^{k}$ to $\mathbf{Set}$ preserving finite products. Well, that category is not a variety (unless $k=\emptyset$) because it does not have a small regular generator. But we have a canonical embedding

$$ev: k \longrightarrow \mathbf{Mod}\,(\mathbf{Set}^{k})$$

assigning to every object x of k the evaluation-at-x functor

$$ev(x): \mathbf{Set}^{k} \longrightarrow \mathbf{Set}\ .$$

And ev has the following universal property:

Let $\mathcal{V}$ be a variety, say $\mathcal{V}=\mathbf{Mod}\,\mathcal{T}$ for a (small) algebraic theory $\mathcal{T}$. We have natural bijections as follows:

$k \xrightarrow{H} \mathcal{V} = \mathbf{Mod}\,\mathcal{T}$	an arbitrary functor
$k \times \mathcal{T} \xrightarrow{H'} \mathbf{Set}$	preserves finite products in the 2^{nd} coordinate
$\mathcal{T} \xrightarrow{H''} \mathbf{Set}^{\,k}$	preserves finite products
$\mathbf{Mod}\,(\mathbf{Set}^{\,k}) \xrightarrow{\mathbf{Mod}\,H''} \mathbf{Mod}\,\mathcal{T} = \mathcal{V}$	algebraically exact functor

Let us explain the last step, in which $\mathbf{Mod}\,H''$ is the functor $F \longmapsto F \cdot H''$:

(1) The functor $\mathbf{Mod}\,H''$ is algebraically exact.

In fact, it is a domain-codomain restriction of the functor

$$- \cdot H'' : \mathbf{Set}^{\,\mathbf{Set}^{\,k}} \to \mathbf{Set}^{\,\mathcal{T}}$$

(which, obviously, preserves limits and colimits). Now $\mathbf{Mod}\,\mathcal{T}$ is clearly closed under limits and sifted colimits in $\mathbf{Set}^{\,\mathcal{T}}$ – recall that sifted colimits commute in $\mathbf{Set}$ with finite products, thus, they commute with them in $\mathbf{Set}^{\,\mathcal{T}}$ too. And $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$ is closed in $\mathbf{Set}^{\,\mathbf{Set}^{\,k}}$ under limits and sifted colimits. Thus our restricted functor preserves limits and sifted colimits.

(2) Every algebraically exact functor $F : \mathbf{Mod}\,(\mathbf{Set}^{\,k}) \to \mathcal{V}$ is naturally equivalent to $\mathbf{Mod}\,H''$ where $H = F \cdot ev$. In fact, every object of $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$, i.e., accessible, finite-product preserving functor $D : \mathbf{Set}^{\,k} \to \mathbf{Set}$, is a sifted colimit of representables, see 3.8; say

$$D = \operatorname*{colim}_{i\in I} \mathbf{Set}^{\,k}(X_i, -) \qquad (I \text{ small, sifted})\,.$$

And every $X_i \in \mathbf{Set}^{\,k}$ is, of course, a small colimit of representables

$$X_i = \operatorname*{colim}_{j\in J_i} k(x_{ij}, -) \qquad (J_i \text{ small})\,.$$

Since $ev(x_{i,j}) \cong \mathbf{Set}^{\,k}\big(k(x_{ij}, -), -\big)$, we obtain

$$\begin{aligned} D &\cong \operatorname*{colim}_{i\in I} \lim_{j\in J} \mathbf{Set}^{\,k}\big(k(x_{i,j}, -), -\big) \\ &\cong \operatorname*{colim}_{i\in I} \lim_{j\in J_i} ev(x_{ij})\,. \end{aligned}$$

Since F preserves sifted colimits and limits, we conclude

$$FD \cong \operatorname*{colim}_{i\in I} \lim_{j\in J_i} Hx_{ij}\,.$$

It is easy to verify the same formula for $\mathbf{Mod}\,H''$:

$$\mathbf{Mod}\,H''D \cong \operatorname*{colim}_{i\in I} \lim_{j\in J_i} Hx_{ij}\,.$$

Consequently, $F \cong \mathbf{Mod}\,H''$.

