

# TRACED \*-AUTONOMOUS CATEGORIES ARE COMPACT CLOSED

TAMÁS HAJGATÓ AND MASAHITO HASEGAWA

ABSTRACT. We show that any traced \*-autonomous category is compact closed.

## 1. Introduction

Suppose that  $\mathbb{C} = (\mathbb{C}, I, \otimes, -\circ, \perp, Tr)$  is a traced \*-autonomous category; here we understand that a \*-autonomous category is a symmetric monoidal closed category  $(\mathbb{C}, I, \otimes, -\circ)$  (we write  $A -\circ B$  for the internal hom from  $A$  to  $B$ ) equipped with a dualizing object  $\perp$  [Barr, 1979], and that the trace

$$Tr_{A,B}^X : \mathbb{C}(A \otimes X, B \otimes X) \longrightarrow \mathbb{C}(A, B)$$

is given on the symmetric monoidal structure in the sense of Joyal, Street and Verity [Joyal *et al.*, 1996] (rather than the trace for linearly distributive categories with MIX by Blute, Cockett and Seely [Blute *et al.*, 2000]).

In  $\mathbb{C}$ , we have the trace of the evaluation map

$$\frac{(X -\circ (\perp \otimes X)) \otimes X \xrightarrow{ev} \perp \otimes X}{X -\circ (\perp \otimes X) \xrightarrow{Tr^X ev} \perp}$$

which gives rise to a morphism  $t_X : I \longrightarrow X \otimes (X -\circ I)$  via the isomorphism

$$(X -\circ (\perp \otimes X)) -\circ \perp \simeq X \otimes (X -\circ I).$$

It is then natural to ask if  $t_X$  satisfies the equations

$$X \xrightarrow{t_X \otimes X} X \otimes (X -\circ I) \otimes X \xrightarrow{X \otimes ev_{X,I}} X = id_X \tag{1}$$

and

$$X -\circ I \xrightarrow{(X -\circ I) \otimes t_X} (X -\circ I) \otimes X \otimes (X -\circ I) \xrightarrow{ev_{X,I} \otimes (X -\circ I)} X -\circ I = id_{X -\circ I} \tag{2}$$

The first author was partly supported by the European Union and the European Regional Development Fund under the grant agreement TÁMOP-4.2.2/B-10/1-2010-0012. The second author was partly supported by the Grant-in-Aid for Scientific Research (C) 23500016.

Received by the editors 2013-02-21 and, in revised form, 2013-03-18.

Transmitted by Ross Street. Published on 2013-04-10.

2010 Mathematics Subject Classification: 18D10, 18D15.

Key words and phrases: symmetric monoidal closed categories, traced monoidal categories, \*-autonomous categories, compact closed categories.

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which mean that  $X \multimap I$  is a (left) dual of  $X$ , hence  $\mathbb{C}$  is compact closed [Kelly and Laplaza, 1980]. Below we see that this is the case.

Before proceeding to the proof, let us explain how we came across this observation. In the sequel, we write  $\widehat{f} : A \rightarrow B \multimap C$  for the transpose of  $f : A \otimes B \rightarrow C$  in a symmetric monoidal closed category. In a traced symmetric monoidal closed category, we have a family of morphisms

$$\tau_B^X = \text{Tr}_{X \multimap (B \otimes X), B}^X(\text{ev}_{X, B \otimes X}) : X \multimap (B \otimes X) \rightarrow B.$$

It is easy to see that  $\tau$ 's are sufficient to determine trace of any  $f : A \otimes X \rightarrow B \otimes X$  as

$$\text{Tr}_{A, B}^X f = A \xrightarrow{\widehat{f}} X \multimap (B \otimes X) \xrightarrow{\tau_B^X} B.$$

In the case of traced \*-autonomous categories, we can further restrict our attention to  $\tau$ 's with  $B = \perp$ , from which  $\tau_B^X$  for any  $B$  is recovered as

$$\begin{aligned} X \multimap (B \otimes X) &\simeq X \multimap ((B \wp \perp) \otimes X) \\ &\xrightarrow{X \multimap \delta} X \multimap (B \wp (\perp \otimes X)) \\ &\simeq B \wp (X \multimap (\perp \otimes X)) \\ &\xrightarrow{B \wp \tau_{\perp}^X} B \wp \perp \\ &\simeq B \end{aligned}$$

where  $\delta$  denotes linear distributivity [Cockett and Seely, 1997].

