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ANALYTIC SPECTRUM OF RIG CATEGORIES

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ABSTRACT. We define the analytic spectrum of a rig category $(\mathcal{A}, \oplus, \otimes)$, and equip it with a sheaf of categories of rational functions. If the category is additive, we define a sheaf of categories of analytic functions. We relate this construction to Berkovich's analytic spaces, to Durov's generalized schemes and to Haran's $\mathbb{F}$-schemes. We use these relations to define analytic versions of Arakelov compactifications of affine arithmetic varieties.

1. Introduction

The notion of place of a number field (or, more generally, of a ring) plays a central role in number theory. Algebraic geometry gives a nice formal geometrical setting for the study of non-archimedean places. Berkovich has defined in [Ber90] a notion of global analytic space, that allows to think about all places as points of some new kind of space. Points of this geometry are given by multiplicative seminorms on rings. These spaces were further studied by Poineau (see [Poi10] and [Poi13]). The relation of this geometry with the ideas of Arakelov geometry, formalized categorically by Durov [Dur07], remains mysterious. We use Haran's ideas [Har07] to give a setting that relates global analytic spaces to the ideas of Arakelov geometry. The central point is to replace rings by rig categories, and to define a convenient notion of seminorm on such an object. To compactify the analytic spectrum of $\mathbb{Z}$, one wants to add a point at infinity, that corresponds to the map

$$|.|_\infty^\infty = |.|_{\infty,0} : \mathbb{Z}_\infty := [-1, 1] \to \mathbb{R}_+$$

sending ± 1 to 1 and all the rest to 0. One may naturally view this map as a multiplicative seminorm by promoting the set $\mathbb{Z}_\infty = [-1, 1]$ to a rig category.

In this article, we define and study the analytic spectrum of a general rig category. This allows us to define analytic spectra both for Haran's $\mathbb{F}$-rings and for Durov's generalized rings. We describe the relation of these new analytic spectra to Haran's and Durov's generalized schemes. We also describe the relation of these objects with Berkovich's global analytic spaces, and use them to define analytic versions of algebraic Arakelov compactifications.

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2. Seminorms on rig categories

2.1. DEFINITION. *A* rig category *is a category $\mathcal{A}$ equipped with two symmetric monoidal structures $\oplus$ and $\otimes$ (see [ML98]) together with two distributivity isomorphisms*

$$d_l : (M \oplus N) \otimes P \to M \otimes P \oplus N \otimes P$$
$$d_r : M \otimes (N \oplus P) \to M \otimes N \oplus M \otimes P$$

and two absorption isomorphisms $a_l : M \otimes 0 \to 0$ and $a_r : 0 \otimes M \to 0$ satisfying a set of coherence laws given in [Lap72]. A rig category $\mathcal{A}$ is called pointed *if* 0 *is an initial and final object of $\mathcal{A}$. A rig category $\mathcal{A}$ is called* pre-additive *if it is pointed, $\oplus$ is its coproduct and the natural morphism $i_{M,N} : M \oplus N \to M \times N$ is an isomorphism for all M, N.*

Note that if $\mathcal{A}$ is pre-additive, there is a natural zero morphism

$$0_{M,N} : M \to 0 \to N$$

and an addition operation $+_{M,N} : \mathrm{Hom}(M,N) \times \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N)$ on the morphism spaces, given for $f, g : M \to N$ by the morphism $f+g$ given by the composition:

$$M \xrightarrow{\Delta} M \times M \xrightarrow{i^{-1}_{M,M}} M \oplus M \xrightarrow{f \oplus g} N \oplus N \xrightarrow{\nabla} N,$$

where Δ is the diagonal morphism and ∇ is the codiagonal morphism (given by the fact that $\oplus$ is the coproduct). One calls $\mathcal{A}$ additive if all these laws are abelian group laws.

2.2. DEFINITION. *A pre-additive rig category $\mathcal{A}$ is called* additive *if the laws $+_{M,N}$ are abelian group laws.*

If $\mathcal{A}$ is additive, then $A = \mathrm{End}(\mathbb{1})$ is naturally equipped with a commutative ring structure, whose multiplication is given by $\otimes$.

In any case, the tensor product gives a natural multiplicative operation of $A = \mathrm{End}(\mathbb{1})$ on every morphism space $\mathrm{Hom}(M,N)$.