4.2. Corollary. *Ranked k-ary operations on* VAR *are encoded by* $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$ *as follows:*

(i) *every object D of $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$ defines a k-ary operation*

$$\widehat{D} : U^k \to U$$

whose components are given by

$$\widehat{D}_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V} : H \longmapsto \mathbf{Mod}\,H''(D) \ (H : k \to \mathcal{V})$$

(ii) *every morphism $d : D_1 \to D_2$ of $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$ defines a morphism*

$$\widehat{d} : \widehat{D}_1 \Rightarrow \widehat{D}_2$$

of k-ary operations, i.e., a pseudonatural transformation

$$\widehat{d} : U^k \to U^{\to}$$

whose components are

$$\widehat{d}_{\mathcal{V}} : (\widehat{D}_1)_{\mathcal{V}} \to (\widehat{D}_2)_{\mathcal{V}}\,, \quad H \longmapsto (D_1 H'' \xrightarrow{dH''}) D_2 H''$$

(iii) *Every k-ary operation of rank is equal (in the sense of 2.7) to some $\widehat{D}$ for $D \in \mathbf{Mod}\,(\mathbf{Set}^{\,k})$. That is, given*

$$\omega : U^k \to U \quad \textit{of rank} \quad \lambda$$

we can find D with $\omega \approx \widehat{D}$.

The last statement would be trivial if $\mathcal{V} = \mathbf{Mod}\,(\mathbf{Set}^{\,k})$ would be a variety: then we could evaluate $\omega_{\mathcal{V}} : \mathcal{V}^k \to \mathcal{V}$ in $ev \in \mathcal{V}^k$ and obtain $D = \omega_{\mathcal{V}}(ev)$. The equality $\omega \approx \widehat{D}$ then follows from the fact that for every $H \in \mathcal{V}^k$ the functor $-.H'' : \mathbf{Mod}\,(\mathbf{Set}^{\,k}) \to \mathcal{V}$, being algebraically exact, must preserve ω.

To overcome the difficulty, denote by

$$E_\lambda : \mathbf{Set}^{\,k}_{\,\lambda} \to \mathbf{Set}^{\,k} \qquad\qquad (\lambda \in \text{ Card})$$

the inclusion functor of the subcategory of all functors in $\mathbf{Set}^{\,k}$ preserving λ-filtered colimits. Since each $\mathbf{Set}^{\,k}_{\,\lambda}$ is essentially small, $\mathcal{V}_\lambda = \mathbf{Mod}\,(\mathbf{Set}^{\,k}_{\,\lambda})$ is a variety. And $\mathbf{Mod}\,(\mathbf{Set}^{\,k})$ is a limit of these varieties, with a limit cone

$$\mathbf{Mod}\,E_\lambda : \mathbf{Mod}\,(\mathbf{Set}^{\,k}) \to \mathbf{Mod}\,(\mathbf{Set}^{\,k}_{\,\lambda})$$

formed by algebraically exact functors (and the connecting morphisms $\mathbf{Mod}\,(\mathbf{Set}^{\,k}_{\,\lambda}) \to \mathbf{Mod}\,(\mathbf{Set}^{\,k}_{\,\mu})$ for $\mu \le \lambda$ are all algebraically exact). We obtain objects

$$D_\lambda = \omega_{\mathcal{V}_\lambda}(\mathbf{Mod}\,E_\lambda \cdot ev) \quad \text{in} \quad \mathcal{V}_\lambda\,,$$

and if ω has a rank λ_0, then in fact $D_\lambda = D_{\lambda_0}$ for all $\lambda \ge \lambda_0$, and $\omega \approx \widehat{D}_{\lambda_0}$.

