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ON THE MONAD OF INTERNAL GROUPOIDS

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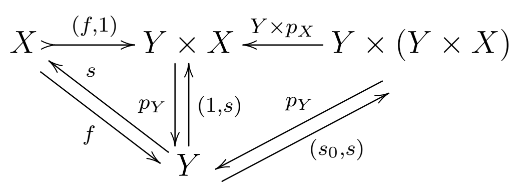
Abstract. We deeply analyse the structural organisation of the fibration of points and of the monad of internal groupoids. From that we derive: 1) a new characterization of internal groupoids among reflexive graphs in the Mal'cev context; 2) a setting in which a Mal'cev category is necessarily a protomodular category.

Introduction

This article is the result of the effort to understand the universal property of the split epimorphism $(p_0^Y, s_0^Y) : Y \times Y \leftrightarrows Y$ among all the split epimorphisms, a question that arises from many aspects of Mal'cev and protomodular categories, where it plays a pivotal role. For that, we had to analyse the remarkable structural organisation of the fibration of points $\P_{\mathbb{E}} : Pt\mathbb{E} \to \mathbb{E}$ (the codomain functor). It was already known that there is on $Pt\mathbb{E}$ a monad (T, λ, μ) whose category of algebras Alg^T is nothing but the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$, see [4]. It appears, here, that the functor $J : \mathbb{E} \to Pt\mathbb{E}$ defined by $J(Y) = (p_0^Y, s_0^Y)$ is precisely the right adjoint of the functor $\P_{\mathbb{E}}.T : Pt\mathbb{E} \to \mathbb{E}$ which is nothing but the *domain* functor. What is unexpected is that this functor J is monadic (where is originated the conceptual reason why the functor $(\ )_0 = \P_{\mathbb{E}}.U^T : Alg^T = Grd\mathbb{E} \to \mathbb{E}$ is still a fibration) and that it allows a *localization* of the monad (T, λ, μ) into a monad (T_Y, λ_Y, μ_Y) on the fibre $Pt_Y\mathbb{E}$ which is described by the following diagram:



Of this last monad we could get out three considerations of the highest interest in the context of a Mal'cev category $\mathbb{C}$:
1) any reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ in $\mathbb{C}$ is a groupoid if and only if the following

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two subobjects commute in the unital fibre $Pt_{X_0}\mathbb{C}$:

$$\begin{array}{ccccc} X_1 & \xrightarrow{(d_0,1)} & X_0 \times X_1 & \xleftarrow{(d_1,1)} & X_1 \\ & {\scriptstyle d_0}\searrow \nwarrow{\scriptstyle s_0} & {\scriptstyle p_{X_0}}\downarrow \uparrow{\scriptstyle (1,s_0)} & {\scriptstyle s_0}\nearrow \swarrow{\scriptstyle d_1} & \\ & & X_0 & & \end{array}$$

which is a localization of the classical characterization $[R[d_0], R[d_1]] = 0$,
2) when a pointed Mal'cev category $\mathbb{C}$ satisfies the property of *algebraic exponentiation* (see [13] and [8]) for any map, it is necessarily protomodular,
3) when the Mal'cev category $\mathbb{C}$ is not pointed and regular, if it satisfies the property of *algebraic exponentiation* for any map, the full subcategory $\mathbb{C}_\sharp$ whose objects have global support is protomodular.
Section 1 is devoted to the fibration of points and to the monad of internal groupoids, Section 2 to the "localization" (T_Y, λ_Y, μ_Y) and to the T_Y-algebras, and Section 3 to the applications in the Mal'cev context.
We should like to thank S. Lack for a very useful bibliographical suggestion.

1. The monad of internal groupoids

1.1. The first observation. Let (T, λ, μ) be any monad on a category $\mathbb{E}$ and let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{T} & \mathbb{E} \\ & {\scriptstyle W}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathbb{F} & \end{array}$$

1.2. Proposition. *Suppose that the functor $W = V.T$ has a right adjoint $\tilde{W}$, then the functor $\overline{W}$ in the following commutative diagram:*

$$\begin{array}{ccc} Alg^T & \xrightarrow{U^T} & \mathbb{E} \\ & {\scriptstyle \overline{W}}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathbb{F} & \end{array}$$

has a right adjoint $\hat{W}$, and it is such that: $U^T.\hat{W} \simeq \tilde{W}$.

Proof. This is a mere corollary of Theorem 3.7.2 in [2]. Consider the following commutative diagram:

$$\begin{array}{ccc} Alg^T & \xrightarrow{\overline{W}=V.U^T} & \mathbb{F} \\ & {\scriptstyle F^T}\nwarrow \quad \nearrow{\scriptstyle V.T} & \\ & \mathbb{E} & \end{array}$$

