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FREE OBJECTS OVER POSEMIGROUPS IN THE CATEGORY $\mathbf{POSGR}_{\vee}$

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ABSTRACT. As we all know, the complete lattice $I_D(S)$ of all D-ideals of a meet-semilattice S is precisely the injective hull of S in the category of meet-semilattices. In this paper, we consider *sm*-ideals of posemigroups which can be regarded as a generalization of D-ideals of meet-semilattices. Unfortunately, the quantale $\mathscr{R}(S)$ of all *sm*-ideals of a posemigroup S is in general not an injective hull of S. However, $\mathscr{R}(S)$ can be seen as a new type of quantale completions of S. Further, we can see that $\mathscr{R}(S)$ is also a free object over S in the category $\mathbf{PoSgr}_{\vee}$ of posemigroups with *sm*-distributive join homomorphisms.

1. Introduction

Bruns and Lakser investigated the injective objects in the category of meet-semilattices with meet-semilattice homomorphisms, and they also gave the concrete form of injective hulls of meet-semilattices in [Bruns and Lakser (1970)]. Since partially ordered monoids (pomonoids for short) can be regarded as a generalization of meet-semilattices, it is natural to consider the injectivity in the category **PoMon** of pomonoids with pomonoid homomorphisms. However, injective objects in **PoMon** are trivial [Lambek et al. (2012)]. Hence, Lambek et al. changed a few of the definitions of pomonoid homomorphisms, chose submultiplicative order-preserving maps as morphisms of the category $\mathbf{PoMon}_{\leq}$ of pomonoids to investigate the injectivity in $\mathbf{PoMon}_{\leq}$, where a *submultiplicative map* between pomonoids is a map $f\colon (A,\cdot,\leq) \to (B,*,\leq)$ such that $f(a)*f(a') \leq f(a\cdot a')$ for all $a,a' \in A$. Also, they proved that every pomonoid has an injective hull and gave the concrete form of the injective hulls. Based on the work of Lambek et al., dropping out the unit, Zhang and Laan considered the injective hulls in the category $\mathbf{PoSgr}_{\leq}$ of partially ordered semigroups (posemigroups for short) and their submultiplicative order-preserving maps , and constructed the injective hulls for a certain class of posemigroups with respect to a class of order embedding. In [Xia et al. (2017)], Xia, Han and Zhao proved the existence of injective hulls and gave the concrete form of the injective hull of an arbitrary posemigroup in the category $\mathbf{PoSgr}_{\leq}$.

As we all know, injective hulls are unique up to isomorphism. In the category of meet-semilattices, the injective hull of a meet-semilattice S is precisely the complete lattice $I_D(S)$ of all D-deals of S with inclusion order [Bruns and Lakser (1970)]. Motivated by the con-

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struction of the injective hulls of meet-semilattices, Lambek wanted to carry over the way of the construction of injective hulls from meet-semilattices to pomonoids. Hence, the candidate for the injective hulls of pomonoids (posemigroups) may be the object $\mathscr{R}(S)$ consisting of all down-closed sets that were closed under distributive joins. Unfortunately, the object $\mathscr{R}(S)$ is in general not the injective hull of a pomonoid (posemigroup) S [Lambek et al. (2012)]. Then we naturally ask for the question whether $\mathscr{R}(S)$ is possible to be some kind of special objects in the category of posemigroups, which is the main motivation of this paper.

