Theory and Applications of Categories, Vol. 32, No. 41, 2017, pp. 1454–1484.

TOPOLOGICAL PROPERTIES OF NON-ARCHIMEDEAN APPROACH SPACES

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ABSTRACT. In this paper we give an isomorphic description of the category of non-Archimedian approach spaces as a category of lax algebras for the ultrafilter monad and an appropriate quantale. Non-Archimedean approach spaces are characterised as those approach spaces having a tower consisting of topologies. We study topological properties p, for p compactness and Hausdorff separation along with low-separation properties, regularity, normality and extremal disconnectedness and link these properties to the condition that all or some of the level topologies in the tower have p. A compactification technique is developed based on Shanin's method.

1. Introduction

In monoidal topology [Hofmann, Seal, Tholen (eds.), 2014] starting from the quantale $\mathsf{P}_+ = \big([0,\infty]^{\mathrm{op}}, +, 0\big)$ where the natural order $\leq$ of the extended real halfline $[0,\infty]$ is reversed so that $0 = \top$ and $\infty = \perp$, based on Lawvere's description of quasi-metric spaces as categories enriched over the extended real line [Lawvere 1973], one obtains an isomorphism between the category $q\mathsf{Met}$ of extended quasi-metric spaces with non-expansive maps and P_+-Cat, the category of P_+-spaces (X,a), where X is a set and $a : X \rightarrow\!\!\!\!\!\mapsto X$ is a transitive and reflexive P_+-relation, meaning it is a map $a : X \times X \to \mathsf{P}_+$ satisfying $a(x,y) \leq a(x,z) + a(z,y)$ and $a(x,x) = 0$ for all x, y, z in X, with morphisms $f : (X,a) \to (Y,b)$ of P_+-Cat, maps satisfying $b(f(x), f(x')) \leq a(x,x')$ for all x, x' in X.

Our interest goes to the larger category App of approach spaces and contractions, which contains $q\mathsf{Met}$ as coreflectively embedded full subcategory and which is better behaved than $q\mathsf{Met}$ with respect to products of underlying topologies. A comprehensive source on approach spaces and their vast field of applications is [Lowen, 2015] in which several equivalent descriptions of approach spaces are formulated in terms of (among others) distances, limit operators, towers and gauges. A first lax-algebraic description of approach spaces was established by Clementino and Hofmann in [Clementino, Hofmann, 2003]. The construction involves the ultrafilter monad $\beta = (\beta, m, e)$, the quantale P_+ and an extension of the ultrafilter monad to P_+-relations. The so constructed category (β, P_+)-Cat and its isomorphic description App became an important example in the development of monoidal topology [Hofmann, Seal, Tholen (eds.), 2014]. More details on the construction will be

Received by the editors 2017-07-17 and, in final form, 2017-11-01.

Transmitted by Walter Tholen. Published on 2017-11-13.

2010 Mathematics Subject Classification: 18C15, 18C20, 54A05, 54B30, 54E99.

Key words and phrases: Lax algebra, quantale, non-Archimedean approach space, quasi-ultrametric space, initially dense object, topological properties in $(\beta, \mathsf{P}_\vee)$-Cat, compactification.

recalled in section 2.

In this paper we consider the non-Archimedian counterparts of the previous well known constructions. These are obtained by switching the quantale to $\mathsf{P}_{\vee}$. As before we consider the extended real halfline $[0,\infty]$ as a complete lattice with respect to the natural order $\leq$ and we reverse its order, so that $0 = \top$ is the top and $\infty = \bot$ is the bottom element. Since $[0,\infty]^{\mathrm{op}}$ is a chain, it is a frame, and we may consider it a quantale with its meet operation which, according to our conventions on the reversed order, is the supremum with respect to the natural order of $[0,\infty]$ and as in approach theory this operation will be denoted by $\vee$ (note that here we deviate from the notation in [Hofmann, Seal, Tholen (eds.), 2014]). As stated in exercise III.2.B of [Hofmann, Seal, Tholen (eds.), 2014] the category $\mathsf{P}_{\vee}$-Cat is isomorphic to the category $q\mathsf{Met}^u$ of extended quasi-ultrametric spaces (X,d) with non-expansive maps, where an extended quasi-metric is called an extended quasi-ultrametric if it satisfies the strong triangular inequality $d(x,z) \leq d(x,y) \vee d(y,z)$, for all $x,y,z \in X$.

In section 2, using the ultrafilter monad $\beta = (\beta, m, e)$, with $\mathsf{P}_{\vee}$ as quantale and an extension of the ultrafilter monad to $\mathsf{P}_{\vee}$-relations we show that the category of associated lax algebras $(\beta, \mathsf{P}_{\vee})$-$\mathsf{Cat}$ is isomorphic to the reflectively embedded full subcategory NA-App of App, the objects of which are called non-Archimedean approach spaces. In section 3 we give equivalent descriptions of non-Archimedean approach spaces in terms of distances, limit operators, towers and gauges. The characterization in terms of towers can be easily expressed: an approach space is non-Archimedean if and only if its tower $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ consists of topologies. We show that NA-App can be generated as the concretely reflective hull by one particular extended quasi-ultrametric.

Following the general philosophy to consider lax algebras as spaces, it is our main purpose in this paper to study topological properties p like compactness and Hausdorff separation, along with low-separation properties, regularity, normality and extremal disconnectedness for non-Archimedean approach spaces. In section 6 we introduce these properties $(\beta, \mathsf{P}_{\vee})$-$p$ as an application to $(\beta, \mathsf{P}_{\vee})$-$\mathsf{Cat}$ of the corresponding $(\mathbb{T}, \mathcal{V})$-properties for arbitrary $(\mathbb{T}, \mathcal{V})$-$\mathsf{Cat}$ as developed in V.1 and V.2 of [Hofmann, Seal, Tholen (eds.), 2014]. For each of the properties p we characterise $(\beta, \mathsf{P}_{\vee})$-$p$ in the context of NA-App. On the other hand we make use of the well known meaning of these properties in the setting of Top and for a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ we compare the property $(\beta, \mathsf{P}_{\vee})$-$p$ to the properties X *has* p *at level* 0, meaning the topological space $(X, \mathcal{T}_0)$ has p in Top, X *almost strongly has* p, meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}_0^+$ and X *strongly has* p, meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}^+$. For p the Hausdorff property, low separation properties or regularity the conditions strongly p, almost strongly p and $(\beta, \mathsf{P}_{\vee})$-$p$ are all equivalent and different from p at level 0. For p compactness strongly p is equivalent to p at level 0, and almost strongly p is equivalent to $(\beta, \mathsf{P}_{\vee})$-$p$. For p normality or extremal disconnectedness strongly p implies almost strongly p which implies $(\beta, \mathsf{P}_{\vee})$-$p$. Moreover strongly p implies p at level 0. We give counterexamples showing that there are no other valid implications.

In the last sections of the paper we develop a compactification technique and prove that every non-Archimedean approach space X given by its tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ can be densely embedded in a compact (at level 0) non-Archimedean approach space Y given by its tower of topologies $(\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ in such a way that at level 0 the compact space $(Y, \mathcal{S}_0)$ is the Shanin compactification of $(X, \mathcal{T}_0)$. We give necessary and sufficient conditions for the compactification to be Hausdorff at level 0 and we prove that a space X that is Hausdorff at level 0, $(\beta, \mathsf{P}_\vee)$-regular and $(\beta, \mathsf{P}_\vee)$-normal has such a Hausdorff compactification.

2. $\mathsf{P}_\vee$-Cat and $(\beta, \mathsf{P}_\vee)$-Cat

We follow definitions and notations from [Hofmann, Seal, Tholen (eds.), 2014], unless stated otherwise. In this section we adapt the constructions of P_+-Cat and (β, P_+)-Cat in [Hofmann, Seal, Tholen (eds.), 2014] to the quantale $\mathsf{P}_\vee$. Consider the extended real halfline $[0,\infty]$ which is a complete lattice with respect to the natural order $\leq$. We reverse its order, so that $0 = \top$ is the top and $\infty = \bot$ is the bottom element. When working with $([0,\infty]^{\mathrm{op}}, \leq^{\mathrm{op}})$ and forming infima or suprema we will denote these by $\inf^{\mathrm{op}}, \bigwedge^{\mathrm{op}}, \sup^{\mathrm{op}}$ or $\bigvee^{\mathrm{op}}$. It means that we will deviate from the conventions made in [Hofmann, Seal, Tholen (eds.), 2014] since we will use both the symbols inf and $\bigwedge$ when forming infima and sup and $\bigvee$ when forming suprema, referring to the natural order on $([0,\infty], \leq)$.

Since $[0,\infty]^{\mathrm{op}}$ is a chain, it is a frame, and we may consider it a quantale $\mathsf{P}_\vee = \big([0,\infty]^{\mathrm{op}}, \vee, 0\big)$ with its meet operation (which, according to our conventions, is the supremum with respect to the natural order of $[0,\infty]$ and will be denoted by $\vee$).
The map $a \vee (-) : \mathsf{P}_\vee \longrightarrow \mathsf{P}_\vee$ is left adjoint to the map $a{-\!\bullet}(-) : \mathsf{P}_\vee \longrightarrow \mathsf{P}_\vee$ defined by

$$a{-\!\bullet} b = \inf\{v \in [0,\infty] \mid b \leq a \vee v\} = \begin{cases} 0 & a \geq b, \\ b & a < b. \end{cases}$$

The category $\mathsf{P}_\vee$-Rel has sets as objects and $\mathsf{P}_\vee$-relations as morphisms. A $\mathsf{P}_\vee$-relation $r : X \nrightarrow Y$ from X to Y is represented by a map $r : X \times Y \to \mathsf{P}_\vee$. Two $\mathsf{P}_\vee$-relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ can be composed in the following way

$$(s \cdot r)(x,z) = \inf\big\{r(x,y) \vee s(y,z) \mid y \in Y\big\}.$$

2.1. Definition. *$\mathsf{P}_\vee$-Cat is the category of $\mathsf{P}_\vee$-spaces (X,a), where X is a set and $a : X \nrightarrow X$ is transitive and reflexive $\mathsf{P}_\vee$-relation, meaning it is a map $a : X \times X \to \mathsf{P}_\vee$ satisfying*

$$a(x,y) \leq a(x,z) \vee a(z,y) \text{ and } a(x,x) = 0$$

for all x,y,z in X. A morphism $f : (X,a) \to (Y,b)$ of $\mathsf{P}_\vee$-Cat is a map satisfying $b(f(x_1), f(x_2)) \leq a(x_1,x_2)$ for all x_1, x_2 in X.

The map $\varphi : \mathsf{P}_\vee \longrightarrow \mathsf{P}_+$ with $\varphi(v) = v$, for all $v \in \mathsf{P}_\vee$, is a lax homomorphisms of quantales. This induces a lax-functor $\varphi : \mathsf{P}_\vee$-Rel $\longrightarrow \mathsf{P}_+$-Rel, which leaves objects unaltered and sends $r : X \times Y \longrightarrow \mathsf{P}_\vee$ to $\varphi \cdot r : X \times Y \longrightarrow \mathsf{P}_+$. Obviously φ is compatible with the identical lax extension of the identity monad $\mathbb{I}$ to $\mathsf{P}_\vee$-Rel and P_+-Rel. Hence, this lax-functor induces a change-of-base functor $B_\varphi : \mathsf{P}_\vee\text{-}\mathsf{Cat} \longrightarrow \mathsf{P}_+\text{-}\mathsf{Cat}$. Moreover, this change-of-base functor is an embedding. Whereas $\mathsf{P}_+\text{-}\mathsf{Cat} \cong q\mathsf{Met}$, the category of extended quasi-metric spaces and non-expansive maps, it is known that $\mathsf{P}_\vee\text{-}\mathsf{Cat} \cong q\mathsf{Met}^u$, the category of quasi-ultrametric spaces. An early systematic study of ultrametric spaces can be found in the work of Monna [Monna] and of de Groot [de Groot] in the 50's, but ever since the amount of literature has become extensive as (quasi)-ultrametrics became important tools in a wide range of fields, like combinatorics, domain theory, linguistics, solid state physics, taxonomy and evolutionary tree constructions, to name only a few.

For a set X let βX be the set of all ultrafilters on X and for a map $f : X \longrightarrow Y$, let $\beta f : \beta X \longrightarrow \beta Y : \mathcal{U} \mapsto \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{U}\}$. Then $\beta : \mathsf{Set} \longrightarrow \mathsf{Set}$ is a functor. To avoid overloaded notations, we will often write $f(\mathcal{U})$ instead of $\beta f(\mathcal{U})$. Now consider the ultrafilter monad $\mathbb{B} = (\beta, m, e)$ on Set with components $m_X : \beta\beta X \to \beta X$ defined by the Kowalsky sum of $\mathfrak{X} \in \beta\beta X$

$$m_X\mathfrak{X} = \Sigma\mathfrak{X} = \big\{A \subseteq X \mid \{\mathcal{U} \in \beta X \mid A \in \mathcal{U}\} \in \mathfrak{X}\big\},$$

and $e_X : X \longrightarrow \beta X$ defined by $e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\}$ for $x \in X$.

Analogous to the P_+ situation, for a $\mathsf{P}_\vee$-relation $r : X \relbar\!\!\mapsto Y$ and $\alpha \in [0, \infty]$, we can define the relation $r_\alpha : X \relbar\!\!\mapsto Y$ by $x \; r_\alpha \; y \Leftrightarrow r(x, y) \leq \alpha$. For $A \subseteq X$ we put $r_\alpha(A) = \{y \in Y \mid \exists x \in A : x \; r_\alpha \; y\}$, and for $\mathcal{A} \subseteq \mathcal{P}(X)$ we let $r_\alpha(\mathcal{A}) = \{r_\alpha(A) \mid A \in \mathcal{A}\}$. Recall that $\beta : \mathsf{Set} \to \mathsf{Set}$ can be extended to a 2-functor $\overline{\beta} : \mathsf{Rel} \to \mathsf{Rel}$ where for any sets X and Y, any relation $r : X \relbar\!\!\mapsto Y$ and ultrafilters $\mathcal{U} \in \beta X$ and $\mathcal{W} \in \beta Y$ we have

$$\mathcal{U} \; (\overline{\beta} r) \; \mathcal{W} \Leftrightarrow r(\mathcal{U}) \subseteq \mathcal{W}.$$

Then, for $r : X \relbar\!\!\mapsto Y$ a $\mathsf{P}_\vee$-relation, we let

$$\overline{\beta} r(\mathcal{U}, \mathcal{W}) = \inf\{\alpha \in [0, \infty] \mid \mathcal{U}\overline{\beta}(r_\alpha)\mathcal{W}\},$$

for $\mathcal{U} \in \beta X$ and $\mathcal{W} \in \beta Y$. Analogous to the P_+ situation, the extension of the ultrafilter monad $\mathbb{B}$ is a flat and associative lax extension to $\mathsf{P}_\vee$-Rel, which we again denote by $\overline{\mathbb{B}} = (\overline{\beta}, m, e)$.