2.3. EXAMPLE. If A is a commutative unital ring, the category $\mathbb{F}(A)$ of free finitely generated modules over A of the form A^n, equipped with its natural tensor product and direct sum operations is an additive rig category (called by Haran the $\mathbb{F}$-ring of A, see [Har07]).

2.4. EXAMPLE. More generally, an important source of rig categories is given by generalized rings in the sense of Durov [Dur07], i.e, commutative algebraic monads on SETS. If A is a generalized ring, then the category $\mathbb{F}(A)$ of free finitely generated modules over A of the form $A(n)$ is naturally equipped with a rig category structure given by the direct sum $\oplus$ and tensor product $\otimes$ of free modules.

If $\mathcal{A}$ is a rig category and $S \subset A = \mathrm{End}(\mathbb{1})$ is a multiplicative subset, one may define, following Haran [Har07], the rig category $\mathcal{A}[S^{-1}]$ as the category with the same objects as that of $\mathcal{A}$ and with morphisms given by the space $\mathrm{Hom}_{\mathcal{A}}(M,N)[S^{-1}]$ of fractions f/s, $f \in \mathrm{Hom}_{\mathcal{A}}(M,N)$ and $s \in S$. It is naturally equipped with a rig category structure induced by that of $\mathcal{A}$.

2.5. Definition. *A* seminorm $|.| : \mathcal{A} \to \mathbb{R}_+$ *on a pointed rig category* $(\mathcal{A}, \oplus, \otimes)$ *is the datum of a family of maps* $|.|_{M,N} : \mathrm{Hom}(M,N) \to \mathbb{R}_+$ *indexed by pairs of objects of* $\mathcal{A}$ *satisfying:*

$$\begin{array}{rcll} |f \oplus g| & = & \max(|f|, |g|) & \\ |f \circ g| & \leq & |f|.|g| & \\ |f \otimes g| & \leq & |f|.|g| & \\ |0| & = & 0 & \\ |\mathrm{id}_M| & = & 1 & \text{if } 0_M \neq \mathrm{id}_M \end{array}$$

A seminorm is called multiplicative *if it satisfies*

$$|f \otimes g| = |f|.|g|.$$

First note that if $|.|$ is a multiplicative seminorm, then $|.|^t$ is also a multiplicative seminorm for every $t \geq 0$.

2.6. Example. Let A be a ring equipped with a non-archimedean multiplicative seminorm, i.e., a map $|.| : A \to \mathbb{R}_+$ such that

$$\begin{array}{rcl} |0| & = & 0 \\ |1| & = & 1 \\ |ab| & = & |a|.|b| \\ |a+b| & \leq & \max(|a|, |b|). \end{array}$$

We consider on each free finitely generated module A^n on A the seminorm $\|x\| = \max(|x_i|)$. For $f : A^n \to A^m$ a morphism, we set

$$|f| = \min\{C \geq 0,\ \ \forall x \in A^n,\ \|f(x)\| \leq C\|x\|\}.$$

Then one shows that if $(a_{i,j})$ is the matrix of f, one has

$$|f| = \max(|a_{i,j}|).$$

This has the properties needed to be a multiplicative seminorm on $\mathbb{F}(A)$.

2.7. Example. One defines similarly the operator norm on $\mathbb{F}(\mathbb{R})$ (resp. $\mathbb{F}(\mathbb{C})$) by putting on each $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) the standard euclidean (resp. hermitian) form and using the induced norm on linear maps. These give multiplicative seminorms on $\mathbb{F}(\mathbb{R})$ and $\mathbb{F}(\mathbb{C})$ (the so-called operator norms). If $\mathcal{A}$ is a pointed rig category and $f : \mathcal{A} \to \mathbb{F}(\mathbb{C})$ is a morphism, one gets an associated multiplicative seminorm on $\mathcal{A}$.

2.8. Definition. *Let* $\mathcal{A}$ *be a pointed rig category and* $|.|$ *be a multiplicative seminorm on* $\mathcal{A}$*. The* valuation rig category $\mathcal{A}_{|.|}$ *of* $|.|$ *is the rig category with the same class of objects but with morphisms given by the set of* $f \in \mathrm{Hom}(M,N)$ *such that* $|f| \leq 1$.