Therefore, giving  $\tau_{\perp}^X : (X \multimap (\perp \otimes X)) \rightarrow \perp$  for each  $X$  is enough to determine a trace. With some efforts of spelling out how the trace can be recovered directly from  $\tau_{\perp}^X$ 's, we noticed that  $\tau_{\perp}^X$  actually determines the unit map  $t_X : I \rightarrow X \otimes (X \multimap I)$  of the duality between  $X$  and  $X \multimap I$ .

To make the proof short and readable, we use some basic results on (enriched) extraordinary natural transformations of Eilenberg and Kelly [Eilenberg and Kelly, 1966, Kelly, 1982], though it is also possible to derive the result by direct calculation from scratch.

## 2. A Characterization of Compact Closedness

There are many ways of characterizing compact closed categories as special symmetric monoidal closed categories [Day, 1977, Kelly and Laplaza, 1980]. In our development, the following characterization turns out to be useful:

**2.1. PROPOSITION.** *Suppose that  $\mathbb{C}$  is a symmetric monoidal closed category such that there is an extraordinary  $\mathbb{C}$ -natural transformation  $t_X : I \rightarrow X \otimes (X \multimap I)$  with  $t_I$  invertible. Then  $\mathbb{C}$  is compact closed.*

PROOF. Recall that, for a symmetric monoidal closed category  $\mathbb{V}$ ,  $\mathbb{V}$ -categories  $\mathbb{A}$ ,  $\mathbb{B}$ , a  $\mathbb{V}$ -functor  $F : \mathbb{A}^{\text{op}} \otimes \mathbb{A} \rightarrow \mathbb{B}$  and an object  $B$  of  $\mathbb{B}$ , a family of morphisms  $\alpha_X : B \rightarrow F(X, X)$  is said to be an extraordinary  $\mathbb{V}$ -natural transformation [Kelly, 1982] when the following diagram commutes for all  $X, X'$ .

$$\begin{array}{ccc}
 & \mathbb{A}(X, X') & \\
 F(X, -) \swarrow & & \searrow F(-, X') \\
 \mathbb{B}(F(X, X), F(X, X')) & & \mathbb{B}(F(X', X'), F(X, X')) \\
 \mathbb{B}(\alpha_X, id) \searrow & & \swarrow \mathbb{B}(\alpha_{X'}, id) \\
 & \mathbb{B}(B, F(X, X')) &
 \end{array}$$

In the proposition, the assumption that  $t_X : I \rightarrow X \otimes (X \multimap I)$  is extraordinarily  $\mathbb{C}$ -natural means that the following diagram commutes for all  $X$  and  $X'$ :

$$\begin{array}{ccc}
 & X \multimap X' & \\
 (X \multimap X') \otimes t_X \swarrow & & \searrow t_{X'} \otimes (X \multimap X') \\
 (X \multimap X') \otimes X \otimes (X \multimap I) & & X' \otimes (X' \multimap I) \otimes (X \multimap X') \\
 ev \otimes (X \multimap I) \searrow & & \swarrow X' \otimes comp \\
 & X' \otimes (X \multimap I) &
 \end{array}$$

By letting  $X$  be  $I$  in the diagram above, we see that (modulo some obvious simplifications)

$$X' \xrightarrow{t_{X'} \otimes X'} X' \otimes (X' \multimap I) \otimes X' \xrightarrow{X' \otimes ev} X'$$

agrees with

$$X' \xrightarrow{X' \otimes t_I} X' \otimes I \otimes (I \multimap I) \xrightarrow{\simeq} X'.$$