2.2. Definition. *Let* $(\mathbb{B}, \mathsf{P}_\vee)$-$\mathsf{Cat}$ *be the category of lax algebras* (X, a) *for the* $\mathsf{P}_\vee$-Rel-*extension of the ultrafilter monad, where* X *is a set and* $a : \beta X \relbar\!\!\mapsto X$ *is a* $\mathsf{P}_\vee$-*relation that is a map* $a : \beta X \times X \to \mathsf{P}_\vee$ *that is reflexive, meaning*

$$a(\dot{x}, x) = 0$$

for all $x \in X$ *and transitive, meaning*

$$a(m_X(\mathfrak{X}), x) \leq \overline{\beta} a(\mathfrak{X}, \mathcal{U}) \vee a(\mathcal{U}, x)$$

for all $\mathfrak{X} \in \beta\beta X$, *for all* $\mathcal{U} \in \beta X$ *and for all* $x \in X$.
A morphism $f : (X, a) \to (Y, b)$ *of* $(\mathbb{B}, \mathsf{P}_\vee)$-$\mathsf{Cat}$ *is a map satisfying*

$$b(f(\mathcal{U}), f(x)) \leq a(\mathcal{U}, x)$$

for all x in X and $\mathcal{U} \in \beta X$.

It is clear that the lax-homomorphism of quantales $\varphi : \mathsf{P}_{\vee} \longrightarrow \mathsf{P}_{+}$ is compatible with the lax-extension of the ultrafilter monad β to $\mathsf{P}_{\vee}$-Rel and P_{+}-Rel. Hence, this induces another change-of-base functor

$$C_{\varphi} : (\beta, \mathsf{P}_{\vee})\text{-}\mathsf{Cat} \longrightarrow (\beta, \mathsf{P}_{+})\text{-}\mathsf{Cat}.$$

Again, this change-of-base functor is an embedding. The category (β, P_{+})-Cat is known to be isomorphic to App . This result was first shown in [Clementino, Hofmann, 2003] and can be found in section III.2.4 in [Hofmann, Seal, Tholen (eds.), 2014]. Both proofs go via distances. We will proceed by giving an isomorphic description of the category $(\beta, \mathsf{P}_{\vee})$-Cat and in order to do so we introduce the concept of non-Archimedean approach spaces by strengthening axiom (LU*) in [Lowen, 2015].

2.3. Definition. *Let* NA-App *be the full subcategory of* App *consisting of approach spaces (X, λ), with a limit operator $\lambda : \beta X \to \mathsf{P}_{\vee}^X$ satisfying the strong inequality*

($L\beta^*_{\vee}$) For any set J, for any $\psi : J \longrightarrow X$, for any $\sigma : J \longrightarrow \beta X$ and for any $\mathcal{U} \in \beta J$

$$\lambda \Sigma \sigma(\mathcal{U}) \leq \lambda \psi(\mathcal{U}) \vee \sup_{U \in \mathcal{U}} \inf_{j \in U} \lambda \sigma(j)\big(\psi(j)\big).$$

We will call these objects non-Archimedean approach spaces and the structure λ will be called a non-Archimedean limit operator.

Most of the time we will be working with ultrafilters. However we should mention that one can equivalently define NA-App as a full subcategory of App by strengthening axiom (L*) in [Lowen, 2015] for a limit operator on the set $\mathsf{F}X$ of all filters on X, again by replacing the $+$ on the righthand side of the inequality by $\vee$. In terms of filters the condition can be proven to be equivalent to property $[F_s]$ in [Brock, Kent, 1998].

2.4. Theorem. *The category $(\beta, \mathsf{P}_{\vee})$-Cat of lax algebras for the $\mathsf{P}_{\vee}$-Rel-extension of the ultrafilter monad is isomorphic to* NA-App.

Proof. The isomorphism directly links an approach space (X, λ) with $\lambda : \beta X \to \mathsf{P}_{\vee}^X$ to (X, a) where a is the corresponding map $a : \beta X \times X \to \mathsf{P}_{\vee}$ and vice versa. As NA-App is a subcategory of App, the limit operator λ satisfies $\lambda(\dot{x})(x) = 0$, for all $x \in X$. So this corresponds to reflexivity of a. In order to show that the axiom ($L\beta^*_{\vee}$) for λ corresponds to transitivity of a, we can use similar techniques as in the proof of theorem 12.7, $(\beta, \mathsf{P}_{+})\text{-}\mathsf{Cat} \cong \mathsf{App}$, which can be found in [Lowen, 2015]. Finally that through the identification of lax algebraic structures and non-Archimedean limit operators, also morphisms in both categories coincide, follows from the characterization of contractions in App via ultrafilters. ∎

3. Equivalent descriptions of non-Archimedean approach spaces

In this section we define various equivalent characterizations for non-Archimedean approach spaces, in terms of distances, towers and gauges.

3.1. Non-Archimedean distances. First we associate to a limit operator λ of an approach space X, its unambiguously defined distance $\delta : X \times 2^X \longrightarrow [0, \infty]$. This operator provides us with a notion of distance between points and sets. The smaller the value of $\delta(x, A)$ the closer the point x is to the set A. When starting with a non-Archimedean limit operator, we get a non-Archimedean distance, i.e. a distance which satisfies a stronger triangular inequality. Since the proof of the next proposition is a modification of the proof in [Lowen, 2015], replacing + by $\vee$ and since an analogous result for $[F_s]$ appears in [Brock, Kent, 1998], we omit it here.

3.2. Proposition. *If $\lambda : \beta X \longrightarrow \mathsf{P}_{\vee}^X$ is the limit operator of a non-Archimedean approach space X, then the associated distance, given by*

$$\delta : X \times 2^X \longrightarrow \mathsf{P}_{\vee} : (x, A) \mapsto \inf_{\mathcal{U} \in \beta A} \lambda \mathcal{U}(x),$$

satisfies the strong triangular inequality (D4$_\vee$)

$$\delta(x, A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \epsilon,$$

for all $x \in X$, $A \subseteq X$ and $\varepsilon \in \mathsf{P}_{\vee}$, with $A^{(\varepsilon)} := \{x \in X \mid \delta(x, A) \leq \epsilon\}$.

If $\delta : X \times 2^X \longrightarrow \mathsf{P}_{\vee}$ is a distance of an approach space X satisfying the strong triangular inequality (D4$_\vee$), then the associated limit operator, given by

$$\lambda : \beta X \longrightarrow \mathsf{P}_{\vee}^X : \mathcal{U} \mapsto \sup_{U \in \mathcal{U}} \delta_U,$$

is a non-Archimedean limit operator.

Distances satisfying the strong triangular inequality (D4$_\vee$) are called *non-Archimedean distances*. The following inequality will be useful later on and has a straightforward proof.

3.3. Proposition. *If $\delta : X \times 2^X \longrightarrow \mathsf{P}_{\vee}$ is a non-Archimedean distance, then the following inequality holds.*

$$\delta(x, A) \leq \delta(x, B) \vee \sup_{b \in B} \delta(b, A),$$

for all $x \in X$ and $A, B \subseteq X$.

We give an example of a non-Archimedean approach space on $\mathsf{P}_{\vee}$ which will play an important role later on in Section 5.

3.4. Example. Define $\delta_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times 2^{\mathsf{P}_\vee} \longrightarrow \mathsf{P}_\vee$ by

$$\begin{aligned}\delta_{\mathsf{P}_\vee}(x, A) &:= \begin{cases} \sup A \multimap x & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases} \\ &= \begin{cases} 0 & A \neq \emptyset \text{ and } x \leq \sup A, \\ x & A \neq \emptyset \text{ and } x > \sup A, \\ \infty & A = \emptyset. \end{cases}\end{aligned}$$

Then $\delta_{\mathsf{P}_\vee}$ is a non-Archimedean distance on $\mathsf{P}_\vee$. The associated non-Archimedean limit operator is determined as follows. Given an ultrafilter $\mathcal{U}$ on $\mathsf{P}_\vee$ and an element $x \in X$, then

$$\begin{aligned}\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) &= \sup_{U \in \mathcal{U}}(\sup U \multimap x) \\ &= \begin{cases} 0 & \forall U \in \mathcal{U} : x \leq \sup U, \\ x & \exists U \in \mathcal{U} : x > \sup U. \end{cases}\end{aligned}$$

3.5. Non-Archimedean towers. To a distance of an approach space X, we now associate its unambiguously defined tower $(\mathsf{t}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ of closure operators. Linking the limit operator on filters satisfying $[F_s]$ to the tower, an analogous result appears in [Brock, Kent, 1998], so here we omit the proof.

3.6. Proposition. *If $\delta : X \times 2^X \longrightarrow \mathsf{P}_\vee$ is the distance of a non-Archimedean approach space X, then all levels of the associated tower $(\mathsf{t}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, with*

$$\mathsf{t}_\varepsilon(A) = A^{(\varepsilon)}, \quad \forall A \subset X, \forall \varepsilon \in \mathbb{R}^+,$$

are topological closure operators.

If $(\mathsf{t}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ is the tower of an approach space X, where all levels are topological closure operators, then the associated distance

$$\delta : X \times 2^X \longrightarrow \mathsf{P}_\vee : (x, A) \mapsto \inf\{\varepsilon \in \mathbb{R}^+ \mid x \in \mathsf{t}_\varepsilon(A)\}$$

is a non-Archimedean distance on X.

Towers of approach spaces that satisfy the stronger condition of the previous proposition are called *non-Archimedean towers*. At each level the closure operator t_ε defines a topology $\mathcal{T}_\varepsilon$, so we will denote the structure also by $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, when working with open sets at each level, or $(\mathcal{C}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$, when using closed sets.

From the characterisation of approach spaces in terms of towers given in [Lowen, 2015] we now have the following.

3.7. Corollary. *A collection $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of topologies on a set X defines a tower for some non-Archimedean approach space if and only if it satisfies the coherence condition*

$$\mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon} \mathcal{T}_\gamma,$$

where the supremum is taken in Top.

We include more examples of non-Archimedean approach spaces that will be useful in section 6. The construction of the non-Archimedean approach spaces in the following examples is based on 3.7.

3.8. Example. Let X be a set and $\mathcal{S}$ a given topology on X. Let $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be defined by

$$\mathcal{T}_\varepsilon = \begin{cases} \mathcal{P}(X) & \text{whenever } 0 \leq \varepsilon < 1 \\ \mathcal{S} & \text{whenever } 1 \leq \varepsilon < 2 \\ \{X, \emptyset\} & \text{whenever } 2 \leq \varepsilon \end{cases}$$

Clearly the coherence condition is satisfied and so $(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+})$ defines a non-Archimedean approach space which we will denote by $X_\mathcal{S}$.

3.9. Example. Let $X =]0,\infty[$, endowed with a topology $\mathcal{T}$ with neighborhood filters $(\mathcal{V}(x))_{x\in X}$ and assume $\mathcal{T}$ is finer than the right order topology. We define $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_\varepsilon$ at level $0 < \varepsilon$ having a neighborhood filter

$$\mathcal{V}_\varepsilon(x) = \begin{cases} \{X\} & \text{whenever } x \leq \varepsilon \\ \mathcal{V}(x) & \text{whenever } \varepsilon < x \end{cases}$$

at $x \in X$. For a fixed level ε and $\varepsilon < x$ the set $\{G \in \mathcal{T} \mid x \in G, G \subseteq]\varepsilon,\infty[\}$ is an open base of $\mathcal{V}_\varepsilon(x)$. So clearly $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a decending chain of topologies. To check the other inclusion of the coherence condition, let $0 \leq \varepsilon$ and $x \in]0,\infty[$. Either $x \leq \varepsilon$ and then $\mathcal{V}_\varepsilon(x) = \{X\} \subseteq \mathcal{V}_\gamma(x)$ for every γ. Or $\varepsilon < x$, then choose γ with $\varepsilon < \gamma < x$. We have $\mathcal{V}_\varepsilon(x) = \mathcal{V}_\gamma(x) = \mathcal{V}(x)$. So $(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+})$ defines a non-Archimedean approach space.

3.10. Non-Archimedean gauges. Finally, we look at the gauges. To a distance operator δ of an approach space X, we associate its unambiguously defined gauge

$$\mathcal{G} = \{d \in q\mathsf{Met}(X) \mid \forall A \subseteq X, \forall x \in X : \inf_{a\in A} d(x,a) \leq \delta(x,A)\},$$

where $q\mathsf{Met}(X)$ is the collection of all extended quasi-metrics on X. Moreover the distance can be recovered from the gauge by the formula

$$\delta(x,A) = \sup_{d\in\mathcal{G}} \inf_{a\in A} d(x,a).$$

When the approach space X is given by its tower of closure operators $(\mathfrak{t}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ the gauge can also be described as

$$\mathcal{G} = \{d \in q\mathsf{Met}(X) \mid \forall \varepsilon \geq 0 : \mathfrak{t}_\varepsilon \leq \mathfrak{t}^d_\varepsilon\},$$

where $(\mathfrak{t}^d_\varepsilon)_{\varepsilon\geq 0}$ is the tower of the approach space (X, δ_d).

We recall a few definitions from approach theory [Lowen, 2015]. Given a collection $\mathcal{H} \subseteq q\mathsf{Met}(X)$ and a quasi-metric $d \in q\mathsf{Met}(X)$ we say that d is *locally dominated by* $\mathcal{H}$ if for all $x \in X, \varepsilon > 0$ and $\omega < \infty$ there exists a $d^{\varepsilon,\omega}_x \in \mathcal{H}$ such that

$$d(x, \cdot) \wedge \omega \leq d^{\varepsilon,\omega}_x(x, \cdot) + \epsilon. \tag{1}$$

A subset $\mathcal{H}$ of $q\mathsf{Met}(X)$ is called *locally directed* if for any $\mathcal{H}_0 \subseteq \mathcal{H}$ finite, we have that $\sup_{d\in\mathcal{H}_0} d$ is locally dominated by $\mathcal{H}$. Given a subset $\mathcal{H} \subseteq q\mathsf{Met}(X)$ we define

$$\widehat{\mathcal{H}} := \{d \in q\mathsf{Met}(X) \mid \mathcal{D} \text{ locally dominates } d\}.$$

Given an approach space X and $\mathcal{G}$ its gauge, $\mathcal{H} \subseteq q\mathsf{Met}(X)$ is called a *basis for the gauge* $\mathcal{G}$ if $\widehat{\mathcal{H}} = \mathcal{G}$. If $\mathcal{H}$ is locally directed, then $\widehat{\mathcal{H}}$ is a gauge of an approach space with $\mathcal{H}$ as a basis.