4.3. Remark. Observe that for small categories k,

$$(\mathbf{Set}^{k})^{\mathrm{op}} \cong \mathit{Lim}\ k$$

is a free limit-completion, and since this category is locally presentable, its free completion under sifted colimits is precisely $\mathbf{Mod}\,(-)$, see 3.7:

$$\mathbf{Mod}\,(\mathbf{Set}^{k}) \cong \mathit{Sind}\,(\mathit{Lim}\ k) \qquad k \text{ small}\,.$$

Thus we see that "free varieties" are precisely the free algebras of the composite pseudomonad *Sind* ∘ *Lim* of Section 3. Moreover, for k small, all ranked k-ary operations are encoded by *Sind* (*Lim* k), thus, they are the sifted-colimits operations applied to the limit operations. It follows that the appropriate homomorphisms are the functors preserving limits and sifted colimits – but these are precisely the homomorphisms (1-cells) of the 2-quasicategory of algebras of *Sind* ∘ *Lim* . All this proves that the algebras of *Sind* ∘ *Lim* form an equational hull of VAR w.r.t. ranked operations. Well, except that, strictly speaking, we have not verified that this equational hull is presented by a pseudomonad; this will be done in the last section, where we introduce that pseudomonad as the "ranked core" of the pseudomonad representing the equational hull w.r.t. all operations. But modulo this result, we conclude the following

4.4. Corollary. *The 2-quasicategory of algebras of the pseudomonad Sind ∘ Lim is an equational hull of* VAR *w.r.t. all operations with rank.*

We are now ready to describe the algebras of the pseudomonad *Sind* ∘ *Lim* :

4.5. Definition. *A category $\mathcal{K}$ is called* algebraically exact *if it is complete, has sifted colimits, and these distribute over limits in the following sense: the functor Colim : Sind $\mathcal{K} \to \mathcal{K}$ computing sifted colimits in $\mathcal{K}$ preserves limits.*

4.6. Remark. (1) A more precise definition of *Colim* is: this is the essentially unique functor preserving sifted colimits with

$$\mathit{Colim} \cdot \eta_{\mathcal{K}} = \mathit{Ind}_{\mathcal{K}}\,.$$

(2) It is evident that algebraic exactness is precisely the property of being an algebra of the lifting of *Sind* to the 2-quasicategory of all complete categories. Since the latter is the 2-quasicategory of all algebras of *Lim* , we have just described the algebras of *Sind* ∘ *Lim* , see 3.12.

(3) We denote by

ALG

the 2-quasicategory of all algebraically exact categories, all algebraically exact functors (i.e., functors preserving limits and sifted colimits) and all natural transformations.

This is biequivalent to the quasicategory of all algebras of the composite pseudomonad *Sind* ∘ *Lim* .

5. Properties of Algebraically Exact Categories

5.1. Theorem. *Every algebraically exact category $\mathcal{K}$ has the following properties:*

(a) $\mathcal{K}$ *is complete,*
(b) $\mathcal{K}$ *is exact,*
(c) *filtered colimits commute in $\mathcal{K}$ with finite products,*
(d) *filtered colimits distribute in $\mathcal{K}$ over all products (see 2.8 (2))*
and
(e) *regular epimorphisms in $\mathcal{K}$ are product-stable.*

Proof. Assume at first that $\mathcal{K}$ is a locally small algebraically exact category. The category $\mathbf{Set}$ fulfills (a)-(e) and therefore so does $\mathbf{Set}^{\mathcal{K}^{op}}$. Following 3.11, *Sind* $\mathcal{K}$ is closed under limits and sifted colimits in $\mathbf{Set}^{\mathcal{K}^{op}}$ (see part II. of the proof of 3.11). Therefore *Sind* $\mathcal{K}$ satisfies (a)-(e) as well. Since *Colim* is left adjoint to $\eta_{\mathcal{K}}$, $\mathcal{K}$ is a reflective subcategory of *Sind* $\mathcal{K}$ and the reflector *Colim* preserves all limits. Hence $\mathcal{K}$ satisfies (a)-(e) too.

For general algebraically exact categories $\mathcal{K}$ it is sufficient to prove that they are ∞-filtered colimits of locally small algebraically exact categories with connecting functors algebraically exact. This follows from 1.5 and the observation that given a small subcategory $\mathcal{A}$ of $\mathcal{K}$, then the closure of $\mathcal{A}$ under limits and filtered colimits in $\mathcal{K}$ (with transfinitely many iterations) is locally small and algebraically exact. ■

5.2. Open problem. Is every category with properties (a)-(e) above algebraically exact?

5.3. Examples. (1) Every variety is algebraically exact. In fact, this follows from 3.8 and the next example:

(2) Every complete category of the form *Sind* $\mathcal{K}$ is algebraically exact. In fact, here *Colim* $\cong \mu_{\mathcal{K}}^{Sind}$ preserves limits by 3.11 (3).