In [Han and Zhao (2008)], we investigated the quantale completions of posemigroups, and proved that up to isomorphism the quantale completions of a posemigroup S are completely determined by the topological closures on the power-set quantale $(\mathscr{P}(S), \subseteq, \bullet)$. In [Han and Zhao (2008), Xia et al. (2016)], we gave two classes of standard quantale completions of a posemigroup S—the greatest quantale completion $((\mathscr{P}(S))_{\mathscr{D}}, \subseteq, \otimes)$ and the smallest quantale completion $(\mathbf{S}^{\star}, \subseteq, \otimes)$. Whether there is another type of quantale completions for a posemigroup except two classes of standard quantale completions is an unsolvable question in [Xia et al. (2016)]. In fact, we can also see that $(\mathbf{S}^{\star}, \subseteq, \otimes)$ is exactly the injective hull of a posemigroup (pomonoid) S in the category $\mathbf{PoSgr}_{\leq}$ ($\mathbf{PoMon}_{\leq}$) [Xia et al. (2017)]. In this paper, we shall prove that $\mathscr{R}(S)$ is a new type of quantale completions of a posemigroup S. Moreover, we shall see that $\mathscr{R}(S)$ is also the free object over S in the category $\mathbf{PoSgr}_{\vee}$ of posemigroups with *sm*-distributive join homomorphisms.

A *partially ordered semigroup* (*posemigroup* for short) is a semigroup $(S, \cdot)$ with a partial order $\leq$ on S which is compatible with the multiplication $\cdot$, that is, $x \leq y \Rightarrow x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for any z in S. A *quantale* is a complete lattice Q with an associative binary operation & satisfying $a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i)$ and $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$ for any $a \in Q$ and $\{b_i\}_{i\in I} \subseteq Q$. Since $a\&-$ and $-\&a$ preserve arbitrary joins, they have right adjoints and we shall denote them by $a \rightarrow -$ and $- \leftarrow a$ respectively. Thus $a\&c \leq b$ iff $c \leq a \rightarrow b$ iff $a \leq b \leftarrow c$. A *quantale homomorphism* $f: P \rightarrow Q$ is both a semigroup homomorphism and a complete lattice homomorphism. A *quantic nucleus* j on a quantale Q is a submultiplicative closure operator where a closure operator on Q is an order-preserving, expansive and idempotent map. It is easy to show that $j(a\&b) = j(a\&j(b)) = j(j(a)\&b) = j(j(a)\&j(b))$ for all $a, b \in Q$. Furthermore, we can also prove that the set Q_j of fixed points of j is a quantale with $a \otimes b = j(a\&b)$ and $\bigvee^{Q_j}_{i\in I} a_i = j(\bigvee_{i\in I} a_i)$. A subset S of Q is called a *quantic quotient* of Q if there exists some quantic nucleus j such that $S = Q_j$. Let j, k be two quantic nuclei on Q. Then we have that $j \leq k \Longleftrightarrow Q_k \subseteq Q_j$. Clearly, every quantale is necessarily a posemigroup. For a given semigroup $(S, \cdot)$, we denote by $\mathscr{P}(S)$ the set of all subsets of S. Clearly, $(\mathscr{P}(S), \bullet)$ forms a quantale under the inclusion order, where $A \bullet B = \{a \cdot b : a \in A, b \in B\}$. When $A = \{a\}$, we write $A \bullet B$ and $B \bullet A$ simply as $a \cdot B$ and $B \cdot a$. Then $A \bullet B \subseteq C \Leftrightarrow A \subseteq C \leftarrow B$ and $B \bullet A \subseteq C \Leftrightarrow A \subseteq B \rightarrow C$, where $C \leftarrow B = \{x \in S : x \cdot B \subseteq C\}$ and $B \rightarrow C = \{x \in S : B \cdot x \subseteq C\}$.

For notions and concepts concerned, but not explained here, please refer to [Adámek et al. (2004)], [Lambek et al. (2012)] and [Rosenthal (1990)].

2. Quantale completions of posemigroups

In order to describe the revised semantic for intuitionistic linear logic, Larchey-Wendling and Galmiche introduced the concept of quantale completions of pomonoids in [Larchey-Wendling and Galmiche (2000)]. However, quantales mentioned in [Larchey-Wendling and Galmiche (2000)] are called *right semiquantales* in [Rosenthal (1990)], which are not the real quantales we usually speak of. Similar to the work of Larchey-Wendling and Galmiche, Han and Zhao considered the concept of quantale completions of posemigroups, and proved that up to isomorphism the quantale completions of a posemigroup S are completely determined by the topological closures on the power-set quantale $\mathscr{P}(S)$ [Han and Zhao (2008)].