Note that the axioms of a multiplicative seminorm imply that $\mathcal{A}_{|.|}$ is still a pointed rig category.

If $\mathcal{A}$ is a pointed rig category and $|.|$ is a multiplicative seminorm, one defines the residue seminorm $|.|^\infty$ on $\mathcal{A}_{|.|}$ as the limit

$$|f|^\infty := \lim_{t\to\infty} |f|^t.$$

This seminorm sends morphisms of norm strictly smaller than 1 to 0, and morphisms of norm 1 to 1.

2.9. Example. If $|.| : A \to \mathbb{R}_+$ is a ring equipped with a non-archimedean multiplicative seminorm, and $|.| : \mathbb{F}(A) \to \mathbb{R}_+$ is the associated multiplicative seminorm, there is a natural isomorphism

$$\mathbb{F}(A_{|.|}) \xrightarrow{\sim} \mathbb{F}(A)_{|.|},$$

where $A_{|.|} = \{a \in A,\ |a| \leq 1\}$ is the classical valuation ring of the given seminorm. The residue seminorm is then associated to the trivial seminorm on the residue ring $A/\mathrm{Ker}(|.|)$.

2.10. Example. Let $\mathbb{F}(\mathbb{R})$ be equipped with the usual archimedean seminorm $|.|_\infty$, given by the euclidean operator norms on matrices. Then the valuation rig category $\mathbb{F}(\mathbb{R})_{|.|_\infty}$ is denoted $\mathbb{Z}_\eta$ by Haran, and called the archimedean valuation ring. In $\mathbb{Z}_\eta$, one has $\mathrm{End}(\mathbb{1}) = [-1, 1]$, so that $\mathbb{Z}_\eta$ promotes $[-1, 1]$ to a rig category. Furthermore, the residual seminorm on $\mathbb{Z}_\eta$ induces the map

$$|.|_\infty^\infty : [-1, 1] \to \mathbb{R}_+$$

that sends ± 1 to 1 and everything else to 0.

3. Spectrum of rig categories

3.1. Definition. *The set of multiplicative seminorms on a pointed rig category* $(\mathcal{A}, \oplus, \otimes)$ *is denoted* $\mathcal{M}(\mathcal{A})$ *and is called the* analytic spectrum *of* $\mathcal{A}$*. It is equipped with the coarsest topology that makes the map*

$$|.| \mapsto |f|$$

continuous for $f \in \mathrm{Hom}(M, N)$*.*

3.2. Relation to Berkovich's spectra. We now relate our notion of analytic spectrum to Berkovich's analytic spectrum of a ring, given by multiplicative seminorms. Remark that a multiplicative seminorm $|.| : A \to \mathbb{R}_+$ on a ring fulfills the triangular inequality

$$|f + g| \leq |f| + |g|,$$

which implies (and is actually equivalent to)

$$|f + g| \leq 2 \max(|f|, |g|),$$

so that the seminorms we will now consider are generalizations of those used by Berkovich. The main advantage of considering them is that the inequality

$$|f+g| \leq C \max(|f|,|g|)$$

implies

$$|f+g|^t \leq C^t \max(|f|^t,|g|^t),$$

so that (generalized) seminorms are stable by taking positive real powers, contrary to usual seminorms. For example, the power $|.|_\infty^2 : \mathbb{Z} \to \mathbb{R}_+$ (where $|.|_\infty$ is the usual archimedean seminorm) does not fulfill the triangular inequality.

3.3. Lemma. *If $\mathcal{A}$ is an additive rig category and $|.|$ is a seminorm on $\mathcal{A}$ then for all objects M, N of $\mathcal{A}$, there exists $C_{M,N} \in \mathbb{R}_+$ such that*

$$|f+g| \leq C_{M,N} \max(|f|,|g|).$$

Furthermore, one has $C_{M,M} \geq 1$ if $0_M \neq \mathrm{id}_M$.