(3) Torsion-free abelian groups fulfill (a)-(d) above but not (e), thus they do not form an algebraically exact category.

(4) Essential localizations of varieties are algebraically exact (see [ARV] for a description of these categories). In more generality:

5.4. Proposition. *Let $\mathcal{K}$ be an algebraically exact category. Then all complete localizations of $\mathcal{K}$ (i.e. full reflective subcategories whose reflector preserves limits) are algebraically exact.*

Proof. Let $E : \mathcal{L} \hookrightarrow \mathcal{K}$ be a complete localization with a reflector $R : \mathcal{K} \to \mathcal{L}$; thus

$$R \dashv E \quad \text{and} \quad RE = \text{id} .$$

Each of the following three functors

$$Sind\ \mathcal{L} \xrightarrow{Sind\ E} Sind\ \mathcal{K} \xrightarrow{Colim} \mathcal{K} \xrightarrow{R} \mathcal{L} \tag{7}$$

preserves sifted colimits. Hence the composite functor of (7) preserves sifted colimits. We will show that this composite is a functor computing sifted colimits in $\mathcal{L}$ – for this, it is only necessary to observe that $\eta_{\mathcal{L}}^{Sind}$ composed with (7) yields the identity on $\mathcal{L}$:

$$\begin{aligned}&(R \cdot Colim \cdot Sind\ E) \cdot \eta_{\mathcal{L}}^{Sind}\\ &\cong R \cdot Colim \cdot \eta_{\mathcal{K}}^{Sind} \cdot E\\ &\cong R \cdot E\\ &\cong \mathrm{id}\ .\end{aligned}$$

Thus, we are to prove that the composite functor (7) preserves limits: *Sind E* preserves limits by 3.11, *Colim* does by assumption on $\mathcal{K}$ and R does by assumption on $\mathcal{L}$. ■

5.5. Corollary. *Essential localizations of the category R-**Mod** of right R-modules are algebraically exact categories for every ring R.*

Those essential localizations have been described in [R] as follows: Let I be an idempotent ideal of R. Let $\mathcal{K}_I$ be the full subcategory of R-**Mod** formed by all modules orthogonal to the embedding $I \hookrightarrow R$. Then $\mathcal{K}_I$ is an essential localization of R-**Mod**, and there are no other (up to equivalence of categories).

More about algebraically exact categories can be found in [ARV].

6. A full equational hull of VAR

6.1. In the present section we exhibit a pseudomonad $\mathbb{D}^*$ on CAT, such that the category of algebras of $\mathbb{D}^*$ is an equational hull of VAR w.r.t. all operations. Here, arities are not assumed to be small (nor the operations to have rank, of course). This is based on the formation of dual monads on the cartesian closed 2-quasicategory CAT. The whole procedure is completely analogous to that used in [ALR_2] to describe the full equational hull of LFP. We will first recall, as we have already done in [ALR_2], the concept of a dual monad on a cartesian closed 2-category from [Law_2].

6.2. Enriched Monads in CCC. By an enriched monad $\mathbb{D}$ over a cartesian closed category $\mathcal{C}$ we understand an enriched functor $D : \mathcal{C} \to \mathcal{C}$ together with enriched natural transformations $\eta : 1_{\mathcal{C}} \to D$ and $\mu : DD \to D$ satisfying the usual monad axioms. The category

$$\mathcal{C}^{\mathbb{D}}$$

of (strict) algebras and homomorphisms is then also enriched over $\mathcal{C}$: the internal hom-functor

$$\mathrm{Hom}_{\mathbb{D}} : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \times \mathcal{C}^{\mathbb{D}} \to \mathcal{C}$$

is defined by means of the following equalizer for $\mathbb{D}$-algebras (A, α) and (B, β):

$$\mathrm{Hom}_{\mathbb{D}}((B,\beta),(A,\alpha)) \xrightarrow{e} A^B \xrightarrow{A^\beta} A^{(DB)}, \qquad A^B \xrightarrow{D} (DA)^{(DB)} \xrightarrow{\alpha^{(DB)}} A^{(DB)}$$