2.1. DEFINITION. [Han and Zhao (2008)] A pair (i,Q) is called a *quantale completion* of a posemigroup S if $i\colon S\to Q$ is a posemigroup embedding into a quantale Q and $i(S)$ is join-dense in Q, that is, for any $a\in Q$, there exists $S_a\subseteq S$ such that $a=\bigvee\limits_{s\in S_a} i(s)$.

For convenience of the readers, we shall list some basic concepts and results on quantale completions of posemigroups.

2.2. DEFINITION. [Han and Zhao (2008)] Let S be a posemigroup. A *topological closure* on the power-set quantale $\mathscr{P}(S)$ is a quantic nucleus j on $\mathscr{P}(S)$ such that $j(\{x\})=x{\downarrow}$ for all $x\in S$.

2.3. PROPOSITION. [Han and Zhao (2008)] *Let S be a posemigroup. Then (i_Q,Q) is a quantale completion of S if and only if there exists a topological closure j on $\mathscr{P}(S)$ and a quantale isomorphism $f\colon(\mathscr{P}(S))_j\to Q$ such that $i_Q=f\circ i_{(\mathscr{P}(S))_j}$, where $i_{(\mathscr{P}(S))_j}(s)=j(\{s\})$.*

2.4. COROLLARY. [Han and Zhao (2008)] *Let S be a posemigroup, j be a topological closure on $\mathscr{P}(S)$. Then $(i_{(\mathscr{P}(S))_j},(\mathscr{P}(S))_j)$ is a quantale completion of S.*

In what follows we shall give two classes of standard topological closures—the smallest topological closure $\mathscr{D}$ and the greatest topological closure $(\cdot)^\star$ (see [Han and Zhao (2008), Xia et al. (2016)]).

$$\forall X\in\mathscr{P}(S),\;\; \mathscr{D}(X)=X{\downarrow},\;\; X^\star=: X^{\mathrm{ul}}\cap X^L\cap X^R\cap X^T,$$

where

$$X^{\mathrm{ul}}=:\{s\in S:\forall\, b\in S, X\subseteq b{\downarrow}\Rightarrow s\le b\}$$
$$X^{L}=:\{s\in S:\forall\, a,b\in S, X\cdot a\subseteq b{\downarrow}\Rightarrow s\cdot a\le b\}$$
$$X^{R}=:\{s\in S:\forall\, a,b\in S, a\cdot X\subseteq b{\downarrow}\Rightarrow a\cdot s\le b\}$$
$$X^{T}=:\{s\in S:\forall\, a,b,c\in S, a\cdot X\cdot c\subseteq b{\downarrow}\Rightarrow a\cdot s\cdot c\le b\}.$$

In this paper, we denote by $\mathbf{S}^\star$ the set of fixed points of $(\cdot)^\star$, that is, $\mathbf{S}^\star=\{A\in\mathscr{P}(S):A^\star=A\}$. Let (i_1,Q_1) and (i_2,Q_2) be quantale completions of a posemigroup S. Then there exist two topological closures j_1,j_2 such that $(\mathscr{P}(S))_{j_1}\cong Q_1$ and $(\mathscr{P}(S))_{j_2}\cong Q_2$. If $j_1\le j_2$, then we say that Q_2 is smaller than Q_1 (or Q_1 is greater than Q_2), written as $Q_2\sqsubseteq Q_1$. In this sense, if $Q_2\sqsubseteq Q_1$ and $Q_1\sqsubseteq Q_2$, then $Q_2\cong Q_1$. Hence, up to isomorphism $(\mathscr{P}(S))_{\mathscr{D}}$ is the greatest quantale completion of S and $\mathbf{S}^\star$ is the smallest quantale completion of S. The greatest quantale completion and the smallest quantale completion play an important role in the study of free objects and injective hulls [Kruml and Paseka (2008), Xia et al. (2017)]. Naturally, we consider a question whether there exists another type of quantale completions of posemigroups except two classes of standard quantale completioms, which is another motivation of this paper.