3.11. THEOREM. *Consider a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$. Then the associated gauge has a basis consisting of quasi-ultrametrics.*

PROOF. For $n \in \mathbb{N}_0$ we consider $\{\frac{k}{n} \mid k = 0, \cdots, k = n^2\}$ on $[0, n]$. At level $\frac{k}{n}$ we choose a finite $\mathcal{T}_{\frac{k}{n}}$-open cover $\mathcal{C}_{\frac{k}{n}}$ in such a way that for $k > 0$ the finite $\mathcal{T}_{\frac{k-1}{n}}$-open cover $\mathcal{C}_{\frac{k-1}{n}}$ is a refinement of $\mathcal{C}_{\frac{k}{n}}$. The following notations are frequently used in the setting of quasi-uniform spaces. For $x \in X$ and $k = 0, \cdots, k = n^2$

$$A^x_{\mathcal{C}_{\frac{k}{n}}} = \bigcap\{C \mid C \in \mathcal{C}_{\frac{k}{n}}, x \in C\}$$

and

$$U_{\mathcal{C}_{\frac{k}{n}}} = \bigcup_{x\in X}\{x\} \times A^x_{\mathcal{C}_{\frac{k}{n}}}.$$

We employ a standard technique as used with developments in [Lowen, 2015] to construct a function depending on n and on the choice $\mathcal{C}_{\frac{1}{n}}, \cdots, \mathcal{C}_{\frac{n^2}{n}}$ by letting

$$p_n = \inf_{k=1}^{n^2}\left(\frac{k-1}{n} + \theta_{U_{\mathcal{C}_{\frac{k}{n}}}}\right) \wedge n, \tag{2}$$

where for $Z \subseteq X$, we use the notation

$$\theta_Z : X \to [0, \infty] : x \mapsto \begin{cases} 0 & x \in Z \\ \infty & x \notin Z. \end{cases}$$

Clearly for $k \in \{1, \cdots, n^2\}$ we have $U_{\mathcal{C}_{\frac{k-1}{n}}} \subseteq U_{\mathcal{C}_{\frac{k}{n}}}$ and every $U_{\mathcal{C}_{\frac{k}{n}}}$ is a preorder. This implies that p_n is a quasi-ultrametric on X.

Next we show that each p_n belongs to the gauge $\mathcal{G}$. Fix $\varepsilon \geq 0$ and $\alpha > \varepsilon$. Either $\varepsilon \geq n$ and then the open ball $B_p(x, \alpha) = X \in \mathcal{T}_\varepsilon$, or $\varepsilon \in [\frac{k-1}{n}, \frac{k}{n}[$ for some $k \in \{1, \cdots, n^2\}$.

Then for $y \in A^x_{\mathcal{C}_{\frac{k}{n}}}$ we have $(x,y) \in U_{\mathcal{C}_{\frac{k}{n}}}$ which implies $p_n(x,y) \leq \frac{k-1}{n} \leq \varepsilon < \alpha$. So $A^x_{\mathcal{C}_{\frac{k}{n}}} \subseteq B_p(x,\alpha)$ and again $B_p(x,\alpha) \in \mathcal{T}_\varepsilon$.

Let $\mathcal{H}$ be the collection of all quasi-ultrametrics p_n, for arbitrary choices of n and $\mathcal{C}_{\frac{1}{n}}, \cdots, \mathcal{C}_{\frac{n^2}{n}}$. We prove that $\mathcal{H}$ is a basis for the gauge $\mathcal{G}$. Let $d \in \mathcal{G}$, $x \in X$ and $n \in \mathbb{N}_0$. Consider $\{\frac{k}{n} \mid k = 0, \cdots, k = n^2\}$ on $[0,n]$ and at level $\frac{k}{n}$ choose the cover

$$\mathcal{C}_{\frac{k}{n}} = \{B_d(x, \frac{k}{n} + \frac{1}{2n}), X\}.$$

Since $d \in \mathcal{G}$ the inclusion $\mathcal{T}^d_{\frac{k}{n}} \subseteq \mathcal{T}_{\frac{k}{n}}$ holds, so the cover $\mathcal{C}_{\frac{k}{n}}$ is $\mathcal{T}_{\frac{k}{n}}$-open for every k. Moreover clearly at each level $\mathcal{C}_{\frac{k-1}{n}}$ refines $\mathcal{C}_{\frac{k}{n}}$. Let p_n be the associated quasi-ultrametric as in (2). We show that

$$d(x,\cdot) \wedge n \leq p_n(x,\cdot) + \frac{2}{n}. \tag{3}$$

Let $y \in X$. Either $p_n(x,y) + \frac{2}{n} \geq n$ and then we are done, or $p_n(x,y) + \frac{2}{n} = \alpha \in [\frac{k}{n}, \frac{k+1}{n}[$ for some k. In this case $p_n(x,y) = \alpha - \frac{2}{n} \in [\frac{k-2}{n}, \frac{k-1}{n}[$ which implies $(x,y) \in U_{\mathcal{C}_{\frac{k-1}{n}}}$. So we have $y \in B_d(x, \frac{k-1}{n} + \frac{1}{2n})$, by which $d(x,y) < \frac{k-1}{n} + \frac{1}{2n} < \frac{k}{n} \leq \alpha$. Since (3) holds for every $n \in \mathbb{N}_0$ it now follows that also (1) is fulfilled for every $\varepsilon > 0$ and $\omega < \infty$, so we can conclude that $\widehat{\mathcal{H}} = \mathcal{G}$. ■

3.12. THEOREM. *Consider an approach space X with gauge $\mathcal{G}$, having a basis $\mathcal{H}$ consisting of quasi-ultrametrics. Then the associated distance is a non-Archimedean distance.*

PROOF. The distance δ associated with the gauge $\mathcal{G}$ can be derived directly from the basis $\mathcal{H}$ by

$$\delta(x,A) = \sup_{d\in\mathcal{H}} \inf_{a\in A} d(x,a).$$

We only have to show that this distance satisfies (D4$_\vee$). Take $x \in X, A \subseteq X$ and $\varepsilon \in \mathsf{P}_\vee$ arbitrary. Then, for any $b \in A^{(\varepsilon)}$, $d \in \mathcal{H}$ and $\theta > 0$, there exists $a_d \in A$ such that $d(b,a_d) < \varepsilon + \theta$. Consequently, $d(x,a_d) \leq d(x,b) \vee d(b,a_d) \leq d(x,b) \vee (\varepsilon+\theta)$, which proves that $\inf_{a\in A} d(x,a) \leq \inf_{b\in A^{(\varepsilon)}} d(x,b) \vee (\varepsilon + \theta)$. Since this holds for all $d \in \mathcal{H}$, it follows that $\delta(x,A) \leq \delta\big(x, A^{(\varepsilon)}\big) \vee \varepsilon$. ■

A gauge with a basis consisting of quasi-ultrametrics will be called a *non-Archimedean gauge.*

Based on the characterisation of non-Archimedean approach spaces in terms of non-Archimedean gauges, we can come back to the embedding of NA-App in App corresponding to the change of base functor $C_\varphi : (\beta, \mathsf{P}_\vee)\text{-}\mathsf{Cat} \longrightarrow (\beta, \mathsf{P}_+)\text{-}\mathsf{Cat}$.

3.13. THEOREM. NA-App *is a concretely reflective subcategory of* App. *If X is an approach space with gauge $\mathcal{G}$, then its* NA-App*-reflection $1_X : X \to X^u$ is given by the approach space X^u having $\mathcal{G} \cap q\mathsf{Met}^u(X)$ as basis for its gauge.*

PROOF. Since $\mathcal{G} \cap q\mathsf{Met}^u(X)$ is stable under finite suprema, it is locally directed and therefore $\widehat{\mathcal{G} \cap q\mathsf{Met}^u(X)}$ defines a gauge. Let X^u be the associated approach space. Suppose $f : X \to Y$ is a contraction with Y in $\mathsf{NA\text{-}App}$ with a basis $\mathcal{H}$ consisting of quasi-ultrametrics. Then $d \in \mathcal{H}$ clearly implies $d \circ f \times f \in \mathcal{G} \cap q\mathsf{Met}^u(X)$. By the characterization of a contraction in terms of a gauge basis, we have that $f : X^u \to Y$ is contractive. ∎

4. The embeddings $q\mathsf{Met}^u \hookrightarrow \mathsf{App}$ and $\mathsf{Top} \hookrightarrow \mathsf{App}$.

Restricting the coreflector $\mathsf{App} \to q\mathsf{Met}$ from [Lowen, 2015] to $\mathsf{NA\text{-}App}$, a non-Archimedean approach space X with limit operator λ (distance δ) is sent to its underlying quasi-ultrametric space (X, d_λ) ((X, d_δ)) given by

$$d_\lambda(x, y) = \lambda(\dot{y})(x) = \delta(x, \{y\}) = d_\delta(x, y)$$

for x, y in X. Moreover restricting the embedding $q\mathsf{Met} \hookrightarrow \mathsf{App}$ from [Lowen, 2015] to $q\mathsf{Met}^u$, a quasi-ultrametric space (X, d) is mapped to a non-Archimedean approach space X with limit operator defined by

$$\lambda_d(\mathcal{U})(x) = \sup_{U \in \mathcal{U}} \inf_{u \in U} d(x, u),$$

for all $\mathcal{U} \in \beta X$ and $x \in X$ and distance

$$\delta_d(x, A) = \inf_{a \in A} d(x, a)$$

for all $A \subseteq X$ and $x \in X$. We can conclude that $q\mathsf{Met}^u$ is concretely coreflectively embedded in $\mathsf{NA\text{-}App}$ and the coreflector is the restriction of the well known coreflector $\mathsf{App} \to q\mathsf{Met}$.

Considering the lax extension $\overline{\mathbb{\beta}}$ to $\mathsf{P}_{\triangledown}\text{-}\mathsf{Rel}$, the lax extension of the identity monad $\mathbb{I}$ to $\mathsf{P}_{\triangledown}\text{-}\mathsf{Rel}$ and the associated morphism $(I, \overline{I}) \to (\beta, \overline{\beta})$ of lax extensions, the induced algebraic functor

$$(\mathbb{\beta}, \mathsf{P}_{\vee})\text{-}\mathsf{Cat} \longrightarrow \mathsf{P}_{\vee}\text{-}\mathsf{Cat},$$

sends a $(\mathbb{\beta}, \mathsf{P}_{\vee})$-algebra (X, a) to its underlying $\mathsf{P}_{\vee}$-algebra $(X, a \cdot e_X)$, as introduced in section III.3.4 of [Hofmann, Seal, Tholen (eds.), 2014]. Using the isomorphisms described in section 2 and in Theorem 2.4, this functor sends a $(\mathbb{\beta}, \mathsf{P}_{\vee})$-algebra (X, a), corresponding to a non-Archimedean approach space X with limit operator λ, to its underlying quasi-ultrametric space (X, d_λ^-). This functor has a left adjoint $\mathsf{P}_{\vee}\text{-}\mathsf{Cat} \hookrightarrow (\mathbb{\beta}, \mathsf{P}_{\vee})\text{-}\mathsf{Cat}$, which associates to a $\mathsf{P}_{\vee}$-algebra (X, d), the $(\mathbb{\beta}, \mathsf{P}_{\vee})$-algebra corresponding to (X, λ_{d^-}).

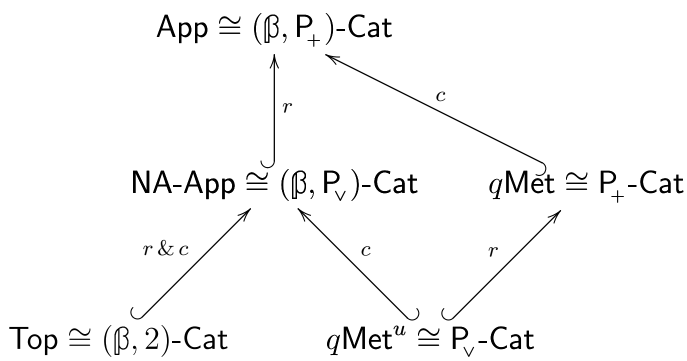
Next consider the lax homomorphism $\iota : 2 \longrightarrow \mathsf{P}_{\vee}$, sending $\top$ to 0 and $\bot$ to ∞, which is compatible with the lax extensions of the ultrafilter monad to Rel and $\mathsf{P}_{\triangledown}\text{-}\mathsf{Rel}$. Analogous to the situation for P_{+} [Hofmann, Seal, Tholen (eds.), 2014] the change-of-base functor associated to the lax homomorphism ι constitutes an embedding $(\mathbb{\beta}, 2)\text{-}\mathsf{Cat} \hookrightarrow (\mathbb{\beta}, \mathsf{P}_{\vee})\text{-}\mathsf{Cat}$. Using the isomorphisms described in section 2, Theorem 2.4, and the well known isomorphism $(\beta, 2)$-$\mathsf{Cat} \cong \mathsf{Top}$ [Barr, 1970] or [Hofmann, Seal, Tholen (eds.), 2014], this gives an embedding of Top in $\mathsf{NA\text{-}App}$.

In terms of the limit operator or the distance the embedding $\mathsf{Top} \hookrightarrow \mathsf{NA\text{-}App}$ associates the limit operator $\lambda_{\mathcal{T}}$ (distance $\delta_{\mathcal{T}}$) to a topological space $(X, \mathcal{T})$ by $\lambda_{\mathcal{T}}\mathcal{U}(x) = 0$ if $\mathcal{U}$ converges to x in $(X, \mathcal{T})$ ($\delta_{\mathcal{T}}(x, A) = 0$ if $x \in \overline{A}$) and with values ∞ in all other cases, for $\mathcal{U} \in \beta X, A \subseteq X, x \in X$. Later on we will also make use of the embedding of Top described in terms of the tower. All levels of the approach tower $(X, (\mathsf{t}_\varepsilon)_{\varepsilon \geq 0})$ associated to a topological space $(X, \mathcal{T})$ coincide, so we have $\mathcal{T}_\varepsilon = \mathcal{T}$ for all $\varepsilon \geq 0$. These formulations of the embedding $\mathsf{Top} \hookrightarrow \mathsf{NA\text{-}App}$ are the codomain restrictions of the embedding of Top in App as described in [Lowen, 2015].

ι has a right adjoint $p : \mathsf{P}_\vee \longrightarrow \mathsf{2}$, where $p(0) = \top$ and $p(v) = \perp$ otherwise, that is again a quantale homomorphism. p is also compatible with the lax extensions of the ultrafilter monad β to Rel and $\mathsf{P}_\vee$-Rel and provides the embedding with a right adjoint $(\beta, \mathsf{P}_\vee)$-$\mathsf{Cat} \longrightarrow (\beta, 2)$-$\mathsf{Cat}$. This functor can also be obtained by restricting the coreflector $\mathrm{T} : \mathsf{App} \longrightarrow \mathsf{Top}$, as described in [Lowen, 2015] to $\mathsf{NA\text{-}App}$ and we will continue in using the notation T. This coreflector sends a non-Archimedean approach space X to a topological space $\mathrm{T}(X)$ in which an ultrafilter $\mathcal{U}$ converges to a point x precisely when $\lambda\mathcal{U}(x) = 0$ or in which a point x is in the closure of a set A precisely when $\delta(x, A) = 0$. In terms of the tower $(\mathcal{T}_\varepsilon)_{\varepsilon \geq 0}$ the topological space $\mathrm{T}(X)$ is precisely $(X, \mathcal{T}_0)$.