Proof. One may use the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{i_{M,M}^{-1}} M \oplus M \xrightarrow{f \oplus g} N \oplus N \xrightarrow{\nabla} N$$

to define the sum $f+g$, and set $C_{M,N} = |i_{M,M}^{-1} \circ \Delta|.|\nabla|$. By compatibility of $|.|$ with composition, and the equality $|f \oplus g| = \max(|f|,|g|)$, one gets the desired inequality. One has

$$1 = |\mathrm{id}_M| = |0_M + \mathrm{id}_M| \leq C_{M,M} \max(|0_M|,|\mathrm{id}|) = C_{M,M} \max(0,1) = C_{M,M}.$$

■

3.4. Corollary. *If $\mathcal{A}$ is an additive rig category and $A = \mathrm{End}(\mathbb{1})$ is the associated ring, supposed to be nonzero, there is a natural continuous map*

$$\mathcal{M}(\mathcal{A}) \to \mathcal{M}(A),$$

where $\mathcal{M}(A)$ is the set of maps $|.| : A \to \mathbb{R}_+$ such that there exists $C \geq 1$ with:

$$\begin{array}{rcl} |0| & = & 0 \\ |1| & = & 1 \\ |f.g| & = & |f|.|g| \\ |f+g| & \leq & C \max(|f|,|g|). \end{array}$$

Note that for A a ring and $\mathcal{A} = \mathbb{F}(A)$, the above map is surjective. Indeed, every non-archimedean seminorm ($C = 1$) has a pre-image, described in Example 2.6. If $|.| : A \to \mathbb{R}_+$ is archimedean ($|2| > 1$, so that $C > 1$), there exists t such that $|.|^t$ fulfils the inequality for $C = 2$, which is equivalent to the triangular inequality

$$|f+g|^t \leq |f|^t + |g|^t.$$

This corresponds to a morphism $A \to \mathbb{C}$ such that $|.|^t$ is induced by the classical norm on $\mathbb{C}$. Then one may take the operator seminorm $\|.\|$ on $\mathbb{F}(A)$ described in Example 2.7 (related to $|.|^t$), and raise it to the power $1/t$, to get back $|.|$ on A.

3.5. Relation to Haran's spectra. If $U \subset \mathcal{M}(\mathcal{A})$ is open, we denote $\tilde{\mathcal{K}}(U) = \mathcal{A}[S_U^{-1}]$, where S_U is the multiplicative set of elements a in $A = \mathrm{End}(\mathbb{1})$ such that $|a| \neq 0$ for all $|.| \in U$. If $x \in \mathcal{M}(\mathcal{A})$ is a point, we denote $\mathcal{O}_{(x)} = \mathcal{A}[S_x^{-1}]$, where S_x is the multiplicative set of elements a in $A = \mathrm{End}(\mathbb{1})$ such that $|a(x)| \neq 0$. An element f of $\mathrm{Hom}_{\tilde{\mathcal{K}}(U)}(M, N)$ may be sent to a function

$$f : U \to \coprod_{x \in U} \mathrm{Hom}_{\mathcal{O}_{(x)}}(M, N)$$

that sends x to $f(x) \in \mathrm{Hom}_{\mathcal{O}_{(x)}}(M, N)$. We denote $\mathcal{K}(U)$ the set of maps f as above that locally come from sections of $\tilde{\mathcal{K}}$.

3.6. Proposition. *The above construction gives a sheaf of rig categories $\mathcal{K}$ on $\mathcal{M}(\mathcal{A})$ called the sheaf of rational functions without poles.*

Proof. This follows from the local definition of $\mathcal{K}(U)$. ∎

Recall from Haran [Har07] the definition of the spectrum of a rig category.

3.7. Definition. *Let $\mathcal{A}$ be a rig category. An H-*ideal *in $\mathcal{A}$ is a subset $\mathfrak{a} \subset \mathrm{End}(\mathbb{1})$ such that for all $a_1, \ldots, a_n \in \mathfrak{a}$ and all $f : \mathbb{1} \to \mathbb{1}^n$ and $g : \mathbb{1}^n \to \mathbb{1}$,*

$$g \circ (a_1 \oplus \cdots \oplus a_n) \circ f \in \mathfrak{a}.$$

An H-ideal $\mathfrak{p}$ is called prime if $\mathrm{End}(\mathbb{1}) - \mathfrak{p}$ is multiplicative. The spectrum *of $\mathcal{A}$ is the set $\mathrm{Spec}(\mathcal{A})$ of prime H-ideals equipped with its Zariski topology generated by subsets $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{A}),\ f \notin \mathfrak{p}\}$ for $f \in \mathrm{End}(\mathbb{1})$.*

3.8. Proposition. *If $|.|$ is a multiplicative seminorm on $\mathcal{A}$, then the subset*

$$\mathrm{Ker}(|.|) = \{a \in \mathrm{End}(\mathbb{1}),\ |a| = 0\}$$

is a prime H-ideal and this gives a natural continuous map

$$\mathcal{M}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{A})$$

that respects the sheaves of rig categories given by rational functions without poles.