3. A new type of quantale completions

In this section, we shall prove that $\mathscr{R}(S)$ can be seen as a new type of quantale completions of a posemigroup S. Furthermore, we can see that $\mathbf{S}^\star \subseteq \mathscr{R}(S) \subseteq (\mathscr{P}(S))_{\mathscr{D}}$. In general, $\mathbf{S}^\star$, $\mathscr{R}(S)$ and $(\mathscr{P}(S))_{\mathscr{D}}$ are not the same (see Example 3.13). First, we shall introduce the concept of *sm*-ideals of posemigroups which can be regarded as a generalization of D-deals of meet-lattices [Bruns and Lakser (1970)].

3.1. DEFINITION. Let $(S,\cdot,\leq)$ be a posemigroup, $A \subseteq S$. Then A is called *left* (*right*) *m-distributive* if $\bigvee A$ exists and A satisfies the left (right) distributive law, that is, $(\bigvee A)\cdot s = \bigvee(A\cdot s)$, $(s\cdot(\bigvee A) = \bigvee(s\cdot A))$ for all $s \in S$. A is called *m-distributive* if A is both left m-distributive and right m-distributive.

3.2. REMARK. Let $(S,\cdot,\leq)$ be a posemigroup. Then

1. If A is a left (right) m-distributive subset of S, then $A\cdot s$ $(s\cdot A)$ is left (right) m-distributive for all $s \in S$.

2. If A is a m-distributive subset of a posemigroup S, then $A\cdot s$ and $s\cdot A$ are not necessarily m-distributive for all $s \in S$.

3.3. DEFINITION. Let A be a m-distributive subset of a posemigroup S. Then A is called *strong m-distributive* (*sm-distributive* for short) provided that $A\cdot s$ is right m-distributive for all $s \in S$.

3.4. LEMMA. *Let A be a m-distributive subset of a posemigroup S. Then A is sm-distributive if and only if $s\cdot A$ is left m-distributive for all $s \in S$.*

PROOF. Let A be *sm*-distributive. Then for all $x,s \in S$, we have that $(\bigvee(s\cdot A))\cdot x = (s\cdot(\bigvee A))\cdot x = s\cdot((\bigvee A)\cdot x) = s\cdot(\bigvee(A\cdot x)) = \bigvee(s\cdot(A\cdot x)) = \bigvee((s\cdot A)\cdot x)$, which implies that $s\cdot A$ is left m-distributive. By symmetry, we can prove the inverse. ■

3.5. REMARK. Let $(S,\cdot,\leq)$ be a posemigroup. Then

1. $\{x\}$, $x\downarrow$ are *sm*-distributive for all $x \in S$.

2. If X is *sm*-distributive, then $X\cdot y$ and $y\cdot X$ are also *sm*-distributive for all $y \in S$.

3. If $(S,\leq)$ is a complete lattice, then every subset of S is m-distributive if and only if S is a quantale. Furthermore, every subset of a quantale is *sm*-distributive.

3.6. DEFINITION. Let $(S,\cdot,\leq)$ be a posemigroup. A lower set X of S is called a *sm-ideal* if it is closed under *sm*-distributive joins, that is, for any subset D of X, $\bigvee D \in X$ whenever D is *sm*-distributive in $(S,\cdot,\leq)$. We denote by $\mathscr{R}(S)$ the set of all *sm*-ideals of S.