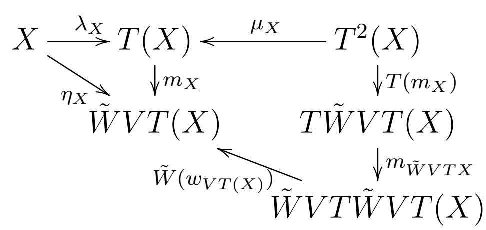
It satisfies the conditions of Theorem 3.7.2 in [2]: since F^T has a right adjoint U^T which is monadic and $V.T$ has a right adjoint $\tilde{W}$, it remains to show that $\overline{W} = V.U^T$ preserves coequalizers of U^T-contractible pairs; it is the case for U^T, and such a coequalizer becomes a contractible coequalizer in $\mathbb{E}$, and thus it is preserved by any functor.
The right adjoint functor $\hat{W}$ can be described in the following way; let us denote by $w : W\tilde{W} \Rightarrow 1_{\mathbb{F}}$ the natural transformation associated with the right adjoint $\tilde{W}$, then the natural transformation $w.V\mu : VT^2\tilde{W} \Rightarrow 1_{\mathbb{F}}$ induces a unique natural transformation: $\beta : T\tilde{W} \Rightarrow \tilde{W}$ such that: $w.VT\beta = w.V\mu\tilde{W}$. It is easy to check that β satisfies the axioms of T-algebras. We have $\hat{W}(Y) = (\tilde{W}(Y), \beta_Y)$, the natural map $\hat{\epsilon} : V.U^T.\hat{W} \Rightarrow 1_{\mathbb{F}}$ being given by $w_Y.V(\lambda_{\tilde{W}(Y)}) : V.\tilde{W}(Y) \to Y$. ∎

When the functor V is the identity on $\mathbb{E}$, then we get the well known result of Eilenberg-Moore [11] according to which *when the endofunctor of a monad* (T, λ, μ) *admits a right adjoint* G*, this functor* G *is underlying a comonad* (G, ϵ, δ) *such that the categories* Alg^T *and* $Coal^C$ *coincide.* Actually we shall need the following precisions:

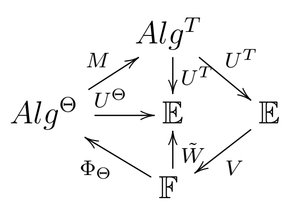
1.3. PROPOSITION. *Assume the assumptions of the previous proposition, and denote by* $(\Theta, \eta, \tilde{W}(w_{VT}))$ *the monad on* $\mathbb{E}$ *induced by the right adjoint* $\tilde{W}$*. There is a unique natural transformation* $m : T \Rightarrow \Theta = \tilde{W}.V.T$ *such that* $w_{VT}.VT(m) = V(\mu)$ *and which is actually a morphism of monads:*

$$(T, \lambda, \mu) \Rightarrow (\Theta, \eta, \tilde{W}(w_{VT}))$$

namely such that the following diagram commutes:



If we denote by $M : Alg^\Theta \to Alg^T$ *the induced comparison functor, the right adjoint* $\hat{W}$ *of* $V.U^T$ *is nothing but the functor* $M.\Phi_\Theta$ *in the following diagram:*



where Φ_Θ *is the natural comparison functor to the category of* Θ*-algebras and any triangle on the left hand side commutes.*

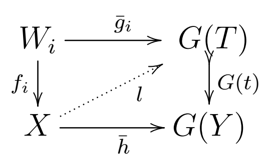
PROOF. By adjunction, the map $V(\mu_X) : V.T^2(X) \to V.T(X)$ determines a unique map $m_X : T(X) \to \tilde{W}.V.T(X)$ such that $w_{VT(X)}.VT(m_X) = V(\mu_X)$. Let us show it is underlying a morphism of monads. First we have $m_X.\lambda_X = \eta_X$ just checking:

$w_{VT(X)}.VT(m_X).VT(\lambda_X) = V(\mu_X.VT(\lambda_X)) = 1_{VT(X)} = w_{VT(X)}.VT(\eta_X)$. Then we have $m_X.\mu_X = \tilde{W}(w_{VT(X)}).m_{\tilde{W}VT(X)}.T(m_X)$ just checking:
$w_{VT(X)}.VT\tilde{W}(w_{VT(X)}).VT(m_{\tilde{W}VT(X)}).VT^2(m_X)$
$= w_{VT(X)}.w_{VT\tilde{W}VT(X)}.VT(m_{\tilde{W}VT(X)}).VT^2(m_X)$
$= w_{VT(X)}.V(\mu_{\tilde{W}VT(X)}).VT^2(m_X) = w_{VT(X)}.VT(m_X).V(\mu_{T(X)})$
$= V(\mu_X).V(\mu_{T(X)}) = V(\mu_X).VT(\mu_X) = w_{VT(X)}.VT(m_X).VT(\mu_X)$
Now we have $M.\Phi_\Theta(Y) = (\tilde{W}(Y), \tilde{W}(w_Y).m_{\tilde{W}(Y)})$, and it is easy to check that the map $\tilde{W}(w_Y).m_{\tilde{W}(Y)}$ is the map β_Y of the end of the proof of the previous proposition. The natural transformation $\hat{\eta} : 1_{Alg^T} \Rightarrow M.\Phi_\Theta.V.U^T$ is given, for any T-algebra (X, ξ) by the unique map $\hat{\eta}(X, \xi) : X \to \tilde{W}V(X)$ satisfying: $w_{V(X)}.VT(\hat{\eta}(X, \xi)) = V(\xi)$. ∎

Let us end this very general section by a relatively straightforward result we shall need later on:

1.4. Proposition. *Let $U : \mathbb{E} \to \mathbb{F}$ be a functor having a right adjoint G. Then U preserves the jointly strongly epic families.*

Proof. Let $W_i \stackrel{f_i}{\to} X;\ i \in I$ be a jointly strongly epic family in $\mathbb{E}$; now consider a map $h : U(X) \to Y$ and a monomorphism $t : T \rightarrowtail Y$ through which any map $h.U(f_i)$ factorizes by a map $g_i : U(W_i) \to T$. The image $G(t)$ by the right adjoint G is a monomorphism $G(T) \rightarrowtail G(Y)$ in $\mathbb{E}$. On the other hand, the map h produces a map $\bar{h} : X \to G(Y)$ in the same way as the maps g_i produce maps $\bar{g}_i : W_i \to G(T)$ such that $G(t).\bar{g}_i = \bar{h}.f_i$. Since the family $(f_i)_{i \in I}$ is jointly strongly epic, then necessarily, we get a factorization l:



The adjunction gives rise to a map $\tilde{l} : U(X) \to T$ such that $t.\tilde{l} = h$. ∎

1.5. The fibration of points. We shall study here a specific example of the situation introduced above. Recall that, $\mathbb{E}$ being any category, we denote by $Pt\mathbb{E}$ the category whose objects are the split epimorphisms in $\mathbb{E}$ with a given splitting and morphisms the commutative squares between these data, and by $¶_{\mathbb{E}} : Pt\mathbb{E} \to \mathbb{E}$ the functor associating its codomain with any split epimorphism. As soon as the category $\mathbb{E}$ has pullbacks, the left exact functor $¶_{\mathbb{E}}$ is a fibration whose cartesian maps are the pullbacks between split epimorphisms. More precisely it is a *fibered reflection* [4], in the sense that it admits a fully faithful right adjoint I defined by $I(Y) = (1_Y, 1_Y)$. The fibre above Y will be denoted $Pt_Y\mathbb{E}$. From now on we shall suppose that any category has finite limits. There is a left exact monad (T, λ, μ) on $Pt\mathbb{E}$ defined by the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{s_1} & R[f] & \xleftarrow{p_2} & R^2[f] \\ f \downarrow\uparrow s & & p_0 \downarrow\uparrow s_0 & & p_0 \downarrow\uparrow s_0 \\ Y & \xrightarrow[s]{} & X & \xleftarrow[p_1]{} & R[f] \end{array}$$

where $T(f, s) = (p_0, s_0)$, and the other arrows in the diagram above are given by the iterated kernel equivalence relations:

$$R^2[d_0] \underset{d_0}{\overset{d_2}{\underset{d_1}{\rightrightarrows}}} \!\!\rightarrow R[f] \underset{d_0}{\overset{d_1}{\rightleftarrows}}\!\!\rightarrow X \underset{f}{\overset{s}{\leftrightarrows}} Y$$

Now consider the diagram:

$$\begin{array}{ccc} Pt\mathbb{E} & \xrightarrow{T} & Pt\mathbb{E} \\ & {\scriptstyle ¶_{\mathbb{E}}.T} \searrow \quad \swarrow {\scriptstyle ¶_{\mathbb{E}}} & \\ & \mathbb{E} & \end{array}$$

Actually, the functor $¶_{\mathbb{E}}.T : Pt\mathbb{E} \to E$ is nothing but the *domain* functor. Finally let us recall that the category Alg^T is nothing but the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$, see [4].

1.6. PROPOSITION. *The functor $¶_{\mathbb{E}}.T : Pt\mathbb{E} \to E$ has a right adjoint J defined by $J(Y) = (p_0, s_0) : Y \times Y \leftrightarrows Y$. In other words the split epimorphism $J(Y)$ is the universal split epimorphism with respect to the domain functor. Moreover this functor J is monadic and consequently conservative.*

PROOF. Let us denote by $w : ¶_{\mathbb{E}}.T.J \Rightarrow 1_{\mathbb{E}}$ the natural transformation defined by $w_Y = p_1^Y : Y \times Y \to Y$. Given any split epimorphism (f', s') and any map $g : ¶_{\mathbb{E}}.T(f', s') = X' \to Y$, the unique factorization $(m, n) : (f', s') \to J(Y)$ in $Pt\mathbb{E}$ such that we get $w_Y.¶_{\mathbb{E}}.T(m, n) = g$ is given by the following diagram:

$$\begin{array}{ccc} X' & \xrightarrow{(g.s'.f', g)} & Y \times Y \\ f' \downarrow\uparrow s' & & p_0 \downarrow\uparrow s_0 \\ Y' & \xrightarrow[g.s']{} & Y \end{array}$$

Finally, it is straightforward that the functor J creates coequalizers of J-contractible pairs. ∎

Accordingly, we are here in the situation of our first observation where moreover any functor is left exact, the endofunctor T as well, and with two very special features. We already noticed the first one: the functor $\tilde{W} = J$ is monadic. On the other hand, given any groupoid $\underline{X}_1$, the natural map $\hat{\eta}_{\underline{X}_1}$ (see Proposition 1.3) is the following one:

$$\begin{array}{ccc} X_1 & \xrightarrow{(d_0,d_1)} & X_0 \times X_0 \\ {\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} & & {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle p_1} \\ X_0 & = & X_0 \end{array}$$

Accordingly, and this is the second special feature, the map $\hat{\eta}_{\hat{W}(Y)}$ is an isomorphism:

$$\begin{array}{ccc} Y \times Y & \xrightarrow{(p_0,p_1)} & Y \times Y \\ {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle p_1} & & {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle p_1} \\ Y & = & Y \end{array}$$

It is the conceptual reason why *the forgetful left exact functor* $(\ )_0 : Alg^T = Grd\mathbb{E} \to \mathbb{E}$ *is a fibered reflection*, namely is such that the natural transformation $\hat{\epsilon}$ (see the end of the proof of Proposition 1.2) is an isomorphism: indeed, since the map $\hat{\eta}_{\hat{W}(Y)}$ is an isomorphism, it is the case for any $\hat{W}(\hat{\epsilon}_X)$; but, since the functor $\tilde{W} = U^T.\hat{W}$ is monadic, the functor $\hat{W}$ is conservative, and consequently $\hat{\epsilon}_X$ is an isomorphism. Rather unexpectedly, the Proposition 1.2 seems to show that the fibered reflection aspect of $(\ )_0$ is independent from the existence of the right adjoint I of $\P_{\mathbb{E}}$, namely from the fact that $\P_{\mathbb{E}}$ is itself a fibered reflection.