Similarly, by letting  $X'$  be  $I$  in the diagram, we have that

$$X \multimap I \xrightarrow{(X \multimap I) \otimes t_X} (X \multimap I) \otimes X \otimes (X \multimap I) \xrightarrow{ev \otimes (X \multimap I)} X \multimap I$$

agrees with

$$X \multimap I \xrightarrow{t_I \otimes (X \multimap I)} I \otimes (I \multimap I) \otimes (X \multimap I) \xrightarrow{\simeq} X \multimap I.$$

Hence

$$t'_X = I \xrightarrow{\simeq} I \otimes (I \multimap I) \xrightarrow{t_I^{-1}} I \xrightarrow{t_X} X \otimes (X \multimap I)$$

satisfies the equations (1) and (2) for making  $X \multimap I$  a left dual of  $X$ . ■

Note that, in the proof above,  $t'_X$  agrees with  $t_X$  if  $t_I : I \rightarrow I \otimes (I \multimap I)$  itself is the canonical isomorphism from  $I$  to  $I \otimes (I \multimap I)$ .

2.2. **REMARK.** In Proposition 2.1, the assumption that  $t_I$  is invertible cannot be dropped. For instance, consider the category  $\omega\mathbf{Cppo}_\perp$  of pointed  $\omega$ -complete partial orders and strict  $\omega$ -continuous functions.  $\omega\mathbf{Cppo}_\perp$  is symmetric monoidal closed, with Sierpinski space as the unit object, smash products as tensor and strict function spaces as internal hom. In  $\omega\mathbf{Cppo}_\perp$ , there is an extraordinary  $\omega\mathbf{Cppo}_\perp$ -natural transformation  $t_X : I \rightarrow X \otimes (X \multimap I)$  given by the constant functions returning the least element. However,  $t_I$  is not invertible, and  $\omega\mathbf{Cppo}_\perp$  is not compact closed.

### 3. Proof of the Main Result

As in the introduction, let us define

$$\tau_B^X = \text{Tr}_{X \multimap (B \otimes X), B}^X(\text{ev}_{X, B \otimes X}) : X \multimap (B \otimes X) \rightarrow B$$

in traced symmetric monoidal closed categories.

3.1. **LEMMA.** *In a traced symmetric monoidal closed category  $\mathbb{C}$  with an object  $B$ ,*

$$\widehat{\tau}_B^X : I \rightarrow (X \multimap (B \otimes X)) \multimap B$$

*is extraordinary  $\mathbb{C}$ -natural in  $X$ .*

**PROOF.** The extranaturality amounts to the commutativity of

$$\begin{array}{ccc}
 & (X \multimap X') \otimes (X' \multimap (B \otimes X)) & \\
 \text{id} \otimes \widehat{\tau}_B^X \swarrow & & \searrow \text{id} \otimes \widehat{\tau}_B^{X'} \\
 (X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) & & (X \multimap X') \otimes (X' \multimap (B \otimes X)) \otimes ((X' \multimap (B \otimes X')) \multimap B) \\
 \text{comp} \otimes \text{id} \downarrow & & \downarrow \text{comp} \otimes \text{id} \\
 (X \multimap (B \otimes X)) \otimes ((X \multimap (B \otimes X)) \multimap B) & & (X' \multimap (B \otimes X')) \otimes ((X' \multimap (B \otimes X')) \multimap B) \\
 \text{ev} \searrow & & \swarrow \text{ev} \\
 & B &
 \end{array}$$

which is a consequence of the sliding property (dinaturality) of trace. ■

3.2. LEMMA. *In a  $*$ -autonomous category  $\mathbb{C}$ , the isomorphism*

$$\varphi_{X,Y} : (X \multimap (\perp \otimes Y)) \multimap \perp \xrightarrow{\cong} X \otimes (Y \multimap I)$$