Now define the map $o : \mathsf{P}_\vee \longrightarrow \mathsf{2}$ by $o(v) = \top$ if and only if $v < \infty$. Analogous to the situation of P_+ in [Hofmann, Seal, Tholen (eds.), 2014] the map o is a lax homomorphism, however it is not compatible with the ultrafilter lax extensions. Nevertheless, given a $(\beta, \mathsf{P}_\vee)$-algebra (X, a), one can still consider the pair (X, oa), where $oa : \beta X \rightarrow\!\!\!\!\!\rightarrow X$ is defined by $\mathcal{U}\, oa\, x$ precisely when $a(\mathcal{U}, x) < \infty$. This structure satisfies the reflexivity but not the transitivity condition. In other words, (X, oa) is a pseudotopological space. Now we can apply the left adjoint of the full reflective embedding $\mathsf{Top} \hookrightarrow \mathsf{PsTop}$ to (X, oa) to obtain a topological space and thereby a left adjoint $(\beta, \mathsf{P}_\vee)$-$\mathsf{Cat} \longrightarrow (\beta, 2)$-$\mathsf{Cat}$ to the embedding $(\beta, 2)$-$\mathsf{Cat} \hookrightarrow (\beta, \mathsf{P}_\vee)$-$\mathsf{Cat}$. This functor can also be obtained by restricting the reflector $\mathsf{App} \longrightarrow \mathsf{Top}$, as described in [Lowen, 2015] to $\mathsf{NA\text{-}App}$, where the Top-reflection of a non-Archimedean approach space (X, δ) is determined by the non-Archimedean distance associated with the topological reflection of the pretopological closure operator cl, defined by $\mathrm{cl}(A) := \{x \in A \mid \delta(x, A) < \infty\}$.

The results in this section can be summarized in the following diagram.



5. Initially dense objects in NA-App

At this point, we can introduce two new examples of non-Archimedean approach spaces.

5.1. Example. Consider the quasi-ultrametrics $d_{\mathsf{P}_\vee}$ and $d^-_{\mathsf{P}_\vee}$ on $\mathsf{P}_\vee$ defined by

$$d_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times \mathsf{P}_\vee \to \mathsf{P}_\vee : (x, y) \mapsto x \multimapdot y,$$

and

$$d^-_{\mathsf{P}_\vee} : \mathsf{P}_\vee \times \mathsf{P}_\vee \to \mathsf{P}_\vee : (x, y) \mapsto y \multimapdot x.$$

The first quasi-ultrametric space is a special case of a $\mathcal{V}$-structure on $\mathcal{V}$ as introduced in [Hofmann, Seal, Tholen (eds.), 2014]. In this section we show that each of the non-Archimedean approach spaces $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ of 3.4 and $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$, as introduced above, are initially dense objects in NA-App. Recall that for a source $(f_i : X \to (X_i, \lambda_i))_{i\in I}$ where (X_i, λ_i) are given approach spaces in terms of their limit operator, the initial lift on X is described by the limit operator

$$\lambda_{in}\mathcal{U}(x) = \sup_{i\in I} \lambda_i f_i(\mathcal{U})(f_i(x))$$

for $\mathcal{U} \in \beta X$ and $x \in X$.

5.2. Proposition. *For any non-Archimedean approach space X with distance δ and for $A \subseteq X$, the distance functional*

$$\delta_A : (X, \delta) \longrightarrow (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) : x \mapsto \delta(x, A)$$

is a contraction.

Proof. Let $x \in X$ and $B \subseteq X$. By application of 3.3 for $B \neq \emptyset$ and $A \neq \emptyset$, we have

$$\begin{aligned} \delta_{\mathsf{P}_\vee}\big(\delta_A(x), \delta_A(B)\big) &= \sup_{b\in B} \delta(b, A) \multimapdot \delta(x, A) \\ &\leq \sup_{b\in B} \delta(b, A) \multimapdot \big(\delta(x, B) \vee \sup_{b\in B} \delta(b, A)\big) \\ &\leq \delta(x, B). \end{aligned}$$

■

5.3. THEOREM. $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ *is initially dense in* NA-App. *More precisely, for any non-Archimedean approach space X, the source*

$$\left(\delta_A : X \longrightarrow (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})\right)_{A \in 2^X}$$

is initial.

PROOF. If λ_{in} stands for the initial non-Archimedean limit operator on X, then we already know by Proposition 5.2 that $\lambda_{in} \leq \lambda$. Conversely, take $\mathcal{U} \in \beta X$ and $x \in X$. Then

$$\begin{aligned}
\lambda_{in}\mathcal{U}(x) &= \sup_{A \in 2^X} \lambda_{\mathsf{P}_\vee}\left(\delta_A(\mathcal{U})\right)\left(\delta_A(x)\right) \\
&= \sup_{A \in 2^X} \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}\left(\delta_A(x), \delta_A(U)\right) \\
&\geq \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}\left(\delta_U(x), \delta_U(U)\right) \\
&= \sup_{U \in \mathcal{U}} \delta_{\mathsf{P}_\vee}(\delta_U(x), \{0\}) \\
&= \sup_{U \in \mathcal{U}} \delta(x, U) \\
&= \lambda\mathcal{U}(x).
\end{aligned}$$

■

5.4. THEOREM. *Both* $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ *and* $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ *are initially dense objects in* NA-App.

PROOF. Since we already know that $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is initially dense in NA-App, it suffices to show that we can obtain it via an initial lift of sources of either of the two objects above. First of all, consider the following source

$$\left(g_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d_{\mathsf{P}_\vee}})\right)_{\alpha \in \mathbb{R}^+},$$

where g_α is defined as follows:

$$g_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d_{\mathsf{P}_\vee}}) : x \mapsto \begin{cases} 0 & x > \alpha, \\ \alpha & x \leq \alpha. \end{cases}$$

To show that g_α is a contraction, for any $\alpha \in \mathbb{R}^+$, take $x \in \mathsf{P}_\vee$ and $B \subseteq \mathsf{P}_\vee$ arbitrary. For $B \neq \emptyset$, we consider two cases. First suppose $x \leq \alpha$, then

$$\delta_{d_{\mathsf{P}_\vee}}\left(g_\alpha(x), g_\alpha(B)\right) = \inf_{b \in B} g_\alpha(x) \multimap g_\alpha(b) = \inf_{b \in B} \alpha \multimap g_\alpha(b) = 0.$$

In case $x > \alpha$, we have

$$\delta_{d_{\mathsf{P}_\vee}}\left(g_\alpha(x), g_\alpha(B)\right) = \inf_{b \in B} g_\alpha(x) \multimap g_\alpha(b) = \inf_{b \in B} 0 \multimap g_\alpha(b).$$

If there exists $b \in B$ such that $b > \alpha$, then $\delta_{d_{\mathsf{P}_\vee}}\left(g_\alpha(x), g_\alpha(B)\right) = 0 \leq \delta_{\mathsf{P}_\vee}(x, B)$. If for all $b \in B$ we have that $b \leq \alpha$ then $\sup B \leq \alpha < x$, and thus $\delta_{d_{\mathsf{P}_\vee}}\left(g_\alpha(x), g_\alpha(B)\right) = \alpha < x = \delta_{\mathsf{P}_\vee}(x, B)$.

It remains to show that the source $(g_\alpha)_{\alpha\in\mathbb{R}^+}$ is initial. Let λ_{in} stands for the initial limit operator on $\mathsf{P}_\vee$, then we know that $\lambda_{in} \leq \lambda_{\mathsf{P}_\vee}$. To prove the other inequality, take $\mathcal{U} \in \beta\mathsf{P}_\vee$ and $x \in \mathsf{P}_\vee$ arbitrary. If $\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) = 0$, then the inequality is clear. In case $\lambda_{\mathsf{P}_\vee}\mathcal{U}(x) = x$ there exists $A \in \mathcal{U}$ such that $x > \sup A$. Take α arbitrary in the interval $[\sup A, x[$. Then

$$\lambda_{d_{\mathsf{P}_\vee}} g_\alpha(\mathcal{U})\big(g_\alpha(x)\big) = \sup_{U\in\mathcal{U}} \inf_{y\in U} d_{\mathsf{P}_\vee}\big(g_\alpha(x), g_\alpha(y)\big) \geq \inf_{y\in A} d_{\mathsf{P}_\vee}\big(g_\alpha(x), g_\alpha(y)\big) = 0 -\!\!\bullet \alpha = \alpha.$$

Hence $\lambda_{in}\mathcal{U}(x) = \sup_{\alpha\in\mathbb{R}^+} \lambda_{d_{\mathsf{P}_\vee}} g_\alpha(\mathcal{U})\big(g_\alpha(x)\big) \geq \sup_{\alpha\in[\sup A,x[} \alpha = x = \lambda_{\mathsf{P}_\vee}\mathcal{U}(x)$. So we can conclude that the source is initial.

In a similar way, we can prove that the source

$$\big(f_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d^-_{\mathsf{P}_\vee}})\big)_{\alpha\in\mathbb{R}^+},$$

with

$$f_\alpha : (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee}) \longrightarrow (\mathsf{P}_\vee, \delta_{d^-_{\mathsf{P}_\vee}}) : x \mapsto \begin{cases} \alpha & x > \alpha, \\ 0 & x \leq \alpha, \end{cases}$$

is initial as well. ■

6. Topological properties on $(\beta, \mathsf{P}_\vee)$-Cat

In this section we explore topological properties in $(\beta, \mathsf{P}_\vee)$-Cat, following the relational calculus, developed in V.1 and V.2 of [Hofmann, Seal, Tholen (eds.), 2014] for $(\mathbb{T}, \mathcal{V})$-properties. We introduce low separation properties, Hausdorffness, compactness, regularity, normality and extremal disconnectedness as an application to $(\beta, \mathsf{P}_\vee)$-Cat of the corresponding $(\mathbb{T}, \mathcal{V})$-properties for arbitrary $(\mathbb{T}, \mathcal{V})$-Cat. For each of the properties p we characterise the property $(\beta, \mathsf{P}_\vee)$-p in the context NA-App. On the other hand we make use of the well known meaning of these properties in the setting of Top $\cong (\beta, 2)$-Cat. For a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ we compare the property $(\beta, \mathsf{P}_\vee)$-p to the properties:

- X *has* p *at level* 0: meaning $(X, \mathcal{T}_0)$ has p in Top
- X *strongly has* p: meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}^+$.
- X *almost strongly has* p: meaning $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}^+_0$.

6.1. Low separation properties and Hausdorffness. We recall the definition of $(\beta, \mathsf{P}_\vee)$-p for p the Hausdorff, T_1 and T_0 properties in $(\beta, \mathsf{P}_\vee)$-Cat by giving the pointwise interpretation through the isomorphism in 2.4 in terms of the limit operator. For a study of low separation properties in App we refer to [Lowen, Sioen 2003].

6.2. Definition. *A non-Archimedean approach space X is*

1. $(\beta, \mathsf{P}_\vee)$*-Hausdorff if* $\lambda\mathcal{U}(x) < \infty$ & $\lambda\mathcal{U}(y) < \infty \Rightarrow x = y$, *for every* $\mathcal{U} \in \beta X$ *and all* $x, y \in X$.
2. X *is* $(\beta, \mathsf{P}_\vee)$*-T1 if* $\lambda\dot{x}(y) < \infty \Rightarrow x = y$, *for all* $x, y \in X$
3. X *is* $(\beta, \mathsf{P}_\vee)$*-T0 if* $\lambda\dot{x}(y) < \infty \quad \& \quad \lambda\dot{y}(x) < \infty \Rightarrow x = y$, *for all* $x, y \in X$.

Remark that $(\beta, \mathsf{P}_\vee)$-Hausdorff ($(\beta, \mathsf{P}_\vee)$-T_1, $(\beta, \mathsf{P}_\vee)$-T_0) is equivalent to (β, P_+)-Hausdorff ((β, P_+)-T_1, (β, P_+)-T_0) as described in [Hofmann, Seal, Tholen (eds.), 2014]

6.3. Theorem. *For a non-Archimedean approach space X the following properties are equivalent*

1. *X is strongly Hausdorff (strongly T_1, strongly T_0 respectively).*
2. *X is almost strongly Hausdorff (almost strongly T_1, almost strongly T_0 respectively).*
3. *X is $(\beta, \mathsf{P}_\vee)$-Hausdorff ($(\beta, \mathsf{P}_\vee)$-T_1, $(\beta, \mathsf{P}_\vee)$-T_0)*

and imply that X is Hausdorff (T_1, T_0 respectively) at level 0.

Proof. (1) $\Leftrightarrow$ (2) This is straightforward.
(1) $\Leftrightarrow$ (3) is based on

$$\lambda\mathcal{U}(x) < \infty \Leftrightarrow \exists \varepsilon \in \mathbb{R}^+, \lambda\mathcal{U}(x) \leq \varepsilon \Leftrightarrow \exists \varepsilon \in \mathbb{R}^+, \mathcal{U} \to x \text{ in } \mathcal{T}_\varepsilon,$$

for $\mathcal{U} \in \beta X$ and $x \in X$.
The proofs of the other cases follow analogously. ∎

That $(\beta, \mathsf{P}_\vee)$-Hausdorff ($(\beta, \mathsf{P}_\vee)$-T_1, $(\beta, \mathsf{P}_\vee)$-T_0) is not equivalent to the Hausdorff (T_1, T_0) property at level 0 follows from example $X_\mathcal{S}$ in 3.8 with $\mathcal{S} = \{X, \emptyset\}$.

6.4. Compactness. The next property we consider is $(\beta, \mathsf{P}_\vee)$-compactness. We recall the definition of $(\beta, \mathsf{P}_\vee)$-compactness in $(\beta, \mathsf{P}_\vee)$-Cat by giving the pointwise interpretation through the isomorphism in 2.4 in terms of the limit operator.

6.5. Definition. *A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-compact if*

$$\inf_{x \in X} \lambda\mathcal{U}(x) = 0,$$

for all $\mathcal{U} \in \beta X$.

This condition coincides with (β, P_+)-compactness studied in (β, P_+)-$\mathsf{Cat} \cong \mathsf{App}$ [Hofmann, Seal, Tholen (eds.), 2014] and is equivalent to what is called 0-compactness in [Lowen, 2015]. The proofs of the following results are straightforward.

6.6. THEOREM. *For a non-Archimedean approach space X the following equivalences hold:*

1. $(\beta, \mathsf{P}_\vee)$*-compact* $\Leftrightarrow$ *almost strongly compact*
2. *compact at level* $0 \Leftrightarrow$ *strongly compact*

That $(\beta, \mathsf{P}_\vee)$-compactness does not imply compactness at level 0 is well known in the setting of App [Lowen, 2015]. That this is neither the case in the setting of NA-App follows from the following example.

6.7. EXAMPLE. Consider the example in 3.9 on $]0, \infty[$ with $\mathcal{T}$ the right order topology. Clearly $\mathcal{T}$ is not compact, whereas the topologies at strictly positive levels are all compact.