The Dual Monad $\mathbb{D}^*$. We now fix a $\mathbb{D}$-algebra

$$DS \xrightarrow{\sigma} S$$

called *dualizer*, and observe that the enriched hom-functor

$$\mathrm{Hom}_{\mathbb{D}}(-,(S,\sigma)) : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}$$

has a left adjoint which can be constructed as follows (see [L]): for every object $C \in \mathcal{C}$ we denote by

$$\mathrm{eval}^*_C : C \to (DS)^{D(S^C)}$$

the composite of the evaluation map $C \to S^{(S^C)}$ and the D-map $S^{(S^C)} \mapsto DS^{D(S^C)}$, and we consider the following adjunctions:

$$\frac{\dfrac{C \xrightarrow{\mathrm{eval}^*_C} (DS)^{D(S^C)}}{C \times D(S^C) \longrightarrow DS}}{D(S^C) \xrightarrow{\varrho_C} (DS)^C}$$

We now define a functor

$$S^{(-)} : \mathcal{C} \to (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}}$$

by assigning to $C \in \mathcal{C}$ the $\mathbb{D}$-algebra

$$D(S^C) \xrightarrow{\varrho_C} (DS)^C \xrightarrow{\sigma^C} S^C$$

and to every morphism $f : C \to \bar{C}$ in $\mathcal{C}$ the $\mathbb{D}$-homomorphism

$$S^f : (S^{\bar{C}}, \sigma^{\bar{C}} \cdot \varrho_{\bar{C}}) \to (S^C, \sigma^C \cdot \varrho_C)\,.$$

Then we get an enriched adjoint situation

$$S^{(-)} \to \mathrm{Hom}_{\mathbb{D}}(-,S)$$

(the argument in this general case is analogous to that in 2.2 above).

Notation. The enriched monad generated by the last adjoint situation is denoted by $\mathbb{D}^* = (D^*, \eta^*, \mu^*)$ and is called the *dual monad* of $\mathbb{D}$ w.r.t. the dualizer (S,σ). Thus,

$$D^*C = \mathrm{Hom}_D(S^C, S)$$

Equational Hull of the category dual to $\mathcal{C}^{\mathbb{D}}$, the category of $\mathbb{D}$-algebras w.r.t. the "forgetful" functor $\mathrm{Hom}_{\mathbb{D}}(-,(S,\sigma)) : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}$) is given by the comparison functor

$$K : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}^{\mathbb{D}^*} .$$

6.3. Doctrine of finite products. We apply the above to $\mathcal{C} = \mathrm{CAT}$ and to the pseudomonad induced by the forgetful 2-functor

$$U : \mathrm{CAT}_{fpc} \longrightarrow \mathrm{CAT}$$

where CAT_{fpc} is the 2-category of

all Cauchy complete categories with finite products (as objects),

all functors preserving finite products (as 1-cells)

and

all natural transformations.

The functor U is pseudomonadic: consider free completions of categories under finite products and limits of idempotents (= equalizers of id_A, $e : A \to A$ for idempotents $e : A \to A$). This, like *Lim* in 3.9, can be considered as a pseudomonad $\mathbb{D}$ on CAT. Obviously, the 2-quasicategory CAT_{fpc} is biequivalent to the 2-quasicategory of algebras for $\mathbb{D}$.

We denote by

$$\mathbb{D}^*$$

the dual pseudomonad with the dualizer **Set**. That is, we consider the 2-adjoint situation

$$\begin{array}{ccc} & \mathrm{CAT}^{\mathrm{op}}_{fpc} & \\ \mathrm{CAT}_{fpc}(-,\mathbf{Set}) & \uparrow \vdash \downarrow & \mathrm{CAT}(-,\mathbf{Set}) \\ & \mathrm{CAT} & \end{array}$$

generating the pseudomonad $\mathbb{D}^* = (D^*, \eta^*, \mu^*)$ over CAT. It means that