given by

$$\begin{aligned} (X \multimap (\perp \otimes Y)) \multimap \perp &\simeq (X \multimap (((\perp \otimes Y) \multimap \perp) \multimap \perp)) \multimap \perp \\ &\simeq (X \multimap ((Y \multimap (\perp \multimap \perp)) \multimap \perp)) \multimap \perp \\ &\simeq (X \multimap ((Y \multimap I) \multimap \perp)) \multimap \perp \\ &\simeq ((X \otimes (Y \multimap I)) \multimap \perp) \multimap \perp \\ &\simeq X \otimes (Y \multimap I) \end{aligned}$$

is  $\mathbb{C}$ -natural in  $X$  and  $Y$ .

PROOF. Each isomorphism involved in  $\varphi$  is  $\mathbb{C}$ -natural. ■

3.3. LEMMA. [Eilenberg and Kelly, 1966, Kelly, 1982] *Assume that  $\mathbb{V}$  is a symmetric monoidal closed category. Let  $\mathbb{A}, \mathbb{B}$  be  $\mathbb{V}$ -categories and  $G, H$  be  $\mathbb{V}$ -functors of the form  $\mathbb{A}^{\text{op}} \otimes \mathbb{A} \rightarrow \mathbb{B}$  and suppose that  $B$  is an object of  $\mathbb{B}$ . If  $\alpha_X : B \rightarrow G(X, X)$  is extraordinary  $\mathbb{V}$ -natural in  $X$  and  $\beta_{X,Y} : G(X, Y) \rightarrow H(X, Y)$  is  $\mathbb{V}$ -natural in  $X$  and  $Y$ , then*

$$\beta_{X,X} \circ \alpha_X : B \rightarrow H(X, X)$$

is extraordinary  $\mathbb{V}$ -natural in  $X$ .

3.4. COROLLARY. *In a traced  $*$ -autonomous category  $\mathbb{C}$ ,  $t_X = \varphi_{X,X} \circ \widehat{\tau}_\perp^X : I \rightarrow X \otimes (X \multimap I)$  is extraordinary  $\mathbb{C}$ -natural in  $X$ .  $\square$*

3.5. LEMMA.  $t_I : I \rightarrow I \otimes (I \multimap I)$  agrees with the canonical isomorphism from  $I$  to  $I \otimes (I \multimap I)$ .

PROOF. A consequence of the vanishing property of trace. ■

Putting Proposition 2.1, Corollary 3.4 and Lemma 3.5 together, we obtain our main result.

3.6. THEOREM. *Any traced  $*$ -autonomous category is compact closed.*

It is possible that a compact closed category is equipped with a dualizing object which is not isomorphic to the unit object (and par not isomorphic to tensor). For instance, the linearly ordered set of integers  $\mathbb{Z}$  is compact closed with unit  $I = 0$  and tensor  $X \otimes Y = X + Y$  and duality  $X^* = -X$ , while any element of  $\mathbb{Z}$  serves as a dualizing object. (The same can be done for any partially ordered Abelian group regarded as a compact closed poset.)

Since a compact closed category has a unique trace (cf. [Hasegawa, 2009]), we have:

3.7. THEOREM. *To give a traced  $*$ -autonomous category is to give a compact closed category with a dualizing object.*

Note that a dualizing object in a compact closed category is just an object  $\perp$  such that the unit morphism  $I \rightarrow \perp \otimes \perp^*$  is invertible, cf. the Abelian group example above.

#### 4. On Linear Distributivity

In a compact closed category with a dualizing object  $\perp$ , linear distributivity [Cockett and Seely, 1997] on the  $*$ -autonomous structure is invertible. To see this, recall that the linear distributivity  $\delta : (A \wp B) \otimes C \longrightarrow A \wp (B \otimes C)$  in a  $*$ -autonomous category (regarded as a symmetric linearly distributive category with negation) amounts to the canonical morphism  $(A^\perp \multimap B) \otimes C \longrightarrow A^\perp \multimap (B \otimes C)$  which is just the associativity isomorphism  $((A^\perp)^* \otimes B) \otimes C \simeq (A^\perp)^* \otimes (B \otimes C)$  in a compact closed category.