6.8. PROPOSITION. *Let X be a non-Archimedean approach space. If X is $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff, then it is a compact Hausdorff topological space.*

PROOF. Suppose that the non-Archimedean approach space X is both $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff. Then $(X, \mathcal{T}_\varepsilon)$ is a compact Hausdorff topological space, for every $\varepsilon > 0$. By the coherence condition of the non-Archimedean tower, we have $\mathcal{T}_\gamma \subseteq \mathcal{T}_\varepsilon$ for $\varepsilon \leq \gamma$ and therefore $\mathcal{T}_\gamma = \mathcal{T}_\varepsilon$, for every $\gamma, \varepsilon > 0$. Moreover, since $\mathcal{T}_0 = \bigvee_{\gamma > 0} \mathcal{T}_\gamma$, all levels of the non-Archimedean tower are equal. This implies that X is topological. ∎

6.9. REGULARITY. Next we investigate the notion of regularity. We recall the definition of $(\beta, \mathsf{P}_\vee)$-p for p the regularity property in $(\beta, \mathsf{P}_\vee)$-Cat, by giving the pointwise interpretation through the isomorphism in 2.4 in terms of the limit operator.

6.10. DEFINITION. *A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-regular if*

$$\lambda\mathcal{U}(x) \leq \lambda m_X(\mathfrak{X})(x) \vee \sup_{\mathcal{A} \in \mathfrak{X}, B \in \mathcal{U}} \inf_{\mathcal{W} \in \mathcal{A}, b \in B} \lambda\mathcal{W}(b),$$

for all $\mathfrak{X} \in \beta\beta X, \mathcal{U} \in \beta X$ and $x \in X$.

A strong version of regularity was introduced in [Brock, Kent, 1998]. In terms of arbitrary filters it states that $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} \vee \gamma$, for every $\mathcal{F} \in \mathsf{F}X$ and $x \in X$ and in that paper it was shown to be equivalent to regularity at each level. In more generality this condition was also considered in [Colebunders, Mynard, Trott, 2014] in the context of contractive extensions.

The following result gives a characterisation of $(\beta, \mathsf{P}_\vee)$-regular in terms of the level topologies.

6.11. THEOREM. *For a non-Archimedean approach space X the following are equivalent:*

1. *X is strongly regular*
2. *X is almost strongly regular*
3. *For all $\mathcal{U}, \mathcal{W} \in \beta X$ and for all $\gamma \geq 0$: $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U} \Rightarrow \lambda\mathcal{U} \leq \lambda\mathcal{W} \vee \gamma$.*
4. *X is $(\beta, \mathsf{P}_\vee)$-regular*

PROOF. (1) $\Leftrightarrow$ (2) is clear from the fact that $\bigcup_{\varepsilon>0}\mathcal{C}_\varepsilon$ is a closed basis for the topology $\mathcal{T}_0$.
(1) $\Leftrightarrow$ (3) is essentially known from [Brock, Kent, 1998].
(3) $\Rightarrow$ (4). Take $\mathfrak{X} \in \beta\beta X$, $\mathcal{U} \in \beta X$ and $x \in X$. Put $\gamma = \lambda m_X(\mathfrak{X})(x)$ and $\varepsilon = \sup_{\mathcal{A}\in\mathfrak{X},B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A},b\in B} \lambda\mathcal{W}(b)$. It is sufficient to assume that both γ and ε are finite. Let $0 < \rho < \infty$ be arbitrary and consider

$$S := \{(\mathcal{G}, y) \mid \lambda\mathcal{G}(y) \leq \varepsilon + \rho\} \subseteq \beta X \times X.$$

Clearly the filterbasis $\mathfrak{X}\times\mathcal{U}$ has a trace on S, so we can choose $\mathcal{R} \in \beta S$ refining this trace. For $A \in m_X(\mathfrak{X})$ and $U \in \mathcal{U}$ there exist $R_1 \in \mathcal{R}$ and $R_2 \in \mathcal{R}$ such that $A \in \bigcap_{z\in R_1} \pi_1 z$ and $U = \pi_2 R_2$, with π_1 and π_2 the projections restricted to S. For $z \in R_1 \cap R_2$ we have $\lambda(\pi_1 z, \pi_2 z) \leq \varepsilon + \gamma$ and $\pi_2 z \in A^{(\varepsilon+\gamma)} \cap U$. Finally we can conclude that $m_X(\mathfrak{X})^{(\varepsilon+\gamma)} \subseteq \mathcal{U}$ w! which implies

$$\lambda\mathcal{U}(x) \leq \lambda m_X(\mathfrak{X})(x) \vee (\varepsilon + \rho) \leq \gamma \vee (\varepsilon + \rho).$$

By arbitrariness of ρ our conclusion follows.
(4) $\Rightarrow$ (1). We use a technique similar to the one used in the proof of Theorem 9 in [Brock, Kent, 1998]. Let $\mathcal{W}$ be an ultrafilter converging to $x \in X$ in $\mathcal{T}_\gamma$ for $\gamma \geq 0$ and let $\mathcal{U} \in \beta X$, such that $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$. We may assume that γ is finite. Let $0 < \rho < \infty$ be arbitrary. Consider

$$S = \{(\mathcal{G}, y) \mid \lambda\mathcal{G}(y) \leq \gamma + \rho\} \subseteq \beta X \times X$$

and $\{S_W \mid W \in \mathcal{W}\}$ with $S_W = \{(\mathcal{G}, y) \in S \mid W \in \mathcal{G}\}$ whenever $W \in \mathcal{W}$ which is a filterbasis on S. Let $\mathcal{S}_\mathcal{W}$ be the filter generated. Using the restrictions π_1 and π_2 of the projections to S, we observe the following facts:

i) $\pi_2\mathcal{S}_\mathcal{W} \subseteq \mathcal{W}^{(\gamma)}$: This follows from the fact that $y \in W^{(\gamma)}$ implies the existence of $\mathcal{G} \in \beta X$ with $W \in \mathcal{G}$ and $\lambda\mathcal{G}(y) \leq \gamma + \rho$.

ii) There exists $\mathcal{R} \in \beta S$ satisfying $\mathcal{S}_\mathcal{W} \subseteq \mathcal{R}$ and $\pi_2\mathcal{R} = \mathcal{U}$: Suppose the contrary, i.e. for every $\mathcal{R} \in \beta S$ with $\mathcal{S}_\mathcal{W} \subseteq \mathcal{R}$ there exists $U_\mathcal{R}$ and $R \in \mathcal{R}$ such that $U_\mathcal{R} \cap \pi_2 R = \emptyset$. We can select a finite number of these sets with $U_{\mathcal{R}_i} \cap \pi_2 R_i = \emptyset$ and such that $\bigcup_i R_i \in \mathcal{S}_\mathcal{W}$. In view of i) there exists an index j such that $\pi_2 R_j \in \mathcal{U}$ which is a contradiction.

iii) With $\mathfrak{X} = \pi_1\mathcal{R}$ we have $m_X(\mathfrak{X}) = \mathcal{W}$ since $W \in \mathcal{W}$ implies $W \in \bigcap_{\mathcal{G}\in\pi_1 S_W} \mathcal{G}$.

Combining these results, we now have:

$$\begin{aligned}
\lambda\mathcal{U}(x) &\leq \lambda m_X(\mathfrak{X})(x) \vee \sup_{\mathcal{A}\in\mathfrak{X},B\in\mathcal{U}} \inf_{\mathcal{V}\in\mathcal{A},b\in B} \lambda\mathcal{V}(b) \\
&= \lambda\mathcal{W}(x) \vee \sup_{R\in\mathcal{R},R'\in\mathcal{R}} \inf_{\mathcal{V}\in\pi_1 R,b\in\pi_2 R'} \lambda\mathcal{V}(b) \\
&\leq \gamma \vee \sup_{R\in\mathcal{R}} \inf_{\mathcal{V}\in\pi_1 R,b\in\pi_2 R} \lambda\mathcal{V}(b) \\
&\leq \gamma \vee \sup_{R\in\mathcal{R}} \inf_{z\in R} \lambda\pi_1(z)(\pi_2(z)) \\
&\leq \gamma \vee (\gamma + \rho).
\end{aligned}$$

By arbitrariness of $\rho > 0$, we can conclude that $\mathcal{U}$ converges to x in $\mathcal{T}_\gamma$. ∎

Recall that in (β, P_+)-$\mathsf{Cat} \cong \mathsf{App}$ the notion (β, P_+)-regularity as described in [Hofmann, Seal, Tholen (eds.), 2014] is equivalent to an approach form of regularity considered in [Robeys, 1992] and [Lowen, 2015] meaning $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F}+\gamma$, for every $\mathcal{F} \in \mathsf{F}X$ and $x \in X$. Other forms of regularity obtained by describing the objects of App as relational algebras were studied in [Colebunders, Lowen, Van Opdenbosch, 2016].
Clearly for non-Archimedean approach spaces we have

$$(\beta, \mathsf{P}_\vee)\text{-regular} \Rightarrow (\beta, \mathsf{P}_+)\text{-regular} \Rightarrow \text{regular at level } 0$$

The following examples based on the construction in 3.8 show that none of the implications is reversible.

6.12. Example. (1) Let $X = \{0, 1\}$ and $\mathcal{S}$ the Sierpinski topology on X with $\{1\}$ open. The approach space $X_\mathcal{S}$ is not (β, P_+)-regular. This can be seen by taking $\mathcal{F} = \dot{1}$ and $\gamma = 1$. For this choice we have $\mathcal{F}^{(\gamma)} = \{X\}$ and $\lambda\{X\}(1) = 2 \not\leq \lambda\dot{1}(1) + 1$. However $\mathcal{T}_0$ is discrete and hence regular.
(2) Let X be infinite and $\mathcal{S}$ the cofinite topology on X. The approach space X_S is not $(\beta, \mathsf{P}_\vee)$-regular. However it is (β, P_+)-regular. To see this let $1 \leq \gamma < 2$. A filter $\mathcal{F}$ on X either contains a finite set and then $\mathcal{F}^{(\gamma)} = \mathcal{F}$ and $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} + \gamma$. Or $\mathcal{F}$ does not contain a finite set. In that case we have $\mathcal{F}^{(\gamma)} = \{X\}$ and $\lambda\mathcal{F}^{(\gamma)} = 2 \leq \lambda\mathcal{F} + \gamma$. For $\gamma < 1$ or $2 \leq \gamma$ the condition $\lambda\mathcal{F}^{(\gamma)} \leq \lambda\mathcal{F} + \gamma$ is clearly also fulfilled.

6.13. Normality. Next we investigate normality. We recall the definition of normality in $(\beta, \mathsf{P}_\vee)$-Cat, by giving the pointwise interpretation in terms of the limit operator.

6.14. Definition. *A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-normal if*

$$\hat{a}(\mathcal{U}, \mathcal{A}) \vee \hat{a}(\mathcal{U}, \mathcal{B}) \geq \inf\{\hat{a}(\mathcal{A}, \mathcal{W}) \vee \hat{a}(\mathcal{B}, \mathcal{W}) \mid \mathcal{W} \in \beta X\},$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X, with $\hat{a}(\mathcal{U}, \mathcal{A}) = \inf\{u \in [0, \infty] \mid \mathcal{U}^{(u)} \subseteq \mathcal{A}\}$.

Turning the $\vee$ in the formula into $+$ we have (β, P_+)-normality as studied in [Hofmann, Seal, Tholen (eds.), 2014]. In case X is a topological (approach) space both notions coincide with

$$\overline{\mathcal{U}} \subseteq \mathcal{A} \;\&\; \overline{\mathcal{U}} \subseteq \mathcal{B} \Rightarrow \exists \mathcal{W} \in \beta X, \overline{\mathcal{A}} \subseteq \mathcal{W} \;\&\; \overline{\mathcal{B}} \subseteq \mathcal{W}$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X. As is shown in [Hofmann, Seal, Tholen (eds.), 2014] this condition coincides with the usual notion of normality on the topological (approach) space.
An ultrametric approach space is $(\beta, \mathsf{P}_\vee)$-normal, since by symmetry of the metric the ultrafilter $\mathcal{W}$ can be taken to be equal to $\mathcal{U}$. Without symmetry we will encounter examples of quasi-ultrametric approach spaces that are $(\beta, \mathsf{P}_\vee)$-normal and others that are not.

Next we give some useful characterisations of $(\beta, \mathsf{P}_\vee)$-normality.

6.15. PROPOSITION. *For a non-Archimedean approach space X, the following properties are equivalent:*

1. *X is $(\beta, \mathsf{P}_{\vee})$-normal*
2. $\hat{a}(\mathcal{U},\mathcal{A}) < v$ & $\hat{a}(\mathcal{U},\mathcal{B}) < v \Rightarrow \exists \mathcal{W} \in \beta X, \hat{a}(\mathcal{A},\mathcal{W}) < v$ & $\hat{a}(\mathcal{B},\mathcal{W}) < v$ *for all $\mathcal{U},\mathcal{A},\mathcal{B}$ ultrafilters on X and $v > 0$*
3. $A^{(v)} \cap B^{(v)} = \emptyset \Rightarrow \forall u < v, \exists C \subseteq X, A^{(u)} \cap C^{(u)} = \emptyset$ & $(X \setminus C)^{(u)} \cap B^{(u)} = \emptyset$ *for all $A, B \subseteq X$ and $v > 0$*

PROOF. That (1) and (2) are equivalent is straightforward.
To show that (2) implies (3), let that $A^{(v)} \cap B^{(v)} = \emptyset$ with $A, B \subseteq X$ for some $v > 0$ and let $u < v$ arbitrary. Suppose on the contrary that for all $C \subseteq X, A^{(u)} \cap C^{(u)} \neq \emptyset$ or $(X \setminus C)^{(u)} \cap B^{(u)} \neq \emptyset$. By Lemma V.2.5.1 in [Hofmann, Seal, Tholen (eds.), 2014] there exist ultrafilters $\mathcal{U},\mathcal{A},\mathcal{B}$ on X satisfying

$$\forall U \in \mathcal{U} : A^{(u)} \cap U^{(u)} \in \mathcal{A} \ \& \ B^{(u)} \cap U^{(u)} \in \mathcal{B}.$$