Proof. Let $a_1, \ldots, a_n \in \mathrm{Ker}(|.|)$ and $f : \mathbb{1} \to \mathbb{1}^n$, $g : \mathbb{1}^n \to \mathbb{1}$ be morphisms. Then

$$|g \circ (a_1 \oplus \cdots \oplus a_n) \circ f| \leq |g|. \max(|a_i|)|f| = 0,$$

so that $\mathrm{Ker}(|.|)$ is an H-ideal. It is prime because $|.|$ is multiplicative on $\mathrm{End}(\mathbb{1})$. ∎

Let $\mathbb{Z}_\eta \subset \mathbb{F}(\mathbb{R})$ be the valuation rig category of the archimedean norm and for $N > 1$, denote $A_{N,\eta}$ the rig category $\mathbb{Z}_\eta \cap \mathbb{F}(\mathbb{Z}[1/N])$. Every multiplicative seminorm on $A_{N,\eta}$ that sends $1/N$ to a non-zero element can be extended to $\mathbb{F}(\mathbb{Z}[1/N]) = A_{N,\eta}[(1/N)^{-1}]$. This allows us to identify $\mathcal{M}(\mathbb{F}(\mathbb{Z}[1/N]))$ with the open subset $|1/N| > 0$ of $\mathcal{M}(A_{N,\eta})$. Note that similarly, $\mathcal{M}(\mathbb{F}(\mathbb{Z}[1/N]))$ identifies with the open subset $|N| > 0$ of $\mathcal{M}(\mathbb{F}(\mathbb{Z}))$. We can thus paste $\mathcal{M}(\mathbb{Z})$ and $\mathcal{M}(A_{N,\eta})$ along $\mathcal{M}(\mathbb{F}(\mathbb{Z}[1/N]))$ to get a space $\overline{\mathcal{M}(\mathbb{Z})}^N$ equipped with a rig category $\mathcal{K}^N$ of rational functions. The projective limit of these spaces gives a space $\overline{\mathcal{M}(\mathbb{F}(\mathbb{Z}))}$ equipped with a rig category $\mathcal{K}$ of rational functions, and the natural morphism

$$\overline{\mathcal{M}(\mathbb{F}(\mathbb{Z}))} \to \overline{\mathrm{Spec}(\mathbb{F}(\mathbb{Z}))}$$

to Haran's compactification is a continuous morphism of spaces that respects the sheaves of rig categories.

3.9. Relation to Durov's spectra. We now explain the relation of seminorm spectra to Durov's spectra for generalized rings, defined in [Dur07]. This will be useful because Durov's approach allows the easy construction of higher dimensional Arakelov compactifications.

Recall that a generalized ring is an algebraic monad on sets that is commutative, meaning that its operations commute with each other. Such a monad

$$A : \textsc{Sets} \to \textsc{Sets}$$

is determined by its values on finite sets (algebraicity).

3.10. Definition. *Let A be a generalized ring. A* prime ideal *in A is a submodule $\mathfrak{p} \subset A(1)$ such that $A(1) - \mathfrak{p}$ is multiplicatively closed. The* prime spectrum *of A is the set* $\mathrm{Spec}(A)$ *of prime ideals equipped with the topology generated by subsets $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A), f \notin \mathfrak{p}\}$ for $f \in A(1)$.*

3.11. Proposition. *Let A be a generalized ring and $\mathbb{F}(A)$ be its rig category. There is a natural continuous morphism*

$$\mathcal{M}(\mathbb{F}(A)) \to \mathrm{Spec}(A).$$

Proof. Let $|.| : \mathbb{F}(A) \to \mathbb{R}_+$ be a multiplicative seminorm. Remark that one has $A(n)^m = \mathrm{Hom}_A(A(m), A(n))$. A submodule of A is a subset $\mathfrak{p}$ of $A(1)$ that is stable by the action of operations in $A(n)$ through the action map

$$A(n) \times A(1)^n \to A(1).$$

Remark that this map may be viewed as the composition map

$$\mathrm{Hom}_A(A(1), A(n)) \times \mathrm{Hom}_A(A(n), A(1)) \xrightarrow{\circ} \mathrm{Hom}_A(A(1), A(1)).$$