$$D^*\mathcal{X} = \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}})$$

for any category $\mathcal{X}$, where $\mathbf{MOD}\,\mathcal{T}$ denotes the category of all set-valued functors on $\mathcal{T}$ preserving finite products. Further,

$$D^*F : H \longmapsto H \cdot \mathbf{Set} \qquad (H \in \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}}))$$

for each functor $F : \mathcal{X} \to \mathcal{Y}$, and

$$(D^*\varphi)_F = H \cdot \mathbf{Set}^{\varphi}$$

for each natural transformation $\varphi : F \to F'$. The unit

$$\eta^*_{\mathcal{K}} : \mathcal{X} \to \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}})$$

is given by

$$X \longmapsto ev_X$$

where ev_X is the functor of evaluation-at-X, i.e.,

$$ev_X : \mathbf{Set}^{\mathcal{X}}(\mathcal{X}(X,-),-)\,.$$

The multiplication

$$\mu^*_{\mathcal{X}} : \mathbf{MOD}\,(\mathbf{Set}^{D^*\mathcal{X}}) \to \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}})$$

is defined by means of the evaluation functor

$$ev : \mathbf{Set}^{\mathcal{X}} \to \mathbf{Set}^{\mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}})}$$

as

$$\mu^*_{\mathcal{X}}(G) = G \cdot ev$$

for each $G : \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}}) \to \mathbf{Set}$ preserving finite products. The pseudomonad $\mathbb{D}^*$ is dual to $\mathbb{D}$ (when using $\mathbf{Set}$ as the dualizing object).

6.4. The comparison functor

$$K : \mathrm{CAT}^{\mathrm{op}}_{fpc} \to \mathrm{CAT}^{\mathbb{D}^*}$$

assigns to every Cauchy-complete category $\mathcal{A}$ with finite products a $\mathbb{D}^*$-algebra whose underlying category is $\mathbf{MOD}\,\mathcal{A}$. For every functor $F : \mathcal{A} \to B$ preserving finite products we have

$$K(F) : K\mathcal{B} \to K\mathcal{A}$$

defined by

$$K(F) : H \longmapsto H \cdot F \qquad (H \in \mathbf{MOD}\,\mathcal{B})$$

We observe that varieties are precisely the categories

$$K\mathcal{A} \qquad \text{for } \mathcal{A} \text{ in } \mathrm{CAT}_{fpc}\,,$$

and in fact the duality of [ALR$_1$] is just a domain-codomain restriction of K to $\mathrm{CAT}^{\mathrm{op}}_{fpc} \longrightarrow$ VAR.

6.5. Observation. The 2-category

$$\mathrm{CAT}^{\mathbb{D}^*}$$

of algebras of the dual pseudomonad $\mathbb{D}^*$ is an equational hull of VAR over CAT. That is, given a pseudomonad $\mathbb{T}$ over CAT, and a functor

$$E : \mathrm{VAR} \to \mathrm{CAT}^{\mathbb{T}}$$

such that the composition $U_{\mathbb{T}} \cdot E : \mathrm{VAR} \to \mathrm{CAT}$ is naturally isomorphic to the forgetful functor $U : \mathrm{VAR} \to \mathrm{CAT}$, then there exists an essentially unique pseudofunctor

$$E^* : \mathrm{CAT}^{\mathbb{D}^*} \to \mathrm{CAT}^{\mathbb{T}}$$

with

$$E \cong KE^* \qquad \text{and} \qquad U_{\mathbb{D}^*} \cong U_{\mathbb{T}} \cdot E^* .$$

The verification is completely analogous to the case LFP (cf. [ALR$_2$], Remark 4.7). Every $\mathbb{D}^*$-algebra is an absolute pseudocolimit of free $\mathbb{D}^*$-algebras (see [LV]) and every free $\mathbb{D}^*$-algebra is a limit of varieties. The second statement follows from

$$\mathbf{MOD}\,(\mathbf{Set}^{\mathcal{X}}) \cong \lim \mathbf{MOD}\,(\mathcal{C})$$

where the limit is induced by expressing $\mathbf{Set}^{\mathcal{X}}$ as a colimit of small Cauchy-complete subcategories closed under finite products.