It follows that $\hat{a}(\mathcal{U},\mathcal{A}) \leq u < v$ and $\hat{a}(\mathcal{U},\mathcal{B}) \leq u < v$. By (2) there exists $\mathcal{W} \in \beta X$ with $\hat{a}(\mathcal{A},\mathcal{W}) < v$ & $\hat{a}(\mathcal{B},\mathcal{W}) < v$. Since $\mathcal{A}^{(v)} \subseteq \mathcal{W}$, $A^{(u)} \in \mathcal{A}$ and $A^{(u)^{(v)}} \subseteq A^{(v)^{(v)}} = A^{(v)}$ we have $A^{(v)} \in \mathcal{W}$. In the same way we have $B^{(v)} \in \mathcal{W}$ which contradicts $A^{(v)} \cap B^{(v)} = \emptyset$.
Next we show that (3) implies (2). Let $\mathcal{U},\mathcal{A},\mathcal{B}$ be ultrafilters on X, and $v > 0$ with $\hat{a}(\mathcal{U},\mathcal{A}) < v$ & $\hat{a}(\mathcal{U},\mathcal{B}) < v$. Choose ε, δ satisfying $\hat{a}(\mathcal{U},\mathcal{A}) < \varepsilon < \delta < v$ & $\hat{a}(\mathcal{U},\mathcal{B}) < \varepsilon < \delta < v$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We claim that for all $C \subseteq X$:

$$A^{(\varepsilon)} \cap C^{(\varepsilon)} \neq \emptyset \text{ or } B^{(\varepsilon)} \cap (X \setminus C)^{(\varepsilon)} \neq \emptyset.$$

Indeed, in view of $\mathcal{U}^{(\varepsilon)} \subseteq \mathcal{A}$ and $\mathcal{U}^{(\varepsilon)} \subseteq \mathcal{B}$, the assertion $\exists C \subseteq X$ with $A^{(\varepsilon)} \cap C^{(\varepsilon)} = \emptyset$ & $B^{(\varepsilon)} \cap (X \setminus C)^{(\varepsilon)} = \emptyset$ would imply $C \notin \mathcal{U}$ and $X \setminus C \notin \mathcal{U}$, which is impossible.
By (3) we have $A^{(\delta)} \cap B^{(\delta)} \neq \emptyset$. So there exists an ultrafilter $\mathcal{W}$ on X refining

$$\{A^{(\delta)} \cap B^{(\delta)} \mid A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Clearly $\mathcal{W}$ satisfies $\mathcal{A}^{(\delta)} \subseteq \mathcal{W}$ and $\mathcal{B}^{(\delta)} \subseteq \mathcal{W}$. So we can conclude that
$\hat{a}(\mathcal{A},\mathcal{W}) < v$ & $\hat{a}(\mathcal{B},\mathcal{W}) < v$. ■

Remark that for a given approach space X, condition (2) in 6.15 implies condition (ii) in Theorem V.2.5.2 of [Hofmann, Seal, Tholen (eds.), 2014] and therefore also (iii) which in [Van Olmen 2005] was shown to be equivalent to approach frame normality of the lower regular function frame of X.

6.16. THEOREM. *For a non-Archimedean approach space X with tower $(\mathcal{T}_\varepsilon)_{\varepsilon \geq 0}$, consider the following properties*

1. *X is strongly normal*

2. X is almost strongly normal

3. X is $(\beta, \mathsf{P}_\vee)$*-normal*

4. X is normal at level 0

The following implications hold: (1) $\Rightarrow$ (4) *and* (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

PROOF. (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (2) are straightforward. We show that (2) $\Rightarrow$ (3). Suppose $(X, \mathcal{T}_u)$ is normal, for every $u > 0$ and let $\hat{a}(\mathcal{U}, \mathcal{A}) < v$ & $\hat{a}(\mathcal{U}, \mathcal{B}) < v$ for some $v > 0$. Take u such that $\hat{a}(\mathcal{U}, \mathcal{A}) < u < v$ & $\hat{a}(\mathcal{U}, \mathcal{B}) < u < v$. Then for the topological space $(X, \mathcal{T}_u)$ we have $\mathcal{U}^{(u)} \subseteq \mathcal{A}$ and $\mathcal{U}^{(u)} \subseteq \mathcal{B}$. By the normality of $(X, \mathcal{T}_u)$ there exists $\mathcal{W} \in \beta X$ satisfying $\mathcal{A}^{(u)} \subseteq \mathcal{W}$ and $\mathcal{B}^{(u)} \subseteq \mathcal{W}$. It follows that $\hat{a}(\mathcal{A}, \mathcal{W}) \leq u < v$ and $\hat{a}(\mathcal{B}, \mathcal{W}) \leq u < v$. ■

There are no other valid implications between the properties considered in the previous theorem. This is shown by the next examples.

6.17. EXAMPLE. On $X =]0, \infty[$, we consider an approach space as in 3.9. We make a particular choice for the topology

$$\mathcal{T} = \{B^c \mid B \subseteq]0, \infty[, \text{ bounded } \} \cup \{\emptyset\},$$

on X which is finer than the right order topology and consider the approach space $(X, (\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+})$. The topology $\mathcal{T}_0$ is not normal since there are no non-empty distinct and disjoint open subsets, although disjoint non-empty closed subsets do exist. So X is not strongly normal either.
Let $\varepsilon > 0$ and consider the topological space $(X, \mathcal{T}_\varepsilon)$. It is a normal topological space since for $x \leq \varepsilon$ we have $x \in A^{(\varepsilon)}$ for every non-empty subset A. So (2) and hence (3) from Theorem 6.16 are satisfied.

6.18. EXAMPLE. Let $X =]0, \infty[$ and let $(\mathcal{T}_\varepsilon)_{\varepsilon \geq 0}$ the tower as defined in 6.17 starting from $\mathcal{T} = \{B^c \mid B \subseteq]0, \infty[, \text{ bounded } \} \cup \{\emptyset\}$. We define another tower $(\mathcal{S}_\gamma)_{\gamma \geq 0}$ on X as follows.

$$\mathcal{S}_\gamma = \begin{cases} \mathcal{P}(X) & \text{whenever } 0 \leq \gamma < 1 \\ \mathcal{T}_{\gamma - 1} & \text{whenever } 1 \leq \gamma. \end{cases}$$

Clearly the tower $(\mathcal{S}_\gamma)_{\gamma \geq 0}$ defines a non-Archimedean approach space on X. For the topology at level 0 we have $\mathcal{T}_0 = \mathcal{P}(X)$ is normal, but the topological space $(X, \mathcal{S}_1) = (X, \mathcal{T})$ is not normal. So (1) and (2) from Theorem 6.16 do not hold. However the approach space is $(\beta, \mathsf{P}_\vee)$-normal. Let A and B be non-empty subsets with $A^{(v)} \cap B^{(v)} = \emptyset$ for some $v > 0$ and let $u < v$. Clearly $v \leq 1$. In that case the level topology for u is discrete. It follows that $C = B$ satisfies the condition in (3) 6.15.

6.19. Example. Consider example (2) in Remark V.2.5.3 in [Hofmann, Seal, Tholen (eds.), 2014]. $X = (\{x, y, z, w\}, d)$ where d is the quasi-ultrametric structure $d(x, z) = d(y, z) = d(w, z) = 1$, $d(w, x) = d(w, y) = 3$, $d(x', x') = 0$ for any $x' \in X$ and $d(x', y') = \infty$ elsewhere. The topology $\mathcal{T}_0$ is discrete and hence normal. However $X = (\{x, y, z, w\}, d)$ is not $(\beta, \mathsf{P}_\vee)$-normal. Let $\mathcal{U} = \dot{z}, \mathcal{A} = \dot{x}$ and $\mathcal{B} = \dot{y}$. Then $\hat{a}(\mathcal{U}, \mathcal{A}) = d(x, z) = 1$ and $\hat{a}(\mathcal{U}, \mathcal{B}) = d(y, z) = 1$. For $\mathcal{W} = \dot{w}$ we have $\hat{a}(\mathcal{A}, \mathcal{W}) = d(w, x) = 3$ and $\hat{a}(\mathcal{B}, \mathcal{W}) = d(w, y) = 3$ and for all other choices of $\mathcal{W}$ we obtain values ∞. So the space $X = (\{x, y, z, w\}, d)$ does not satisfy (1) or (2) from 6.16 either.

Next we prove that the notions $(\beta, \mathsf{P}_\vee)$-normal and (β, P_+)-normal are unrelated. Again this can be shown by looking at finite non-Archimedean approach spaces that are therefore structured by some quasi-ultrametric.

6.20. Example. Let $X = \{x, y, z\}$ endowed with the quasi-ultrametric d defined by $d(y, x) = 2, d(x, z) = 1, d(y, z) = 1$ and $d(x', x') = 0$ for all $x' \in X$ and $d(x', y') = \infty$ elsewhere. Clearly the only inequality that has to be checked is $d(x, z) + d(y, z) = 2 \geq d(y, x)$ which is no longer valid when $+$ is changed into $\vee$. The space is (β, P_+)-normal but not $(\beta, \mathsf{P}_\vee)$-normal.

6.21. Example. Let $X = \{x, y, z, w\}$ endowed with the quasi-ultrametric d defined by $d(x, z) = 1, d(y, z) = 2, d(w, z) = 2, d(w, y) = 2, d(w, x) = 2$ and $d(x', x') = 0$ for all $x' \in X$ and $d(x', y') = \infty$ elsewhere. The space is not (β, P_+)-normal since $d(y, z) + d(x, z) = 3 < 4 = d(w, y) + d(w, x)$. However X is strongly normal, and therefore $(\beta, \mathsf{P}_\vee)$-normal. To see this observe that the level topologies for $0 \leq \varepsilon < 1$ are discrete and hence normal. For levels $1 \leq \varepsilon < 2$ all points are isolated except x for which the smallest neighborhood is $\{x, z\}$. This topology is normal since all closed sets are open. For levels $2 \leq \varepsilon$ we have smallest neighborhoods $V_w = X, V_y = \{y, z\}, V_z = \{z\}$ and $V_x = \{x, z\}$. These levels are normal too since there are no disjoint non-empty closed sets.

6.22. Extremal disconnectedness. Next we investigate extremal disconnectedness. We recall the definition of extremal disconnectedness in $(\beta, \mathsf{P}_\vee)$-Cat, by giving the pointwise interpretation in terms of the limit operator.

6.23. Definition. *A non-Archimedean approach space X is $(\beta, \mathsf{P}_\vee)$-extremally disconnected if*

$$\hat{a}(\mathcal{A},\mathcal{U}) \vee \hat{a}(\mathcal{B},\mathcal{U}) \geq \inf\{\hat{a}(\mathcal{W}, \mathcal{A}) \vee \hat{a}(\mathcal{W}, \mathcal{B}) \mid \mathcal{W} \in \beta X\},$$

for all ultrafilters $\mathcal{U}, \mathcal{A}, \mathcal{B}$ on X, with $\hat{a}(\mathcal{U}, \mathcal{A}) = \inf\{u \in [0, \infty] \mid \mathcal{U}^{(u)} \subseteq \mathcal{A}\}$.

Turning the $\vee$ in the formula into $+$ we have (β, P_+)-extremal disconnectedness as studied in [Hofmann, Seal, Tholen (eds.), 2014]. In case X is a topological (approach) space both notions coincide with

$$\overline{\mathcal{A}} \subseteq \mathcal{U} \;\&\; \overline{\mathcal{B}} \subseteq \mathcal{U} \Rightarrow \exists \mathcal{W} \in \beta X, \overline{\mathcal{W}} \subseteq \mathcal{A} \;\&\; \overline{\mathcal{W}} \subseteq \mathcal{B}$$

for all ultrafilters $\mathcal{U},\mathcal{A},\mathcal{B}$ on X. As is shown in [Hofmann, Seal, Tholen (eds.), 2014] this condition coincides with the usual notion of extremal disconnectedness on the topological (approach) space.

An ultrametric approach space is $(\beta,\mathsf{P}_\vee)$-extremally disconnected, since by symmetry of the metric the ultrafilter $\mathcal{W}$ can be taken to be equal to $\mathcal{U}$. Without symmetry we will encounter examples of quasi-ultrametric approach spaces that are $(\beta,\mathsf{P}_\vee)$-extremally disconnected and others that are not.

6.24. Proposition. *For a non-Archimedean approach space X with tower $(\mathcal{T}_\varepsilon)_{\varepsilon\geq 0}$, the following properties are equivalent:*

1. *X is $(\beta,\mathsf{P}_\vee)$-extremally disconnected*
2. $\hat{a}(\mathcal{A},\mathcal{U}) < v$ & $\hat{a}(\mathcal{B},\mathcal{U}) < v \Rightarrow \exists \mathcal{W}\in\beta X, \hat{a}(\mathcal{W},\mathcal{A}) < v$ & $\hat{a}(\mathcal{W},\mathcal{B}) < v$ *for all $\mathcal{U},\mathcal{A},\mathcal{B}$ ultrafilters on X and $v>0$*
3. *$A\cap B=\emptyset$ with $A,B\in\mathcal{T}_v \Rightarrow \forall u<v, A^{(u)}\cap B^{(u)}=\emptyset$ for all $A,B\subseteq X$ and $v>0$.*

Proof. The equivalence of (1) and (2) is straightforward.

Next we prove that (2) implies (3). If (3) does not hold then there exists $v>0$ and $A,B\in\mathcal{T}_v$ with $A\cap B=\emptyset$ and $u<v$ such that $A^{(u)}\cap B^{(u)}\neq\emptyset$.Take an ultrafilter $\mathcal{U}$ containing both $A^{(u)}$ and $B^{(u)}$. Let $\mathcal{U}_{\mathcal{T}_u}$ be the filter generated by the $\mathcal{T}_u$-open sets in $\mathcal{U}$. Then clearly $\mathcal{U}_{\mathcal{T}_u}\vee\{A\}$ is a proper filter.

We claim that there exists an $\mathcal{A}\in\beta X, \mathcal{U}_{\mathcal{T}_u}\vee\{A\}\subseteq\mathcal{A}$ with $\mathcal{A}^{(u)}\subseteq\mathcal{U}$.

Suppose on the contrary that for every such $\mathcal{A}_i\in\beta X$ with $\mathcal{U}_{\mathcal{T}_u}\vee\{A\}\subseteq\mathcal{A}_i$ there exists $A'_i\in\mathcal{A}_i$ with $A_i'^{(u)}\notin\mathcal{U}$. A finite subcollection of ultrafilters $\{\mathcal{A}_j \mid j\in J\}$ exists, for which the corresponding sets $A'_j\in\mathcal{A}_j$ satisfy $\cup_{j\in J}A'_j\in\mathcal{U}_{\mathcal{T}_u}\vee\{A\}$. Let $U\in\mathcal{U}_{\mathcal{T}_u}$ such that $U\cap A^{(u)}\subseteq\cup_{j\in J}A'_j$. From $A^{(u)}\in\mathcal{U}$ we deduce that $(U\cap A)^{(u)}$ and hence also $\cup_{j\in J}A'_j$ belongs to $\mathcal{U}$. A contradiction follows.

In the same way there exists an $\mathcal{B}\in\beta X, \mathcal{U}_{\mathcal{T}_u}\vee\{B\}\subseteq\mathcal{B}$ with $\mathcal{B}^{(u)}\subseteq\mathcal{U}$. By (2) there exists $\mathcal{W}\in\beta X$, $\mathcal{W}^{(v)}\subseteq\mathcal{A}$ and $\mathcal{W}^{(v)}\subseteq\mathcal{B}$. For every $W\in\mathcal{W}$ we have $W^{(v)}\cap A\in\mathcal{A}$ and $W^{(v)}\cap B\in\mathcal{B}$. By Lemma V.2.5.1 in [Hofmann, Seal, Tholen (eds.), 2014] we get $A\cap C^{(v)}=A\cap C\neq\emptyset$ or $B\cap(X\setminus C)^{(v)}=B\cap(X\setminus C)\neq\emptyset$ which is impossible. Hence we have (3).