3.7. REMARK. Let $(S, \cdot, \leq)$ be a posemigroup. Then

1. $S \in \mathscr{R}(S)$.
2. For all $x \in S$, $x\downarrow \in \mathscr{R}(S)$.
3. If $(S, \cdot, \leq)$ is a meet-semilattice and $\cdot = \wedge$, then A is a D-ideal of S if and only if S is a *sm*-ideal of S.
4. If $(S, \cdot, \leq)$ is a quantale, then for any $A \in \mathscr{R}(S)$, there exists an element $s \in S$ such that $A = s\downarrow$.

3.8. LEMMA. [Rosenthal (1990)] *If Q is a quantale and $S \subseteq Q$, then S is a quantic quotient if and only if S is closed under* infs*, and $a \to s, s \leftarrow a \in S$, whenever $a \in Q$ and $s \in S$.*

3.9. PROPOSITION. *Let $(S, \cdot, \leq)$ be a posemigroup. Then $\mathscr{R}(S)$ is a quantic quotient of the power-set quantale $(\mathscr{P}(S), \subseteq, \bullet)$.*

PROOF. It is easy to show that $\mathscr{R}(S)$ is closed under arbitrary meets, that is, $\bigcap\limits_{i \in I} A_i \in \mathscr{R}(S)$ for all $A_i \in \mathscr{R}(S)$. Let $C \in \mathscr{R}(S)$, $A \in \mathscr{P}(S)$. Then we have that $C \leftarrow A$ and $A \to C$ are in $\mathscr{R}(S)$. Indeed: (1) Let $y \leq x \in C \leftarrow A$. Then $x \cdot a \in C$ for all $a \in A$. Since C is a lower set, one can see that $y \cdot a \leq x \cdot a \in C$ for all $a \in A$, which implies that $y \in C \leftarrow A$, that is, $C \leftarrow A$ is a lower set of S. (2) Let $D \subseteq C \leftarrow A$ be a *sm*-distributive subset. Then $D \bullet A \subseteq C$, which implies that $D \cdot a \subseteq C$ for all $a \in A$. Since $D \cdot a$ is *sm*-distributive for all $a \in A$, we have $\vee(D \cdot a) \in C$, that is, $(\vee D) \cdot a \in C$ for all $a \in A$. Thus, $\vee D \in C \leftarrow A$. By (1) and (2), we have $C \leftarrow A \in \mathscr{R}(S)$. Similarly, we have $A \to C \in \mathscr{R}(S)$. Therefore, by Lemma 3.8 we have that $\mathscr{R}(S)$ is a quantic quotient of $\mathscr{P}(S)$. ■

3.10. REMARK. Let $(S, \cdot, \leq)$ be a posemigroup. Then there exists a quantic nucleus ρ_R such that $\mathscr{R}(S) = (\mathscr{P}(S))_{\rho_R}$ and $\rho_R(\{x\}) = x\downarrow$, that is, ρ_R is a topological closure on $\mathscr{P}(S)$.

By Corollary 2.4 and Remark 3.10, we have the following theorem.

3.11. THEOREM. *Let $(S, \cdot, \leq)$ be a posemigroup. Then $(\eta_S, \mathscr{R}(S))$ is a quantale completion of S, where $\eta_S(s) = \rho_R(\{s\})$.*

3.12. REMARK. If $(S, \cdot, \leq)$ be a quantale, then η_S is a quantale isomorphism.

By Theorem 3.11, we see that $(\eta_S, \mathscr{R}(S))$ is a quantale completion of a posemigroup S. According to the description in Section 2, we have that $\mathbf{S}^\star \subseteq \mathscr{R}(S) \subseteq (\mathscr{P}(S))_{\mathscr{D}}$. The following example will indicate that $\mathscr{R}(S)$ lies strictly between $\mathbf{S}^\star$ and $(\mathscr{P}(S))_{\mathscr{D}}$.