1.7. Strongly split epimorphisms. We shall need also the following, when the category $\mathbb{E}$ is pointed. Any split epimorphism (f, s) determines now a split sequence:

$$Kerf \rightarrowtail^{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

1.8. Definition. *We shall say that the split epimorphism (resp. the split sequence) is a strongly split epimorphism (resp. a strongly split exact sequence), when the pair* (k, s) *is jointly strongly epic.*

This terminology is justified by the following:

1.9. Proposition. *Let* $\mathbb{E}$ *be a pointed category with finite limits. Any strongly split epimorphism* (f, s) *is a normal epimorphism. In other words, any split sequence associated with a strongly split epimorphism is a strongly split exact sequence.*

Proof. Consider the associated split sequence:

$$Kerf \rightarrowtail^{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

We have to show that f is the cokernel of k. Let $g : X \to T$ be a map such that: $g.k = 0$. The only possible factorization $Y \to T$ is $g.s$. So we need to show that $g.s.f = g$. For that let us introduce the equalizer i of the pair $(g, g.s.f)$. It is clear that both k and s factor through i. Consequently i is an isomorphism, and the two maps in question are equal. ■

This result was noticed independently in [15]. According to this proposition, a pointed category $\mathbb{C}$ with finite limits is *protomodular* [3] if and only if any split epimorphism is a strongly split epimorphism.

2. The induced monad on the fibres $Pt_Y\mathbb{E}$

The monad (T, λ, μ) on $Pt\mathbb{E}$ has other very strong properties: the endofunctor T preserves the $¶_{\mathbb{E}}$-cartesian maps, while the natural transformations λ and μ are themselves $¶_{\mathbb{E}}$-cartesian, see [4]. Finally the natural transformation $m : T \Rightarrow \tilde{W}VT$ defined in Proposition 1.3 is given, for any split epimorphism $(f, s) : X \leftrightarrows Y$, by the following map, and consequently, lies inside the fibre $Pt_X\mathbb{E}$:

$$\begin{array}{ccc} R[f] & \xrightarrow{(d_0,d_1)} & X \times X \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle p_0}\downarrow\uparrow{\scriptstyle s_0} \\ X & = & X \end{array}$$

In other words, we have a third special feature, namely: $V(m_X) = 1_{VT(X)}$.

2.1. The monad (T_Y, λ_Y, μ_Y). With all this information, we can derive a monad (T_Y, λ_Y, μ_Y) on the fibre $Pt_Y\mathbb{E}$ from the monad (T, λ, μ) on $Pt\mathbb{E}$ by means of the following fibered construction, where, first, the map $\bar{\lambda}_X$ is the $¶_{\mathbb{E}}$-cartesian map above $V(\lambda_X)$, and $\lambda_Y X$ is the factorization inside the fibre $Pt_Y\mathbb{E}$ induced by m_X; this makes the upper square a pullback:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_X} & T(X) \\ {\scriptstyle \lambda_Y X}\downarrow & & \downarrow{\scriptstyle m_X} \\ T_Y(X) & \xrightarrow{\bar{\lambda}_X} & \tilde{W}VT(X) \\ {\scriptstyle \mu_Y X}\uparrow & & \uparrow{\scriptstyle \tilde{W}(n_X)} \\ T_Y^2(X) & \xrightarrow[\bar{\lambda}_{T_Y X}]{} & \tilde{W}VT(T_Y(X)) \end{array}$$

On the other hand, by adjunction, the map $\bar{\lambda}_X : T_Y(X) \to \tilde{W}VT(X)$ in $Pt\mathbb{E}$ corresponds to a map $n_X : VT(T_Y(X)) \to VT(X)$ in $\mathbb{E}$ such that $n_X.V(\lambda_{T_Y(X)}) = V(\lambda_X)$; indeed we get first: $n_X.VT(\lambda_Y X)) = w_{VT(X)}.VT(\bar{\lambda}_X).VT(\lambda_Y X) = w_{VT(X)}.VT(m_X).VT(\lambda_X) = V(\mu_X).VT(\lambda_X) = 1_{VT(X)}$. Now, from $V(\lambda_{T_Y(X)}) = VT(\lambda_Y X).V(\lambda_X)$, we get:
$n_X.V(\lambda_{T_Y(X)}) = n_X.VT(\lambda_Y X).V(\lambda_X) = V(\lambda_X)$. The map $\mu_Y X$ is the factorization of $\tilde{W}(n_X).\bar{\lambda}_{T_Y X}$ through the $¶_{\mathbb{E}}$-cartesian map $\bar{\lambda}_X$. In spite of appearances, the map μ_X is

involved in the process, since it is involved in the definition of the map m_X. The monad (T_Y, λ_Y, μ_Y) is now precisely described by the following diagram in $\mathbb{E}$:

$$\begin{array}{ccccc} X & \overset{(f,1)}{\rightarrowtail} & Y \times X & \overset{Y \times p_X}{\longleftarrow} & Y \times (Y \times X) \\ & {\scriptstyle f}\searrow \nwarrow{\scriptstyle s} & {\scriptstyle p_Y}\downarrow\uparrow{\scriptstyle (1,s)} & \swarrow{\scriptstyle p_Y} \nearrow{\scriptstyle (s_0,s)} & \\ & & Y & & \end{array}$$

2.2. The T_Y-algebras. We shall determine now the nature of the T_Y-algebra structures on a split epimorphism $(f, s) : X \leftrightarrows Y$ and show how, in some specific context, they are controlling the decompositions of the form $X \simeq Y \times T$ (see also [7]). Actually the monad (T_Y, λ_Y, μ_Y) is generated by an adjunction: let us denote by $Y\backslash\mathbb{E}$ the usual coslice category and by $\Upsilon_Y : Y\backslash\mathbb{E} \to Pt_Y\mathbb{E}$ the functor defined, for any map $t : Y \to T$ by $\Upsilon_Y(t) = (p_Y, (1, t)) : Y \times T \leftrightarrows Y$.