Next we prove (3) implies (2). Let $\mathcal{A},\mathcal{B},\mathcal{U}\in\beta X$ with $\hat{a}(\mathcal{A},\mathcal{U})<\gamma$ & $\hat{a}(\mathcal{B},\mathcal{U})<\gamma$ and choose $u,v>0$ such that $\hat{a}(\mathcal{A},\mathcal{U})<u<v<\gamma$ & $\hat{a}(\mathcal{B},\mathcal{U})<u<v<\gamma$. For $A\in\mathcal{A}\cap\mathcal{T}_v$ and $B\in\mathcal{B}\cap\mathcal{T}_v$ we have $A^{(u)}\cap B^{(u)}\neq\emptyset$. By (3) we have $A\cap B\neq\emptyset$. So

$$\{A\cap B \mid A\in\mathcal{A}\cap\mathcal{T}_v, B\in\mathcal{B}\cap\mathcal{T}_v\}$$

is a filter basis and a finer ultraflter exists. Applying the second equivalence used in the proof of (ii) $\Leftrightarrow$ (iii) in V.2.4.4 of [Hofmann, Seal, Tholen (eds.), 2014] to the topology $\mathcal{T}_v$ we obtain a $\mathcal{W}\in\beta X$ satisfying $\mathcal{W}^{(v)}\subseteq\mathcal{A}$ and $\mathcal{W}^{(v)}\subseteq\mathcal{B}$. So we can conclude that $\hat{a}(\mathcal{W},\mathcal{A})\leq v<\gamma$ & $\hat{a}(\mathcal{W},\mathcal{B})\leq v<\gamma$.

■

6.25. THEOREM. *For a non-Archimedean approach space X with tower $(\mathcal{T}_\varepsilon)_{\varepsilon\geq 0}$, consider the following properties*

1. *X is strongly extremally disconnected*
2. *X is almost strongly extremally disconnected*
3. *X is $(\beta, \mathsf{P}_\vee)$-extremally disconnected*
4. *X is extremally disconnected at level* 0

The following implications hold: (1) ⇒ (2) ⇒ (3) *and* (1) ⇒ (4).

PROOF. (1) ⇒ (4) and (1) ⇒ (2) are straightforward. We show that (2) ⇒ (3) using characterisation (3) in 6.24. Suppose $A \cap B = \emptyset$ with $A, B \in \mathcal{T}_v$ and let $u < v$. since $(X, \mathcal{T}_v)$ is extremally disconnected we have $A^{(v)} \cap B^{(v)} = \emptyset$ and since $\mathcal{T}_v \subseteq \mathcal{T}_u$ the conclusion follows. ■

There are no other valid implications between the properties considered in the previous theorem. This is shown by the next examples.

6.26. EXAMPLE. We use the ultrametric space $X = ([0, \infty], d_M)$ that played a crucial role in the development of the Banaschewski compactification in [Colebunders, Sioen, 2017]. On $[0, \infty]$ let d_M be defined by

$$d_M(x, y) := \begin{cases} x \vee y & x \neq y, \\ 0 & x = y. \end{cases}$$

X clearly is an ultrametric space. When it is considered as an approach space it has a tower $(\mathcal{R}_\varepsilon)_\varepsilon$ of zero dimensional topologies. Since d_M is symmetric the space $([0, \infty], d_M)$ is $(\beta, \mathsf{P}_\vee)$-extremally disconnected. The topology $\mathcal{T}_0$ is discrete in all points $x \neq 0$ and has the usual neighborhood filter in 0. The set $A = \{1/n \mid n \geq 1\}$ is open in $\mathrm{T}X$ but its closure is $A \cup \{0\}$ which clearly is not open. So $\mathcal{T}_0$ is not extremally disconnected and hence X is not strongly extremally disconnected.
Let $\varepsilon > 0$ and consider the level topology $([0, \infty], \mathcal{R}_\varepsilon)$. Consider the open set
$A = \{\varepsilon + 1/n \mid n \geq 1\}$, then its closure $A^{(\varepsilon)} = A \cup [0, \varepsilon]$ which is not open at level ε. So although (3) is fulfilled, none of the other conditions in 6.25 hold.

6.27. EXAMPLE. Let $X = [0, \infty[$, endowed with the topology $\mathcal{T}$ with neighborhood filters $(\mathcal{V}(x))_{x\in X}$ where $\mathcal{V}(0)$ is the usual neighborhood filter in the Euclidean topology, and $\mathcal{V}(x) = \dot{x}$ whenever $x \neq 0$, so at level 0 we use the same topology as in the previous example 6.26. We define $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_\varepsilon$ at level $0 < \varepsilon$ having a neighborhood filter

$$\mathcal{V}_\varepsilon(x) \quad = \quad \begin{cases} \quad \text{stack}\{[0, x]\} \quad \text{whenever } 0 < x \leq \varepsilon \\ \quad \mathcal{V}(x) \quad \text{whenever } \varepsilon < x \text{ or } x = 0 \end{cases}$$

at $x \in X$. The topology at level 0 is $([0, \infty[, \mathcal{T})$. As we know from 6.26 it is not extremally disconnected.

Next we consider level $\varepsilon > 0$ and $([0,\infty[, \mathcal{T}_\varepsilon)$. Let $A \in \mathcal{T}_\varepsilon$ then either there is $x \le \varepsilon$ with $x \in A$. In this case $A^{(\varepsilon)} = [0,\varepsilon] \cup (A \cap]\varepsilon,\infty[)$ which is open, or there is no $x \le \varepsilon$ with $x \in A$. In this case we have $A^{(\varepsilon)} = A$ open. X is extremally disconnected at every strictly positive level, so (2) and (3) from 6.25 are satisfied whereas (1) and (4) are not.

6.28. Example. Let (X,d) be as is 6.19 and now consider (X,d^-). Then we have a space with $\mathcal{T}_0$ extremally disconnected which is not $(\beta, \mathsf{P}_\vee)$-extremally disconnected, so it satisfies (4), but none of the other conditions in 6.25.

To see that the notions $(\beta, \mathsf{P}_\vee)$-extremally disconnected and (β, P_+)-extremally disconnected are unrelated we can consider examples (X,d^-) for each of the spaces described in 6.20 and 6.21.

6.29. Topological properties of the initially dense objects. We come back to the examples in 3.4 and 5.1 and investigate their topological properties.

6.30. Example. First we consider $X = (\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$. Clearly the limit operator satisfies $\lambda\mathcal{U}(0) = 0$, for any $\mathcal{U} \in \beta\mathsf{P}_\vee$. Thus the space is compact at level 0.
$\mathsf{P}_\vee$ is T_0 since $\lambda\dot{x}(y) = 0 = \lambda\dot{y}(x)$ implies $x = y$. However, the space is not $(\beta, \mathsf{P}_\vee)$-T_0. To see this, take $x, y \in [0,\infty[$ such that $x < y$. Then $\lambda\dot{x}(y) = y$ and $\lambda\dot{y}(x) = 0$. Hence both $\lambda\dot{x}(y)$ and $\lambda\dot{y}(x)$ are finite, but $x \ne y$.
Since $\delta_{\mathsf{P}_\vee}(0, A) = 0$ for every nonempty subset A, we have $0 \in \operatorname{cl} A$ in the topology $\mathcal{T}_0$. So clearly $\mathcal{T}_0$ is neither T_1 nor regular. Therefore $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_1 nor $(\beta, \mathsf{P}_\vee)$-regular.
$\mathsf{P}_\vee$ is strongly normal and strongly extremally disconnected. This follows from the fact that at level 0 (and hence at all other levels) all non-empty closed sets contain 0. We conclude that all levels are normal. On the other hand there are no non-empty disjoint open sets at level 0 since $A \cap A' = \emptyset$ and $\infty \in \mathsf{P}_\vee \setminus A$ implies $\mathsf{P}_\vee \subseteq \mathsf{P}_\vee \setminus A$. So all levels are extremally disconnected.

6.31. Example. Next we consider the examples $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ as described in 5.1. For $X = (\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ and for any $\mathcal{U} \in \beta\mathsf{P}_\vee$, we have that $\lambda_{d_{\mathsf{P}_\vee}}\mathcal{U}(\infty) = \inf_{U\in\mathcal{U}} \sup_{y \in U} d_{\mathsf{P}_\vee}(\infty, y)$ $= 0$, so we get that $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is compact at level 0.
For $x, y \in \mathsf{P}_\vee$ arbitrary, we have that $\lambda_{d_{\mathsf{P}_\vee}} \dot{x}(y) = 0$ if and only if $x \le y$. This proves that $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is T_0. However $(\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_0. To see this, take $x, y \in [0,\infty[$ such that $x < y$. Then $\lambda_{d_{\mathsf{P}_\vee}} \dot{x}(y) = 0$, and $\lambda_{d_{\mathsf{P}_\vee}} \dot{y}(x) = y < \infty$.
Since $d_{\mathsf{P}_\vee}(\infty, z) = 0$ for every $z \in \mathsf{P}_\vee$ we have $\infty \in \operatorname{cl} A$ in the topology $\mathcal{T}_0$ for every nonempty subset A. So clearly $\mathcal{T}_0$ is not T_1. Therefore $(\mathsf{P}_\vee, \delta_{\mathsf{P}_\vee})$ is not $(\beta, \mathsf{P}_\vee)$-T_1.
$X = (\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is not (β, P_+)-regular, since the quasi-ultrametric is not symmetric. Moreover as $\infty \in \operatorname{cl} A$ for every subset A, the topology $\mathcal{T}_0$ is not regular either.
$X = (\mathsf{P}_\vee, d_{\mathsf{P}_\vee})$ is strongly normal and strongly extremally disconnected. To see this, take $A \subseteq \mathsf{P}_\vee$ non-empty. We notice that at level 0 (and hence at all other levels) $\infty \in \operatorname{cl}(A)$. So all non-empty closed sets contain ∞. We conclude that all levels are normal. On the other hand all non-empty open sets contain 0. So all levels are extremally disconnected.
The results for $(\mathsf{P}_\vee, d^-_{\mathsf{P}_\vee})$ follow analogously.

7. Compact Hausdorff non-Archimedean approach spaces

In the previous section in 6.8 we proved that a non-Archimedean approach space X that is at the same time $(\beta, \mathsf{P}_\vee)$-compact and $(\beta, \mathsf{P}_\vee)$-Hausdorff is topological. In this section for a space X we consider the axioms compact at level 0 in combination with Hausdorff at level 0, in the sense that the topology $\mathcal{T}_0$ is both compact and Hausdorff. In [Lowen, 2015] these conditions are called compact and Hausdorff and in sections 7 and 8 we will use this simplified terminology. We consider $\mathsf{NA\text{-}App}_2$, the full subcategory of $\mathsf{NA\text{-}App}$ consisting of all Hausdorff non-Archimedean approach spaces and $\mathsf{NA\text{-}App}_{c2}$ the full subcategory of $\mathsf{NA\text{-}App}_2$ consisting of all compact Hausdorff non-Archimedean approach spaces. For terminology used in sections 7 and 8 we refer to [Adámek, Herrlich, Strecker, 1990].

$\mathsf{NA\text{-}App}_2$ is epireflective in $\mathsf{NA\text{-}App}$ and is closed under the construction of finer structures, hence it is a quotient reflective subcategory of $\mathsf{NA\text{-}App}$. So $\mathsf{NA\text{-}App}_2$ is monotopological, i.e. for any family $(X_i)_{i\in I}$ of Hausdorff non-Archimedean approach spaces and any point-separating source $(f_i : X \longrightarrow X_i)_{i\in I}$ there exists a unique initial lift on X. Our next aim in this section is to determine the epimorphisms and extremal monomorphisms in this category.

7.1. THEOREM. *A contraction $f : X \longrightarrow Y$ in $\mathsf{NA\text{-}App}_2$ is an epimorphism in $\mathsf{NA\text{-}App}_2$, if and only if $f(X)$ is $\mathrm{T}\,Y$-dense.*

PROOF. One implication is straightforward. If $f : X \longrightarrow Y$ is a contraction in $\mathsf{NA\text{-}App}_2$ and $f(X)$ is $\mathrm{T}\,Y$-dense then $f : \mathrm{T}\,X \longrightarrow \mathrm{T}\,Y$ is an epimorphism in Haus. This implies f is an epimorphism in $\mathsf{NA\text{-}App}_2$ since for any two contractions $u, v : Y \longrightarrow Z$ in $\mathsf{NA\text{-}App}_2$ with $u \cdot f = v \cdot f$, also $\mathrm{T}\,u \cdot \mathrm{T}\,f = \mathrm{T}\,v \cdot \mathrm{T}\,f$ in Haus.
In order to prove the converse, suppose $f : \mathrm{T}\,X \longrightarrow \mathrm{T}\,Y$ is a contraction in $\mathsf{NA\text{-}App}_2$ with $f(X)$ not $\mathrm{T}\,Y$-dense. Set $M = \mathrm{cl}_{\mathrm{T}\,Y}\big(f(X)\big)$. We use a technique based on the amalgamation, for which we refer to [Dikranjan, Giuli, Tozzi, 1988]. Assume Y has gauge basis $\mathcal{H}$ consisting of quasi-ultrametrics. The gauge basis of the amalgamation $Y \coprod_M Y$ is given by

$$\mathcal{H}_{Y \coprod_M Y} = \{u_{g_d} \mid d \in \mathcal{H}\}\downarrow,$$

where

$$u_{g_d}(\overline{x_i}, \overline{y_j}) = \begin{cases} \inf_{m\in M} d(x,m) \vee d(m,y) & i \neq j; x, y \notin M, \\ d(x,y) & \text{elsewhere.} \end{cases}$$

Moreover since M is $\mathrm{cl}_{\mathrm{T}\,Y}$-closed in Y, the space $Y \coprod_M Y$ is Hausdorff. Both statements follow analogously to the construction and results in [Claes, Colebunders, Gerlo, 2007]. Now, consider the diagram

$$X \xrightarrow{\ f\ } Y \underset{j_2}{\overset{j_1}{\rightrightarrows}} Y \coprod Y \xrightarrow{\ \varphi\ } Y \coprod{}_M Y.$$

Let $u = \varphi \cdot j_1$ and $v = \varphi \cdot j_2$. Then $u \cdot f = v \cdot f$, but $u \neq v$. Hence f is not an epimorphism in $\mathsf{NA\text{-}App}_2$. ∎

We proceed by determining the extremal monomorphisms in NA-App$_2$. Since $\mathrm{cl_T}$ is idempotent and weakly hereditary, NA-App$_2$ is an $(\mathcal{E}^{\mathrm{cl_T}}, \mathcal{M}^{\mathrm{cl_T}})$-category, where $\mathcal{E}^{\mathrm{cl_T}} = \mathrm{Epi}(\text{NA-App}_2)$. The following theorem determines the extremal monomorphisms in NA-App$_2$.