3.13. EXAMPLE. Let S be a posemigroup with the partial order $\leq$ determined by Figure 3.1 and the binary multiplication $\cdot$ defined by (3.1)

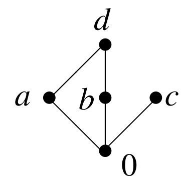

Figure 3.1

$$\forall x,y \in S,\ x \cdot y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ d, & \text{otherwise.} \end{cases} \tag{3.1}$$

One can easily verify that M is a *sm*-distributive subset of S if and only if $\bigvee M$ exists in S. Therefore, we have

$$\mathbf{S}^\star = \{\{0\},\{a,0\},\{b,0\},\{c,0\},\{a,b,d,0\},S\},$$

$$\mathscr{R}(S) = \{\{0\},\{a,0\},\{b,0\},\{c,0\},\{a,c,0\},\{b,c,0\},\{a,b,d,0\},S\},$$

$$(\mathscr{P}(S))_{\mathscr{D}} = \{\varnothing,\{0\},\{a,0\},\{b,0\},\{c,0\},\{a,b,0\},\{a,c,0\},\{b,c,0\},\{a,b,c,0\},\{a,b,d,0\},S\}.$$

4. Free objects over posemigroups in the category **PoSgr**$_\vee$

In this section, we shall show that $\mathscr{R}(S)$ is in fact a free object over S in the category **PoSgr**$_\vee$ of posemigroups with *sm*-distributive join homomorphisms. First, we shall introduce the concept of *sm*-distributive join homomorphisms.

4.1. DEFINITION. Let $(S,\cdot,\leq)$ and $(T,*,\leq)$ be posemigroups. Then a map $f\colon S \to T$ is called a *sm-distributive join homomorphism* if for any *sm*-distributive subset X of S, $f(X)$ is *sm*-distributive with $f(\bigvee X) = \bigvee f(X)$ and f is a semigroup homomorphism.

Let **PoSgr**$_\vee$ the category of posemigroups with *sm*-distributive join homomorphisms. By Remark 3.5(3), we have that the *sm*-distributive join homomorphisms between quantales are exactly the quantale homomorphisms, which implies that the category **Quant** of quantales with quantale homomorphisms is a full subcategory of **PoSgr**$_\vee$.

4.2. LEMMA. *Let $(S,\cdot,\leq)$ be a posemigroup. Then for all $a \in S, X \in \mathscr{R}(S)$, we have that $(-\cdot a)^{-1}(X), (a\cdot -)^{-1}(X) \in \mathscr{R}(S)$.*

PROOF. The proof is straightforward. ∎

In the following, we shall give a characterization for *sm*-distributive join homomorphisms between posemigroups.

4.3. LEMMA. *Let $f\colon (S,\leq,\cdot) \to (T,\leq,*)$ be a semigroup homomorphism between posemigroups. Then f is a sm-distributive join homomorphism if and only if $f^{-1}(X) \in \mathscr{R}(S)$ for all $X \in \mathscr{R}(T)$.*

PROOF. Assume that f is a *sm*-distributive join homomorphism and X is a *sm*-ideal of T, that is, $X \in \mathscr{R}(T)$. One can easily prove that f is order-preserving, which implies that $f^{-1}(X)$ is a lower set of S. For any *sm*-distributive subset A of $f^{-1}(X)$, we have that $f(A) \subseteq X$ and $f(\bigvee A) = \bigvee f(A)$. Since X is a *sm*-ideal and $f(A)$ is a *sm*-distributive subset of X, we have $f(\bigvee A) \in X$, which implies $\bigvee A \in f^{-1}(X)$. Thus, $f^{-1}(X)$ is a *sm*-ideal of S, that is, $f^{-1}(X) \in \mathscr{R}(S)$.

Conversely, it suffices to prove that $f(X)$ is *sm*-distributive with $f(\bigvee X) = \bigvee f(X)$ for any *sm*-distributive set X of S.

(1) f is order-preserving. Let $x, y \in S$ with $x \leq y$. Then from Remark 3.7 it follows that $f^{-1}(f(y)\downarrow) \in \mathscr{R}(S)$. Since $y \in f^{-1}(f(y)\downarrow)$, we have $x \in f^{-1}(f(y)\downarrow)$, which implies that $f(x) \leq f(y)$.