6.6. Open problem. Characterize $\mathbb{D}^*$-algebras. Is the 2-quasicategory $\mathrm{CAT}^{\mathbb{D}^*}$ biequivalent to a (non-full) subcategory of CAT ? That is, is “to be a $\mathbb{D}^*$-algebra” a property of categories ?

6.7. The Pseudomonad $\mathbb{D}^*_{\mathrm{small}}$. We can introduce the small core of the pseudomonad $\mathbb{D}^*$ by defining, for every category $\mathcal{K} = \operatorname{colim}_I \mathcal{K}_i$ (where $\mathcal{K}_i$ are small subcategories of $\mathcal{K}$, and I is ∞-filtered)

$$\mathbb{D}^*_{\mathrm{small}}\mathcal{K} = \operatorname*{colim}_I \mathbb{D}^*\mathcal{K}_i = \operatorname*{colim}_I \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{K}_i}) .$$

In other words, for small categories $\mathcal{K}$ we have $\mathbb{D}^*_{\mathrm{small}}\mathcal{K} = \mathbb{D}^*\mathcal{K} = \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{K}})$. For large categories,

$$\mathbb{D}^*_{\mathrm{small}}\mathcal{K} = \operatorname*{colim}_I \mathbf{MOD}\,(\mathbf{Set}^{\mathcal{K}_i}) .$$

This yields a pseudomonad $\mathbb{D}^*_{\mathrm{small}}$ which preserves ∞-filtered colimits. And the quasicategory

$$\mathrm{CAT}^{\mathbb{D}^*_{\mathrm{small}}}$$

is a small equational hull of VAR over CAT. That is, given a pseudomonad $\mathbb{T}$ over CAT preserving ∞-filtered colimits, then for every functor $E : \mathrm{VAR} \to \mathrm{CAT}^{\mathbb{T}}$ with $U_{\mathbb{T}} \cdot E \cong U$ there exists an essentially unique pseudofunctor $E^* : \mathrm{CAT}^{\mathbb{D}^*_{\mathrm{small}}} \to \mathrm{CAT}^{\mathbb{T}}$ with $E \cong K_{\mathrm{small}} \cdot E^*$ and $U_{\mathbb{D}^*_{\mathrm{small}}} \cong U_{\mathbb{T}} \cdot E^*$ (for the comparison 2-functor K_{small}).

As in 6.6, we have not been able to characterize $\mathrm{CAT}^{\mathbb{D}^*_{\mathrm{small}}}$. Thus, we restrict further:

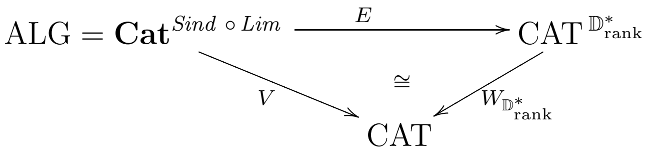
6.8. The Pseudomonad $\mathbb{D}^*_{\mathrm{rank}}$. This is the pseudomonad corresponding to restricting ourselves to operations on VAR with rank. Thus, for small categories $\mathcal{K}$, $\mathbb{D}^*_{rank}\mathcal{K} = \mathbf{Mod}\,(\mathbf{Set}^{\,\mathcal{K}})$ and for large categories $\mathcal{K} = \operatorname*{colim}_I \mathcal{K}_i$ ($\mathcal{K}_i$ small and I ∞-filtered)

$$\mathbb{D}^*_{\mathrm{rank}}\mathcal{K} = \operatorname*{colim}_I \mathbf{Mod}\,(\mathbf{Set}^{\,\mathcal{K}_i})\,.$$

Lemma. *$\mathbb{D}^*_{\mathrm{rank}}$ and Sind $\circ$ Lim are biequivalent pseudomonads, i.e., there exists a biequivalence*

$$E : \mathrm{ALG} = \mathrm{CAT}^{\,Sind\,\circ\,Lim} \to \mathrm{CAT}^{\,\mathbb{D}^*_{\mathrm{rank}}}$$

such that the following triangle



commutes.