7.2. Theorem. *The following classes of morphisms in* NA-App$_2$ *coincide:*

1. *The class of all regular monomorphisms;*
2. *The class of all extremal monomorphisms;*
3. *The class of all* $\mathrm{cl_T}$*-closed embeddings.*

Proof. To prove that extremal monomorphisms are $\mathrm{cl_T}$-closed embeddings, take $\mathcal{E}^{\mathrm{cl_T}}$ the class of all $\mathrm{cl_T}$-dense contractions and $\mathcal{M}^{\mathrm{cl_T}}$ the class of all $\mathrm{cl_T}$-closed embeddings. By the Theorem in section 2.4 in [Dikranjan, Tholen, 1995], we have that NA-App$_2$ is an $(\mathcal{E}^{\mathrm{cl_T}}, \mathcal{M}^{\mathrm{cl_T}})$-category. Since by 7.1 we have $\mathcal{E}^{\mathrm{cl_T}} = \mathrm{Epi}(\text{NA-App}_2)$, it follows that $\mathrm{ExtrMono}(\text{NA-App}_2) \subseteq \mathcal{M}^{\mathrm{cl_T}}$ [Adámek, Herrlich, Strecker, 1990].

In order to prove that every $\mathrm{cl_T}$-closed embedding is a regular monomorphism, take $f : X \longrightarrow Y$ a $\mathrm{cl_{T\,Y}}$-closed embedding in NA-App$_2$. Consider the construction of the amalgamation of Y with respect to $f(X)$.

$$X \xrightarrow{\ f\ } Y \overset{j_1}{\underset{j_2}{\rightrightarrows}} Y \coprod Y \xrightarrow{\ \varphi\ } Y \coprod{}_{f(X)} Y.$$

Since $f(X)$ is $\mathrm{cl_{T\,Y}}$-closed, the amalgamation is Hausdorff. f is the equalizer of the pair $(\varphi \cdot j_1, \varphi \cdot j_2)$. Hence f is a regular monomorphism.

■

We recall the following definitions from section 8.1 in [Dikranjan, Tholen, 1995]. Consider a functor $F : \mathsf{X} \longrightarrow \mathsf{Y}$. For an object $Y \in \mathsf{Y}$, let $F^{-1}Y := \{X \in \mathsf{X} \mid FX = Y\}$ be the class of objects of the fibre of F at Y, and let $\widetilde{F^{-1}}Y := \{X \in \mathsf{X} \mid FX \cong Y\}$ be its *replete closure* in Y. F is called *transportable* if for every $B \in \widetilde{F^{-1}}Y$, there is $A \in F^{-1}Y$ with $A \cong B$.

7.3. Proposition. *The restriction of the coreflector* T : NA-App$_2$ $\longrightarrow$ Haus *is transportable.*

Proof. Take $(Y, \mathcal{T})$ an arbitrary object in Haus. Take $\big(X, (\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big) \in \widetilde{\mathrm{T}^{-1}}(Y, \mathcal{T})$ arbitrary. There exists an isomorphism $f : (X, \mathcal{S}_0) \longrightarrow (Y, \mathcal{T})$ in Haus. For any $\varepsilon > 0$, define the topology on Y by transportation of the topology on X by putting $\mathcal{T}_\varepsilon = \{f(A) \mid A \in \mathcal{S}_\varepsilon\}$ and let $\mathcal{T}_0 = \mathcal{T}$. Then $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a non-Archimedean tower on Y in the fibre of $(Y, \mathcal{T})$ and $f : \big(X, (\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big) \longrightarrow \big(Y, \mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$ is an isomorphism in NA-App$_2$. ■

Now we can conclude the following.

7.4. Theorem. NA-App$_2$ *is cowellpowered.*

Proof. By Theorem 7.1 we have $\mathrm{T}\big(\mathrm{Epi}(\text{NA-App}_2)\big) \subseteq \mathrm{Epi}(\text{Haus})$. Since T is fibre small and transportable 7.3, the Theorem from section 8.1 in [Dikranjan, Tholen, 1995] implies cowellpoweredness of NA-App$_2$. ■

7.5. THEOREM. $\mathsf{NA\text{-}App}_{c2}$ *is an epireflective subcategory of* $\mathsf{NA\text{-}App}_2$.

PROOF. By 3.13 $\mathsf{NA\text{-}App}$ is closed in App under the formation of products and subspaces. Since the class of compact Hausdorff approach spaces is known to be closed under the formation of products and closed subspaces in App [Lowen, 2015] we can conclude that $\mathsf{NA\text{-}App}_{c2}$ is closed under formation of products and closed subspaces in $\mathsf{NA\text{-}App}_2$. Since by 7.2 and 7.4 $\mathsf{NA\text{-}App}_2$ is a cowellpowered (Epi, Extremal mono)-category, $\mathsf{NA\text{-}App}_{c2}$ is epireflective in $\mathsf{NA\text{-}App}_2$. ∎

8. Non-Archimedean Hausdorff Compactifications

Due to the foregoing theorem 7.5, there is a categorical construction of an epireflector $E : \mathsf{NA\text{-}App}_2 \to \mathsf{NA\text{-}App}_{c2}$ with epireflection morphisms $e_X : X \longrightarrow KX$, for every X in $\mathsf{NA\text{-}App}_2$. The question remains whether the epireflection morphisms e_X are embeddings. The following proposition shows that in general, this is not the case.

8.1. PROPOSITION. *A Hausdorff non-Archimedean approach space* X *that can be embedded in a compact Hausdorff non-Archimedean approach space* Y *has a topological coreflection* $\mathrm{T}\,X$ *that is a Tychonoff space.*

PROOF. If $f : X \longrightarrow Y$ is an embedding with $Y \in \mathsf{NA\text{-}App}_{c2}$, then $\mathrm{T}\,f : \mathrm{T}\,X \longrightarrow \mathrm{T}\,Y$ is an embedding in Top with $\mathrm{T}\,Y$ compact Hausdorff topological. So $\mathrm{T}\,X$ is a Tychonoff space. ∎

In theorem 8.3 below we will formulate sufficient conditions on a Hausdorff non-Archimedean approach space X, to ensure that there exists an embedding into a compact Hausdorff non-Archimedean approach space. Our construction is based on Shanin's compactification of topological spaces.

Given $\mathfrak{G}$, a collection of closed sets of a topological space X satisfying the conditions (i) $\emptyset, X \in \mathfrak{G}$ and (ii) $G_1, G_2 \in \mathfrak{G} \Rightarrow G_1 \cup G_2 \in \mathfrak{G}$, a $\mathfrak{G}$-*family* is a non-empty collection of sets belonging to $\mathfrak{G}$ satisfying the finite intersection property (f.i.p.). A $\mathfrak{G}$-family $\mathcal{F}$ is called *vanishing* if $\bigcap_{F\in\mathcal{F}} F = \emptyset$. A *maximal* $\mathfrak{G}$-*family* is a $\mathfrak{G}$-family which is not contained in any $\mathfrak{G}$-family as a proper subcollection. If $\mathfrak{G}$ moreover is a closed basis of X, then a compact topological space $\sigma(X, \mathfrak{G}) = (S, \mathcal{S})$ in which X is densely embedded is constructed in [Nagata, 1968] on the set $S = X \cup X'$ with X' the set of all maximal vanishing $\mathfrak{G}$-families. S is endowed with the topology $\mathcal{S}$ with $\{S(G) \mid G \in \mathfrak{G}\}$ as a closed basis, where $S(G) = G \cup \{p \in X' \mid G \in p\}$.

Given a non-Archimedean approach space X we will construct an extension of X by first considering a special closed base $\mathfrak{G}$ for $\mathrm{T}\,X$ and constructing $\sigma(\mathrm{T}\,X, \mathfrak{G})$.

8.2. THEOREM. *Any non-Archimedean approach space* X, *given by its tower of topological spaces* $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ *and tower of closed sets* $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, *can be densely embedded in a compact non-Archimedean approach space* $\Sigma(X, \mathfrak{G})$, *constructed from the closed basis* $\mathfrak{G} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$ *of* $\mathrm{T}\,X$ *and such that the topological coreflection* $\mathrm{T}\,\Sigma(X, \mathfrak{G})$ *is the Shanin compactification* $\sigma(\mathrm{T}\,X, \mathfrak{G})$ *of the topological coreflection* $\mathrm{T}\,X$.

Proof. Let X be a non-Archimedean approach space given by its tower of topological spaces $(\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and tower of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$, and take $\mathfrak{S} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$. By the coherence condition of the non-Archimedean tower, $\mathfrak{S}$ is a closed basis for the topological space $\mathrm{T}\,X = (X, \mathcal{T}_0)$ and moreover $\mathfrak{S}$ clearly satisfies the assumptions (i) and (ii) made above. Let $\sigma(\mathrm{T}\,X, \mathfrak{S}) = (S, \mathcal{S})$ be Shanin's compactification.

For every $\varepsilon \in \mathbb{R}^+$ the collection $\{S(G) \mid G \in \mathcal{C}_\varepsilon\}$ is a basis for the closed sets $\mathcal{D}_\varepsilon$ of a topology $\mathcal{R}_\varepsilon$ on S. However for $(\mathcal{R}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ the coherence condition from 3.7 need not be satisfied. In order to force the coherence condition we define the following tower of topologies on S; for $\alpha \in \mathbb{R}^+$, let

$$\mathcal{S}_\alpha = \bigvee_{\beta>\alpha} \mathcal{R}_\beta,$$

with the supremum taken in Top. The set S endowed with the tower $(\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ defines a non-Archimedean approach space which we denote by $\Sigma(X, \mathfrak{S})$.

Next we investigate its topological coreflection $\mathrm{T}\,\Sigma(X, \mathfrak{S}) = (S, \mathcal{S}_0)$. By definition, the collection $\bigcup_{\beta>0} \mathcal{D}_\beta$ is a closed basis for $(S, \mathcal{S}_0)$ and therefore the collection

$$\{S(G) \mid G \in \mathcal{C}_\varepsilon,\ \text{for some } \varepsilon > 0\} = \{S(G) \mid G \in \mathfrak{S}\}$$

is a basis for its closed sets too. It follows that $(S, \mathcal{S}_0) = \sigma(TX, \mathfrak{S})$, so $\Sigma(X, \mathfrak{S})$ is a compact non-Archimedean approach space.

Let $j : X \to S$ be the canonical injection. By construction, for $\varepsilon \in \mathbb{R}^+$ the map $j : (X, \mathcal{T}_\varepsilon) \to (S, \mathcal{R}_\varepsilon)$ is an embedding in Top. Since $\mathcal{T}_\varepsilon = \bigvee_{\gamma>\varepsilon} \mathcal{T}_\gamma$ the source $(j : (X, \mathcal{T}_\varepsilon) \to (S, \mathcal{R}_\gamma))_{\gamma>\varepsilon}$ is initial in Top which then implies that $j : (X, \mathcal{T}_\varepsilon) \to (S, \mathcal{S}_\varepsilon)$ is initial too. So finally we have that $j : \big(X, (\mathcal{T}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big) \longrightarrow \big(Y, (\mathcal{S}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}\big)$ is an embedding in $\mathsf{NA\text{-}App}$. That the embedding j is dense follows already from the result at level 0. ∎

In general, this compactification is not Hausdorff. To ensure the Hausdorff property, we need stronger conditions on X.

8.3. Theorem. *Let X be a non-Archimedean approach space given by its tower of closed sets $(\mathcal{C}_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$. The compactification described in Theorem 8.2 is Hausdorff if and only if the following conditions are fulfilled*

i) *X is Hausdorff,*

ii) $\forall\varepsilon > 0 : \forall G \in \mathcal{C}_\varepsilon, \forall x \notin G : \exists\ 0 < \gamma \le \varepsilon, \exists H \in \mathcal{C}_\gamma$ *such that* $x \in H$ *and* $H \cap G = \emptyset$,

iii) $\forall\varepsilon > 0 : \forall F, G \in \mathcal{C}_\varepsilon, F \cap G = \emptyset : \exists\ 0 < \gamma \le \varepsilon, \exists H, K \in \mathcal{C}_\gamma$ *such that* $F \cap H = \emptyset, G \cap K = \emptyset, H \cup K = X$.

Proof. Remark that the collection $\mathfrak{S} = \bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon$ used in the construction of 8.2 is closed under finite intersections. Using this fact, condition (ii) is equivalent to condition C) and (iii) is equivalent to condition D) in [Nagata, 1968]. So Theorem IV.3 C) and D) in [Nagata, 1968] can be applied. ∎

8.4. Corollary. *For a non-Archimedean approach space X that is Hausdorff,* $(\beta, \mathsf{P}_\vee)$*-regular and* $(\beta, \mathsf{P}_\vee)$*-normal, the compactification described in Theorem 8.2 is Hausdorff.*

Proof. Condition (i) in 8.3 is fulfilled. By 6.11 all strictly positive level topologies are regular, so in (ii) of 8.3 we can take $\gamma = \varepsilon$. In order to see that also (iii) is fulfilled let $\varepsilon > 0$ and $F, G \in \mathcal{C}_\varepsilon$. Let $0 < \gamma < \varepsilon$ be arbitrary. Since X is $(\beta, \mathsf{P}_\vee)$-normal, by 6.15 there exists $C \subseteq X$ with $F \cap C^{(\gamma)} = \emptyset$ and $(X \setminus C)^{(\gamma)} \cap G = \emptyset$. Then $H = C^{(\gamma)}$ and $K = (X \setminus C)^{(\gamma)}$, satisfy the conditions needed in (iii). ∎

Remark that in the previous corollary the condition $(\beta, \mathsf{P}_\vee)$-normal can be weakened to Van Olmen's normality, condition V.2.5.2(iii) in [Hofmann, Seal, Tholen (eds.), 2014]. Note that if X is a topological (non-Archimedean approach) space, $\mathcal{T}_\varepsilon = \mathcal{T}_0$ and $\mathcal{C}_\varepsilon = \mathcal{C}_0$, for every $\varepsilon > 0$. In this case condition iii) is equivalent to X being a normal topological space.

We get the following corollary.

8.5. Corollary. *Let X be a non-Archimedean approach space such that the conditions from theorem 8.3 are fulfilled. Then the topological coreflection* $\mathrm{T}\,X$ *is Tychonoff.*

Proof. The conditions from Theorem 8.3 imply that X can be embedded in a compact non-Archimedean approach space $\Sigma(X, \mathfrak{S})$, with $\mathrm{T}\,\Sigma(X, \mathfrak{S})$ the Shanin compactification of $\mathrm{T}\,X$, a compact Hausdorff topological space. ∎

As the Shanin compactification need not be an reflection, we end this section by noting that in general the dense embedding $j : X \to \Sigma(X, \mathfrak{S})$ constructed above, is not a reflection.

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