(2) $f(\bigvee X) = \bigvee f(X)$. Obviously, $f(\bigvee X)$ is an upper bound of $f(X)$. Assume that $t \in T$ is an arbitrary upper bound of $f(X)$, then $X \subseteq f^{-1}(t\downarrow)$. Since $f^{-1}(t\downarrow)$ is a *sm*-ideal and $\bigvee X \in f^{-1}(t\downarrow)$, we have $f(\bigvee X) \leq t$. Thus, $f(\bigvee X) = \bigvee f(X)$.

(3) $f(X)$ is *sm*-distributive. For any $x \in T$, $x * (\bigvee f(X))$ is clearly an upper bound of $x * f(X)$. Assume that $t \in T$ is an arbitrary upper bound of $x * f(X)$, then $x * f(X) \subseteq t\downarrow$. From Lemma 4.2, it follows that $f(X) \subseteq (x * -)^{-1}(t\downarrow) \in \mathscr{R}(T)$. By the assumption of f, we have that $X \subseteq f^{-1}((x*-)^{-1}(t\downarrow))$ and $\bigvee X \in f^{-1}((x*-)^{-1}(t\downarrow))$, which implies that $x * f(\bigvee X) \leq t$. Thus, $x * (\bigvee f(X)) = \bigvee(x * f(X))$. Similarly, $(\bigvee f(X)) * x = \bigvee(f(X) * x)$. So, we have that $f(X)$ is *m*-distributive. In the following we shall prove that for any $x \in T$, $f(X) * x$ is right *sm*-distributive. For any $y \in T$, $y * (\bigvee(f(X) * x))$ is clearly an upper bound of $y * (f(X) * x)$. Suppose that $t \in S$ is an arbitrary upper bound of $y * (f(X) * x)$. Then we have $y * (f(X) * x) \subseteq t\downarrow$ and $X \subseteq f^{-1}((y * -)^{-1}((-*x)^{-1}(t\downarrow)))$. Since X is *sm*-distributive, by Lemma 4.2 and the assumption of f we have that $\bigvee X \in f^{-1}((y*-)^{-1}((-*x)^{-1}(t\downarrow)))$ and $(y * f(\bigvee X)) * x \leq t$, which implies $\bigvee(y * (f(X) * x)) = y * (\bigvee(f(X) * x))$. Thus, $f(X)$ is *sm*-distributive. ∎

4.4. LEMMA. *Let $f\colon (S, \cdot, \leq) \to (T, *, \leq)$ be a sm-distributive join homomorphism between posemigroups. Then*

1. $\rho_R(f(X)) = \rho_R(f(\rho_R(X)))$ *for all* $X \in \mathscr{P}(S)$.
2. $F_f\colon \mathscr{R}(S) \to \mathscr{R}(T)$ *is a quantale homomorphism, where* $F_f(X) = \rho_R(f(X))$.

PROOF. (1) It is obvious to see that $\rho_R(f(X)) \subseteq \rho_R(f(\rho_R(X)))$. From Lemma 4.3, it follows that $f(X) \subseteq \rho_R(f(X)) \Longrightarrow X \subseteq f^{-1}(\rho_R(f(X))) \Longrightarrow \rho_R(X) \subseteq f^{-1}(\rho_R(f(X))) \Longrightarrow f(\rho_R(X)) \subseteq \rho_R(f(X)) \Longrightarrow \rho_R(f(\rho_R(X))) \subseteq \rho_R(f(X))$. Thus, $\rho_R(f(X)) = \rho_R(f(\rho_R(X)))$.