If $\mathfrak{p} \subset A(1)$ is the kernel of $|.|$, and $(p_i) \in \mathfrak{p}^n$, one may see $(p_i) \in \mathrm{Hom}_A(A(n), A(1))$ as the composition of $\oplus_i p_i : A(n) \to A(n)$ with $1^n : A(n) \to A(1)$. This implies that

$$|(p_i)| = |1^n \circ \oplus_i p_i| \leq |1^n|. \max(|p_i|) = 0,$$

so that for every $f \in A(n)$, one has

$$|(p_i) \circ f| \leq |(p_i)|.|f| = 0,$$

and $\mathfrak{p}$ is stable by the operation of elements in $A(n)$, so that it is a submodule. Since $|.|$ is multiplicative, $A(1) - \mathfrak{p}$ is also multiplicatively closed. ∎

Let $\mathbb{Z}_\infty$ be Durov's generalized archimedean valuation ring, given by

$$\mathbb{Z}_\infty(n) = \{\sum a_i\{i\}, \ \sum |a_i| \leq 1\},$$

and let $A_N := \mathbb{Z}_\infty \cap \mathbb{Z}[1/N]$. Then one may use the diagrams

$$\mathbb{F}(A_N) \to \mathbb{F}(\mathbb{Z}[1/N]) \leftarrow \mathbb{F}(\mathbb{Z})$$

for various N to define a compactification of $\mathcal{M}(\mathbb{F}(\mathbb{Z}))$ that projects onto Durov's compactification of $\mathrm{Spec}(\mathbb{Z})$. This construction also allows us to define analytic Arakelov compactifications of affine arithmetic varieties.

Let $f_i \in \mathbb{Z}[T_1, \ldots, T_n]$ be a family of polynomials with associated analytic variety $X = \mathcal{M}(\mathbb{F}(\mathbb{Z}[T_1, \ldots, T_n]/(f_i)))$. There exists $M \in \mathbb{Z}$ such that F_i/M has coefficients in A_N for some N. To the generalized ring $A_N[T_1^{(1)}, \ldots, T_n^{(1)}]/(F_i/M)$ is associated an analytic space X_N, that we may paste over $\mathcal{M}(\mathbb{F}(\mathbb{Z}[1/N]))$ with the zero space X of the given family of polynomials f_i over $\mathbb{Z}$. Taking the limit over all N' divisible by N, we get a space $\overline{X}$ together with a sheaf of rig categories $\mathcal{K}$ and an inclusion $i : X \to \overline{X}$, that will be called the analytic Arakelov compactification of X.

4. Rig categories of analytic functions

Let $\mathcal{A}$ be an additive rig category. Recall that if $A = \mathrm{End}(\mathbb{1})$, one may define, following Berkovich in [Ber90], a sheaf of analytic functions on $\mathcal{M}(A)$ by using local completions for the uniform norm of rational fractions without poles. We will now make the analogous construction for $\mathcal{M}(\mathcal{A})$.

4.1. Definition. *Let $|.|$ be a seminorm on an additive rig category $\mathcal{A}$. The* completion $\hat{\mathcal{A}}$ *of $\mathcal{A}$ with respect to $|.|$ is the rig category with the same class of objects and with morphisms given by the set of Cauchy sequences in* $\mathrm{Hom}(M, N)$ *quotiented by the sequences converging to zero.*

4.2. Proposition. *The above definition indeed gives a rig category.*

PROOF. If (f_p) and (g_p) are Cauchy sequences of morphisms, then $(f_p \circ g_p)$ is also a Cauchy sequence. Indeed, we have

$$|f_p \circ g_p - f_q \circ g_q| \leq |f_p \circ g_p - f_q \circ g_p + f_q \circ g_p - f_q \circ q_q| \leq |f_p - f_q|.|g_p| + |f_q|.|g_p - g_q|,$$

and since Cauchy sequence are bounded, we get the result. Similarly, $(f_p \otimes g_p)$ and $f_p \oplus g_p$ are also Cauchy sequences. Let (f_p) and (g_p) be Cauchy sequences and (z_p) and (w_p) be sequences converging to zero. Then

$$(f_p + z_p) \circ (g_p + w_p) = f_p \circ g_p + z_p \circ g_p + z_p \circ w_p + f_p \circ w_p$$

is equivalent to $f_p \circ g_p$ because the other terms are bounded in norm by a bounded sequence multiplied by a sequence converging to zero. This shows that the composition map is well defined on equivalence classes of Cauchy sequences. The same reasoning applies to the tensor product. ■