2.3. Proposition. *The functor Υ_Y has a left adjoint $\Psi_Y : Pt_Y\mathbb{E} \to Y\backslash\mathbb{E}$ defined by $\Psi_Y(f, s) = s$. It is left exact and produces the monad (T_Y, λ_Y, μ_Y).*

Proof. Straightforward checking. ■

So, there is a canonical left exact comparison functor Γ_Y:

$$\begin{array}{ccc} Y\backslash\mathbb{E} & \xrightarrow{\Gamma_Y} & Alg^{T_Y} \\ & {\scriptstyle \Upsilon_Y}\searrow \quad \swarrow{\scriptstyle U^{T_Y}} & \\ & Pt_Y\mathbb{E} & \end{array}$$

The upper level of the canonical sequence yielded by the adjunction in $Y\backslash\mathbb{E}$:

$$(\Psi_Y.\Upsilon_Y)^2(t) \rightrightarrows \Psi_Y.\Upsilon_Y(t) \longrightarrow t$$

is, in $\mathbb{E}$:

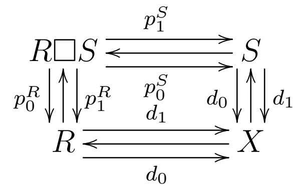
$$Y \times Y \times T \underset{p_0^Y \times T}{\overset{p_1^Y \times T}{\rightrightarrows}} Y \times T \xrightarrow{p_T} T$$

It is clear that, when $\mathbb{E}$ is pointed, it is a coequalizer diagram, and that consequently the factorization Γ_Y is fully faithful. The same property holds, when $\mathbb{E}$ is regular [1] and the object Y has global support.

Let us now characterize the T_Y-algebras. For that let us recall the following: given any pair (R, S) of equivalence relations on an object X consider the following pullback:

$$\begin{array}{ccc} R\square S & \rightarrowtail & S \times S \\ \downarrow & & \downarrow{\scriptstyle (d_0 \times d_0, d_1 \times d_1)} \\ R \times R & \underset{(d_0,d_1)\times(d_0,d_1)}{\rightarrowtail} & X^4 \end{array}$$

which determines the *double parallelistic relation* associated to (R, S) (see [6] and also [10]):

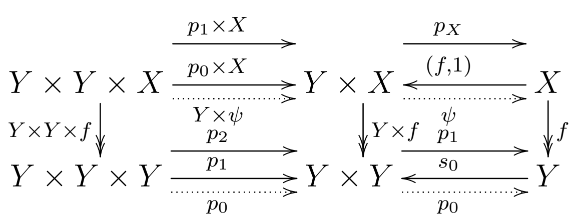


2.4. Definition. *We say that the pair (R, S) is strictly centralizing when one of the downward and rightward commutative square is a pullback.*

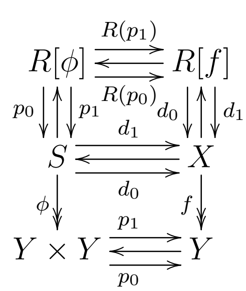
In this circumstance, all the other commutative squares are pullbacks since the one in question determines a discrete fibration between groupoids. In the set theoretical context this means that for any triple $xRySz$, there is a unique t such that $xStRz$. In any category $\mathbb{E}$, and for any pair (Y, T) of objects the pair $(R[p_Y], R[p_T])$ is a strictly centralizing pair on the object $Y \times T$. A strictly centralizing pair (R, S) is necessarily such that: $R \cap S = \Delta_X$.

2.5. Proposition. *A T_Y-algebra structure $\psi : Y \times X \to X$ on (f, s) is equivalent to the data of an equivalence relation S on X such that the pair $(R[f], S)$ is strictly centralizing and the equivalence relation S satisfies: $f(S) = \nabla_Y$.*

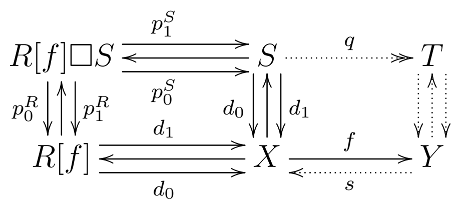
Proof. The map ψ satisfies: $f.\psi = p_Y$, $\psi.(f, 1) = 1_X$ and $\psi.p_0 \times X = \psi.Y \times \psi$. Notice that the map s is not involved. Actually the previous equations show that the map ψ completes the following diagram into a vertical discrete fibration between groupoids:



This implies that the right hand side dotted square is a pullback. Since the lower groupoid is an equivalence relation, so is the upper one which produces an equivalence relation S on X, which is such that: $f(S) = \nabla_Y$ since f and $Y \times f$ are (split) epimorphisms. If we complete the previous diagram by the kernel equivalence relations:



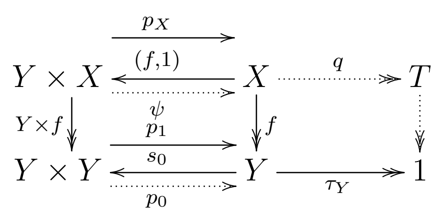
the upper part of the diagram is the parallelistic double relation associated with the pair $(S, R[f])$. The lower squares being pullbacks, so are the upper ones, and the pair $(S, R[f])$ is strictly centralizing.
Conversely, suppose there is an equivalence relation S satisfying the previous conditions. Let us consider the following diagram:



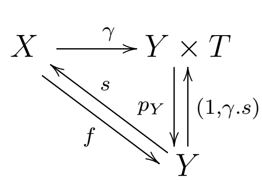
Since we have, on the left hand side, discrete fibrations between equivalence relations and since f is a split epimorphism, we have the quotient T of the upper horizontal equivalence relation, which produces the dotted right hand side pullbacks. Since $R[f] \cap S = \Delta_X$, the vertical right hand side groupoid T is actually an equivalence relation on Y. Now we have $T = f(S) = \nabla_Y$ as equivalence relations, and consequently $T \simeq Y \times Y$ as objects. Accordingly we have $S \simeq Y \times X$, whence the splitting $\psi : Y \times X \simeq S \xrightarrow{d_0} X$ (and the T_Y-algebra) we were looking for. ∎

2.6. Corollary. *When $\mathbb{E}$ is pointed, any comparison functor Γ_Y is an equivalence of categories; in other words, we have $Alg^{T_Y} \simeq Y\backslash\mathbb{E}$. When $\mathbb{E}$ is exact [1] (or even efficiently regular), it is the case provided that the object Y has global support.*

Proof. We know that, in both situations, the comparison Γ_Y is fully faithful. We are going to show that it is essentially surjective. Given any T_Y-algebra ψ let us consider the following diagram where the left hand side part is a vertical discrete fibration between equivalence relations:



In both cases the upper equivalence relation is effective and we can complete the diagram with its quotient map q which produces the right hand side pullback. Accordingly we get an isomorphism γ in $Pt_Y\mathbb{E}$:



In the pointed case, the splitting of the terminal map τ_Y imposes (actually by construction) $T \simeq Kerf$. ∎

2.7. Special T_Y-algebras. But it does not impose that we have $\gamma.s = 0$. We already noticed that the section s of f is not directly involved in the equations defining a T_Y-algebra.

2.8. Definition. *A T_Y-algebra ψ is said to be special when, in addition, it satisfies: $\psi.Y \times s = s.p_0$.*

Accordingly a T_Y-algebra ψ is special, when the section s of f determines a section of the discrete fibration described in the proof of Proposition 2.5; it consequently determines a section of the terminal map $\tau_T : T \to 1$ in the subsequent corollary. Whence the straightforward following:

2.9. Proposition. *Suppose the category $\mathbb{E}$ is pointed. Then a split epimorphism $(f, s) : X \leftrightarrows Y$ is isomorphic to the split epimorphism $(p_Y, \iota_Y) : Y \times Kerf \leftrightarrows Y$ if and only if it is endowed with the structure of a special T_Y-algebra.*

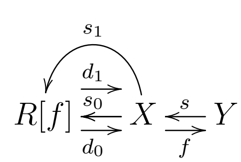
3. Mal'cev context

Mal'cev categories were introduced in [9] and [10] as those categories $\mathbb{C}$ in which any reflexive relation is an equivalence relation. They are characterized by the fact that any fibre $Pt_Y\mathbb{C}$ of the fibration $¶_{\mathbb{C}}$ is unital, a context in which there is an intrinsic notion of commutation [3]. On the other hand, in this same context, there is also an intrinsic notion of commutation of equivalence relations, see [16] and [6]; thanks to Definition 2.4, we are allowed to say now that *two equivalence relations R and S on an object X are strictly centralizing if and only if we have: $R \cap S = \Delta_X$.*

3.1. Mal'cev monads. Recall also that a monad (T, λ, μ) on any category $\mathbb{E}$ is said to be *a Mal'cev monad*, when the pair $(\lambda T, T\lambda)$ is jointly strongly epic [5]. The main property of a Mal'cev monad is that any T-algebra is characterized by the only unit axiom. First we get the following:

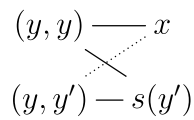
3.2. Proposition. *Let $\mathbb{C}$ be a Mal'cev category. Then the monads (T, λ, μ) on $Pt\mathbb{C}$ and (T_Y, λ_Y, μ_Y) on $Pt_Y\mathbb{C}$ for any Y are Mal'cev monads.*

Proof. 1) The first statement comes from the fact that, in a Mal'cev category, any kernel equivalence relation of a split epimorphism:



is such that the pair (s_0, s_1) is jointly strongly epic.
2) Suppose, now, we have a subobject $W \rightarrowtail Y \times Y \times X$ containing $s_0 \times X$ and $Y \times (f, 1)$. This can be understood as a relation $(y, y')Wx$ such that $(y, y)Wx$ for all (y, x) and

$(y, f(x))Wx$ for all (y, x). Let us recall that, in a Mal'cev category, any relation is difunctional. Consider, for any triple (y, y', x), the following diagram:

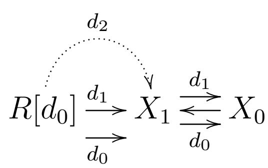


It shows that $((y, y'), x)$ is in W, and that w is an isomorphism. ■

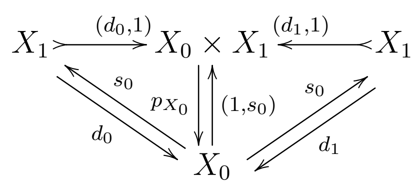
Accordingly, any splitting $\psi : Y \times X \to X$ of the map $(f, 1)$ such that $f.\psi = p_Y$ determines a T_Y-algebra and produces the conditions of Proposition 2.5 which can be reformulated:

3.3. Proposition. *Let $\mathbb{C}$ be a Mal'cev category. A T_Y-algebra structure $\psi : Y \times X \to X$ on (f, s) is equivalent to the data of an equivalence relation S on X such that we have: $R[f] \cap S = \Delta_X$ and $f(S) = \nabla_Y$.*

3.4. Internal groupoids in the Mal'cev context. Since (T, λ, μ) becomes a Mal'cev monad, a reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ is a groupoid if and only if there is a map d_2:



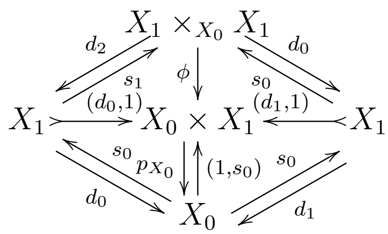
such that $d_0.d_2 = d_1.d_0$ and $d_2.s_1 = 1_{X_1}$.
On the other hand it is well known [10] that, in this context, any internal category is a groupoid. Now, some aspects of the monad T_Y on $Pt_Y\mathbb{C}$ will allow us to shed a new light on another well known property of Mal'cev categories concerning what is called *multiplicative* reflexive graph in [10]. From any reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ in $\mathbb{C}$, we get, by the monad T_{X_0}, two subobjects in $Pt_{X_0}\mathbb{C}$:



We can assert now the following:

3.5. Proposition. *Let $\mathbb{C}$ be a Mal'cev category. The reflexive graph in question is a groupoid if and only if these two subobjects commute in $Pt_{X_0}\mathbb{C}$.*

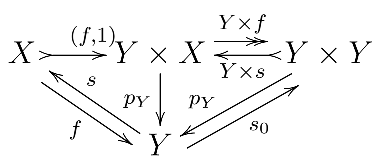
Proof. These two subobjects commute in the fibre $Pt_{X_0}\mathbb{C}$ when they have a cooperator $\phi : X_1 \times_{X_0} X_1 \to X_0 \times X_1$ (i.e. a map satisfying: $\phi.s_0 = (d_1, 1)$ and $\phi.s_1 = (d_0, 1)$):



where the whole quadrangle is a pullback. The map ϕ is consequently a pair $(d_0.d_2, d_1)$, where $d_1 : X_1 \times_{X_0} X_1 \to X_1$ is such that: $d_1.s_0 = 1_{X_1}$ and $d_1.s_1 = 1_{X_1}$. This map d_1 with these two identities makes *multiplicative* in the sense of [10] the reflexive graph in question. And, according to Theorem 2.2 in [10], in a Mal'cev category, any reflexive-multiplicative graph is a groupoid. ∎

It is well known also that, in this context of Mal'cev categories, an internal reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ in $\mathbb{C}$ is actually an internal groupoid if and only if the equivalence relations $R[d_0]$ and $R[d_1]$ commute in $\mathbb{C}$, i.e. if and only if we have $[R[d_0], R[d_1]] = 0$, see [10] and [6]. So, in the Mal'cev context, a reflexive graph is a groupoid if and only if either the equivalence relations $R[d_0]$ and $R[d_1]$ commute in $\mathbb{C}$ or the subobjects $(d_0, 1)$ and $(d_0, 1)$ commute in the fibre $Pt_{X_0}\mathbb{C}$. Further properties of internal groupoids in a Mal'cev categories are developed in [12].

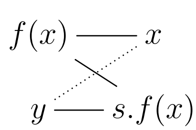
3.6. When Mal'cev categories are protomodular. The unit of the monad T_Y on the fibre $Pt_Y\mathbb{E}$ is actually the kernel of a split epimorphism in this pointed category:



In the Mal'cev context this split sequence is actually a split exact sequence:

3.7. Proposition. *Let $\mathbb{C}$ be any Mal'cev category. Then the previous split sequence in $Pt_Y\mathbb{C}$ is a strongly split exact sequence in $Pt_Y\mathbb{C}$, or equivalently it produces a strongly split epimorphism $Pt_Y\mathbb{C}$ (see Definition 1.8).*

Proof. Let $w : W \rightarrowtail Y \times X$ be a subobject containing $(f, 1)$ and $Y \times s$. Since $\mathbb{C}$ is Mal'cev, the relation W on $Y \times X$ is difunctional. Consider, for any pair $(y, x) \in Y \times X$, the following diagram:



It shows that (y, x) is in W, and that w is an isomorphism. ∎

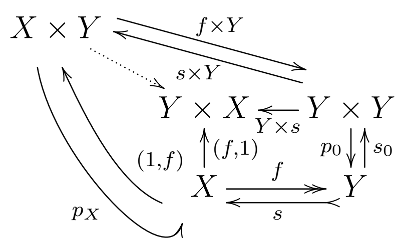
We have now the following characterization:

3.8. Lemma. *Let $\mathbb{C}$ be a Mal'cev category and $(f,s) : X \leftrightarrows Y$ any split epimorphism. A map $\psi : Y \times X \to X$ is a special T_Y-algebra if and only if we have $\psi.(f,1) = 1_X$ and $\psi.Y \times s = s.p_0$. Accordingly there is at most one special T_Y-algebra structure on it.*

Proof. These conditions are necessary. Conversely it remains to show that $\psi.f = p_Y$ which is a consequence of the fact that, thanks to the asserted conditions, the compositions of these two maps by the strongly epic pair $((f,1), Y \times s)$ are the same. ∎

3.9. Proposition. *Let $\mathbb{C}$ be a Mal'cev category. The two subobjects $(f,1)$ and $Y \times s$ commute in $Pt_Y\mathbb{C}$ if and only if the split epimorphism (f,s) is endowed with the structure of a special T_Y-algebra. When the category $\mathbb{C}$ is pointed, the only split epimorphisms inducing this commutation are the trivial ones $(p_Y, \iota_Y) : Y \times T \leftrightarrows T$.*

Proof. Suppose the two subobjects $(f,1)$ and $Y \times s$ commute in $Pt_Y\mathbb{C}$, and denote by $(\alpha,\beta) : X \times Y \to Y \times X$ the associated cooperator:



such that $(\alpha,\beta).(1,f) = (f,1)$ and $(\alpha,\beta).s \times Y = Y \times s$. This means that $\beta.(1,f) = 1_X$ and $\beta.s \times Y = s.p_1$. If we denote by $tw_{Y,X} : Y \times X \to X \times Y$ the twisting isomorphism, we get $(\beta.tw_{Y,X}).(f,1) = 1_X$ and $(\beta.tw_{Y,X}).Y \times s = s.p_0$. According to the previous lemma, the map $\psi = \beta.tw_{Y,X}$ is then a special T_Y-algebra. Conversely if ψ is a special T_Y-algebra structure on (f,s), then $\beta = \psi.tw_{X,Y}$ is the cooperator which makes commute the pair $((f,1), Y \times s)$. The last assertion is a straightforward consequence of Proposition 2.9. ∎

In [13], was made the remarkable observation that, in the category Gp of groups, any change of base functor with respect to the fibration of points has a right adjoint. The same property holds in the category R-Lie of Lie R-algebras, where R is a commutative ring, see [14]. Such a kind of category was called *locally algebraically cartesian closed* in [8]. The specific cohesion generated by this local algebraic cartesian closedness can be measured by the following:

3.10. Theorem. *Let $\mathbb{C}$ be a pointed Mal'cev category. When, in addition, it is locally algebraically cartesian closed, then it is protomodular.*

Proof. Let $(f,s) : X \leftrightarrows Y$ be a split epimorphism in $\mathbb{C}$. In the pointed Mal'cev fibre $Pt_Y\mathbb{C}$ we get the previous strongly split exact sequence. Now the change of base functor α_Y^* along the initial map, having a right adjoint, preserves the jointly strongly epic pairs, see Proposition 1.4. Accordingly the image by α_Y^* of this strongly split exact sequence, namely the following one:

$$Ker f \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

is still a strongly split exact sequence, which is the characterization of a pointed protomodular category [3]. ■

Now, what does happen in the non-pointed case? The answer is the following:

3.11. THEOREM. *Let $\mathbb{C}$ be a regular Mal'cev category. When, in addition, it is locally algebraically cartesian closed, then the full subcategory $\mathbb{C}_\sharp$ of $\mathbb{C}$ whose objects are those which have global support is protomodular.*

PROOF. Let $y : Y' \to Y$ be any map in $\mathbb{C}$; we have to show that the change of base functor $h^* : Pt_Y\mathbb{C} \to Pt_{Y'}\mathbb{C}$ is conservative. Since h^* is left exact, it is enough to check it on monomorphisms. So let $u : (\bar{f}, \bar{s}) \rightarrowtail (f, s)$ be a monomorphism in $Pt_Y\mathbb{C}$. We get a morphism of strong split exact sequences in this fibre:

$$\begin{array}{ccccc}
\bar{X} & \overset{(\bar{f},1)}{\rightarrowtail} & Y \times \bar{X} & \underset{Y\times\bar{s}}{\overset{Y\times\bar{f}}{\rightleftarrows}} & Y \times Y \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle Y\times u} & & \| \\
X & \underset{(f,1)}{\rightarrowtail} & Y \times X & \underset{Y\times s}{\overset{Y\times f}{\rightleftarrows}} & Y \times Y
\end{array}$$

Suppose that $h^*(u)$ is an isomorphism. Since $\mathbb{C}$ is lacc, the change of base functor h^* preserves the strong split exact sequences. Accordingly the monomorphism $h^*(Y \times u)$ is an isomorphism as well in the fibre $Pt_{Y'}\mathbb{C}$; that means that the monomorphism $Y' \times u : Y' \times \bar{X} \rightarrowtail Y' \times X$ is an isomorphism in $\mathbb{C}$. Now, when Y' has global support, the following square has vertical regular epimorphisms:

$$\begin{array}{ccc}
Y' \times \bar{X} & \overset{Y'\times u}{\rightarrowtail} & Y' \times X \\
{\scriptstyle p_{X'}}\downarrow & & \downarrow{\scriptstyle p_X} \\
\bar{X} & \underset{u}{\rightarrowtail} & X
\end{array}$$

Consequently, if $Y' \times u$ is an isomorphism, so is u. ■

The two previous results are even more striking, if we recall that, according to Theorem 4.3 in [8], any locally algebraically cartesian closed protomodular category is necessarily strongly protomodular.

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