(2) Define a map $G\colon \mathscr{R}(T) \to \mathscr{R}(S)$ as follows $G(Y) = f^{-1}(Y)$. From Lemma 4.3, it follows that G is well defined. Since $F_f(X) \subseteq Y \Leftrightarrow \rho_R(f(X))) \subseteq Y \Leftrightarrow f(X) \subseteq Y \Leftrightarrow X \subseteq f^{-1}(Y) \Leftrightarrow X \subseteq G(Y)$ for all $X \in \mathscr{R}(S), Y \in \mathscr{R}(T)$, we have that F is the left adjoint of G, which implies that F_f preserves arbitrary joins. For all $A, B \in \mathscr{R}(S)$, by (1) we have $F_f(A \otimes B) = \rho_R(f(A \otimes B)) = \rho_R(f(\rho_R(A \bullet B))) = \rho_R(f(A \bullet B)) = \rho_R(f(A) \bullet f(B)) = \rho_R(f(A)) \otimes \rho_R(f(B)) = F_f(A) \otimes F_f(B)$. Thus, F_f is a quantale homomorphism. ∎

4.5. THEOREM. *$\mathscr{R}(S)$ is a free object over a posemigroup S in the category* $\mathbf{Posgr}_{\vee}$.

PROOF. As we have seen above, **Quant** is a full subcategory of **Posgr**$_\vee$. Let S be a posemigroup. Then by Proposition 3.9 we see that $\mathscr{R}(S)$ is a quantale. Further, by Theorem 3.11 we have that $\eta_S\colon S \to \mathscr{R}(S)$ is a semigroup homomorphism which is order-preserving. By Remark 3.5(3) and Remark 3.7, we have that for all $A \in \mathscr{P}(S)$, $\eta_S(A)$ is *sm*-distributive in $\mathscr{R}(S)$. For any *sm*-distributive subset A of S, we claim that $\eta_S(\bigvee A) = \bigvee\limits_{a\in A} \eta_S(a)$. It is easy to see that $\eta_S(a) \subseteq \eta_S(\bigvee A)$ for all $a \in A$. Let $B \in \mathscr{R}(S)$ with $\eta_S(a) \subseteq B$ for all $a \in A$. Then $A \subseteq B$. Since $B \in \mathscr{R}(S)$ and A is *sm*-distributive, we have that $\bigvee A \in B$ and $\eta_S(\bigvee A) \subseteq B$, which implies $\eta_S(\bigvee A) = \bigvee\limits_{a\in A} \eta_S(a)$. Thus, η_S is a *sm*-distributive join homomorphism. In what follows we shall prove that $\eta_S\colon S \to \mathscr{R}(S)$ has the universal property.

Let $f\colon S \to Q$ be a *sm*-distributive join homomorphism into a quantale Q. We let $\bar{f} = (\eta_Q)^{-1} \circ F_f$. First, we shall prove $\bar{f} \circ \eta_S = f$. By Remark 3.12 and Lemma 4.4, we have that $\bar{f}$ is a quantale homomorphism. For all $x \in S$, we have $\bar{f} \circ \eta_S(x) = (\eta_Q)^{-1} \circ F_f(x\downarrow) = (\eta_Q)^{-1}(f(x)\downarrow) = f(x)$. Next, we shall prove the uniqueness of $\bar{f}$. Suppose that $g\colon \mathscr{R}(S) \to Q$ is another quantale homomorphism such that $g \circ \eta_S = f$, then $g(x\downarrow) = \bar{f}(x\downarrow)$ for any $x \in S$. By Lemma 4.4(2), we have that for any $X \in \mathscr{R}(S)$, $\bar{f}(X) = \bar{f}(\overset{\mathscr{R}(S)}{\bigvee\limits_{x\in X}} x\downarrow) = \overset{Q}{\bigvee\limits_{x\in X}} \bar{f}(x\downarrow) = \overset{Q}{\bigvee\limits_{x\in X}} g(x\downarrow) = g(\overset{\mathscr{R}(S)}{\bigvee\limits_{x\in X}} x\downarrow) = g(X)$, which implies that $g = \bar{f}$. ■

4.6. COROLLARY. **Quant** *is a full reflective subcategory of* **Posgr**$_\vee$.

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