This follows easily from the monadicity of the two categories, ALG and $\mathrm{CAT}^{\,\mathbb{D}^*_{\mathrm{rank}}}$, over CAT and the fact that the free algebras over $\mathcal{K}$ coincide for all small categories $\mathcal{K}$:

$$(Sind \circ Lim\,)\mathcal{K} \cong Sind\,(\mathbf{Set}^{\,\mathcal{K}})^{\mathrm{op}} \cong \mathbf{Mod}\,(\mathbf{Set}^{\,\mathcal{K}})$$

by 3.8 (2).

6.9. ALG is not an equational hull of VAR. Let us finally observe that ALG is not equivalent to $\mathrm{CAT}^{\,\mathbb{D}^*}$ nor to $\mathrm{CAT}^{\,\mathbb{D}^*_{\mathrm{small}}}$ provided that we assume that **Set** contains arbitrarily large measurable cardinals. In fact, as recalled in 2.4, this assumption implies

$$\mathbf{Mod}\,(\mathbf{Set}\,) \subsetneqq \mathbf{MOD}\,(\mathbf{Set}\,)\,,$$

i.e., there exists a non-accessible functor $H : \mathbf{Set} \to \mathbf{Set}$ preserving finite products. Denote by ω^H the corresponding unary operation on VAR, i.e., for every variety $\mathcal{V} = \mathbf{Mod}\,\mathcal{T}$ we have the operation functor

$$(\omega^H)_{\mathcal{V}} : \mathbf{Mod}\,\mathcal{T} \to \mathbf{Mod}\,\mathcal{T}\,, \quad F \mapsto HF\,.$$

Lemma. *The operation ω^H cannot be extended from* VAR *to* ALG.

Proof. Let $\mathcal{K}_i$ $(i \in I)$ be an ∞-filtered collection of full subcategories of **Set** closed under finite products such that $\mathbf{Set} = \operatorname{colim}\mathcal{K}_i$. Each $\mathcal{V}_i = \mathbf{Mod}\,\mathcal{K}_i$ is a variety, and if $k_i : \mathcal{K}_i \to \mathbf{Set}$ denotes the inclusion, then $\mathbf{Mod}\,(\mathbf{Set}\,)$ is a limit of $\mathcal{V}_i$ $(i \in I)$ with the limit-cone $\mathbf{Mod}\,k_i$ $(F \mapsto F/k_i)$.

Assuming that ω^H has an extension to ALG, then for the algebraically exact category $\mathcal{K} = \mathbf{Mod}\,(\mathbf{Set}\,)$ we have an operation-functor (ω^H) such that the squares

$$\begin{array}{ccc} \mathbf{Mod}\,(\mathbf{Set}\,) & \xrightarrow{(\omega^H)_{\mathcal{K}}} & \mathbf{Mod}\,(\mathbf{Set}\,) \\ {\scriptstyle \mathbf{Mod}\,k_i}\downarrow & \cong & \downarrow{\scriptstyle \mathbf{Mod}\,k_i} \\ \mathbf{Mod}\,\mathcal{K}_i & \xrightarrow{(\omega^H)_{\mathcal{V}_i}} & \mathbf{Mod}\,\mathcal{K}_i \end{array}$$

commute up to isomorphism. Applied to the object $Id : \mathbf{Set} \to \mathbf{Set}$ of $\mathbf{Mod}\,(\mathbf{Set}\,)$, we get

$$H \cdot k_i \cong (\omega^h)_{\mathcal{K}}(Id)/k_i \qquad \text{for all } i \in I$$

and we conclude that

$$H \cong (\omega^H)_{\mathcal{K}}(Id)\,.$$

This is a contradiction since H is not accessible and $(\omega^H)_{\mathcal{K}}$ takes values in $\mathbf{Mod}\,(\mathbf{Set}\,)$, thus, $(\omega^H)_{\mathcal{K}}(Id)$ is accessible. ∎

REMARK. If $\mathbf{MOD}\,(\mathbf{Set}^{\mathcal{K}}) = \mathbf{Mod}\,(\mathbf{Set}^{\mathcal{K}})$ would hold for all small categories $\mathcal{K}$, then, of course, ALG would be small-equational hull of VAR. As remarked in 1.4 we do not know whether this assumption is consistent with set theory.

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