If $x \in \mathcal{M}(\mathcal{A})$ is a point, recall that we denoted $\mathcal{O}_{(x)} := \mathcal{A}[S_x^{-1}]$, where S_x is the multiplicative set of elements a in A such that $|a(x)| \neq 0$. We denote $\mathcal{O}_x$ the completion of $\mathcal{A}[S_x^{-1}]$ with respect to the multiplicative seminorm induced by x. If $U \subset \mathcal{M}(\mathcal{A})$ is an open subset, an analytic morphism $f \in \mathrm{Hom}_{\mathcal{O}(U)}(M, N)$ between two objects M and N of $\mathcal{A}$ on U is a map

$$f : U \to \coprod_{x \in U} \mathrm{Hom}_{\mathcal{O}_x}(M, N)$$

such that $f(x) \in \mathrm{Hom}_{\mathcal{O}_x}(M, N)$ and f is locally on U a uniform limit of elements of $\mathrm{Hom}_{\mathcal{K}}(M, N)$.

4.3. PROPOSITION. *The above defined correspondence $U \mapsto \mathcal{O}(U)$ defines a sheaf of additive rig categories on $\mathcal{M}(\mathcal{A})$.*

PROOF. This follows from the fact that $\mathcal{K}$ is a sheaf of rig categories and from the local definition of analytic morphisms. ■

4.4. EXAMPLES. Let $|.|_\infty : \mathbb{F}(\mathbb{Q}) \to \mathbb{R}_+$ be the archimedean operator norm. Then the completion of $\mathbb{F}(\mathbb{Q})$ for $|.|_\infty$ is simply given by $\mathbb{F}(\mathbb{R})$. Similarly, if $|.|_p : \mathbb{F}(\mathbb{Q}) \to \mathbb{R}$ is the non-archimedean p-adic operator norm, then the completion of $\mathbb{F}(\mathbb{Q})$ for $|.|_p$ is given by $\mathbb{F}(\mathbb{Q}_p)$. The completion of $\mathbb{F}(\mathbb{Z})$ for the p-adic residue norm $|.|_{p,0}$ gives $\mathbb{F}(\mathbb{F}_p)$.

There is a natural morphism

$$(\mathcal{M}(\mathcal{A}), \mathcal{O}) \to (\mathcal{M}(\mathcal{A}), \mathcal{K}).$$

It is not possible to define directly analytic functions on the spectrum of a non-additive rig category, because the completion process is only possible in the additive setting. However, one may define analytic functions on Arakelov compactifications.

4.5. Definition. *Let $i : X \to \overline{X}$ be the analytic Arakelov compactification of an affine arithmetic variety over $\mathbb{Z}$. Let $U \subset \bar{X}$ be an open subset. One defines $\mathcal{O}_{lb}(U)$ as the rig category given by those morphisms f in $\mathcal{O}(i^{-1}(U))$ such that for every point $x \in U$, there exists an open neighborhood $V(x)$ of x in $\bar{X}$ and a constant C such that $|f(y)| \leq C$ for all $y \in i^{-1}(V(x))$. Sections of $\mathcal{O}_{lb}$ are called* locally bounded *analytic functions on $\bar{X}$.*

The natural morphism of spaces

$$i : X \to \overline{X}$$

is compatible with the given sheaves of rig categories. Remark that we don't know that analytic spectra are locally compact, so that locally bounded analytic functions on X are, in principle, less general than analytic functions.

4.6. Example. If we work with $i : \mathcal{M}(\mathbb{F}(\mathbb{Z})) \to \overline{\mathcal{M}(\mathbb{F}(\mathbb{Z}))}$, a real number $r \in \mathcal{O}(|2| > 1)$ such that $|r|_\infty > 1$ is not locally bounded at infinity because $|r|_\infty^\infty = +\infty$ and there is no neighborhood of the residue archimedean norm such that r is bounded around it.

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