Conversely, a  $*$ -autonomous category with invertible linear distributivity is compact closed. We have

$$A \multimap B \simeq A^\perp \wp B \simeq A^\perp \wp (I \otimes B) \stackrel{\delta^{-1}}{\simeq} (A^\perp \wp I) \otimes B \simeq (A \multimap I) \otimes B$$

In particular, the canonical map  $(A \multimap I) \otimes A \longrightarrow A \multimap A$  is invertible, and it follows that the category is compact closed (cf. [Day, 1977]).

Together with Theorem 3.7, we have that the following three structures are essentially the same:

- a traced  $*$ -autonomous category,
- a compact closed category equipped with a dualizing object, and
- a  $*$ -autonomous category with invertible linear distributivity.

4.1. REMARK. As noted in [Cockett and Seely, 1997], in a symmetric linearly distributive category with invertible linear distributivity and also equipped with a tensor-inverse of  $\perp$  (an object  $\perp^*$  such that there is an isomorphism  $I \simeq \perp \otimes \perp^*$  subject to a coherence axiom), the par  $A \wp B$  is isomorphic to the “ $\perp$ -shifted tensor”  $A \otimes \perp^* \otimes B$ . This is the case for  $*$ -autonomous categories with invertible linear distributivity (equivalently: traced  $*$ -autonomous categories, or compact closed categories with a dualizing object), in which  $\perp^* = \perp \multimap I$  serves as a tensor-inverse of  $\perp$  and we have  $A \wp B \simeq A \otimes (\perp \multimap I) \otimes B$ .

ACKNOWLEDGEMENTS. The second author is grateful to Naohiko Hoshino for helpful discussions on  $*$ -autonomous categories and traced monoidal categories.

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*Department of Computer Science  
University of Szeged  
Szeged, Hungary*

*Research Institute for Mathematical Sciences  
Kyoto University  
Kyoto, Japan*

Email: hajgato@inf.u-szeged.hu  
hassei@kurims.kyoto-u.ac.jp

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Tom Leinster, University of Edinburgh, [Tom.Leinster@ed.ac.uk](mailto:Tom.Leinster@ed.ac.uk)

Ieke Moerdijk, University of Utrecht: [moerdijk@math.uu.nl](mailto:moerdijk@math.uu.nl)

Susan Niefield, Union College: [niefiels@union.edu](mailto:niefiels@union.edu)

Robert Paré, Dalhousie University: [pare@mathstat.dal.ca](mailto:pare@mathstat.dal.ca)

Jiri Rosicky, Masaryk University: [rosicky@math.muni.cz](mailto:rosicky@math.muni.cz)

Giuseppe Rosolini, Università di Genova: [rosolini@disi.unige.it](mailto:rosolini@disi.unige.it)

Alex Simpson, University of Edinburgh: [Alex.Simpson@ed.ac.uk](mailto:Alex.Simpson@ed.ac.uk)

James Stasheff, University of North Carolina: [jds@math.upenn.edu](mailto:jds@math.upenn.edu)

Ross Street, Macquarie University: [street@math.mq.edu.au](mailto:street@math.mq.edu.au)

Walter Tholen, York University: [tholen@mathstat.yorku.ca](mailto:tholen@mathstat.yorku.ca)

Myles Tierney, Rutgers University: [tierney@math.rutgers.edu](mailto:tierney@math.rutgers.edu)

Robert F. C. Walters, University of Insubria: [robert.walters@uninsubria.it](mailto:robert.walters@uninsubria.it)

R. J. Wood, Dalhousie University: [rjwood@mathstat.dal.ca](mailto:rjwood@mathstat.dal.ca)