Theory and Applications of Categories, Vol. 24, No. 12, 2010, pp. 288–301.

LAX PRESHEAVES AND EXPONENTIABILITY

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ABSTRACT. The category of **Set**-valued presheaves on a small category B is a topos. Replacing **Set** by a bicategory **S** whose objects are sets and morphisms are spans, relations, or partial maps, we consider a category $\mathrm{Lax}(B, \mathbf{S})$ of **S**-valued lax functors on B. When $\mathbf{S} = \mathbf{Span}$, the resulting category is equivalent to $\mathbf{Cat}/B$, and hence, is rarely even cartesian closed. Restricting this equivalence gives rise to exponentiability characterizations for $\mathrm{Lax}(B, \mathbf{Rel})$ in [9] and for $\mathrm{Lax}(B, \mathbf{Par})$ in this paper. Along the way, we obtain a characterization of those B for which the category $\mathbf{UFL}/B$ is a coreflective subcategory of $\mathbf{Cat}/B$, and hence, a topos.

1. Introduction

If B is a small category, then the category $\mathbf{Set}^B$ of functors $B \to \mathbf{Set}$ and natural transformations is a topos (c.f., [6]). But, what happens if we replace **Set** by the bicategories **Rel**, **Par**, and **Span**, whose objects are sets and morphisms are relations, partial maps, and spans, respectively? Given a bicategory **S**, we consider the categories $\mathrm{Lax}(B, \mathbf{S})$ and $\mathrm{Pseudo}(B, \mathbf{S})$ whose objects are lax functors and pseudo functors $B \to \mathbf{S}$, respectively, and morphisms are function-valued op-lax transformation. We will see that $\mathrm{Lax}(B, \mathbf{S})$ is rarely cartesian closed, when **S** is **Rel**, **Par**, and **Span**, whereas $\mathrm{Pseudo}(B, \mathbf{Span})$ is often a topos, but rarely in the other two cases.

Although it appears that exponentiability in $\mathrm{Lax}(B, \mathbf{Span})$ has not explicitly been considered, $\mathrm{Lax}(B, \mathbf{Span})$ is known to be equivalent to the slice category $\mathbf{Cat}/B$, and exponentiable objects in $\mathbf{Cat}/B$ were characterized independently by Giraud [5] and Conduché [4] as those functors satisfying a factorization lifting property, sometimes called the Giraud-Conduché condition. More recently, using the equivalence with $\mathbf{Cat}/B$ with category $\mathrm{Lax}(B, \mathbf{Prof})$ of normal lax functors $B \to \mathbf{Prof}$ and map-valued op-lax transformations, Street [11] showed that $X \to B$ is exponentiable in **Cat** if and only if the corresponding lax functor is a pseudo functor. Inspired by Street's note and a preprint of Stell [10] using lax functors $B \to \mathbf{Rel}$ to model data varying over time, we established an equivalence between $\mathrm{Lax}(B, \mathbf{Rel})$ and the full subcategory $\mathbf{Cat}_f/B$ of $\mathbf{Cat}/B$ consisting of faithful functors. Using this, we showed that a faithful functor $p\colon X \to B$ is exponentiable if and only if the corresponding lax functor $B \to \mathbf{Rel}$ preserves composition up to isomorphism if and only if p satisfies a weak factorization lifting condition WFL [9].

Received by the editors 2008-09-09 and, in revised form, 2010-07-06.
Transmitted by Walter Tholen. Published on 2010-07-19.
2000 Mathematics Subject Classification: 18A22, 18A25, 18A40, 18B10, 18B25, 18D05, 18F20.
Key words and phrases: span, relation, partial map, topos, cartesian closed, exponentiable, presheaf.

We begin this paper, in Section 2, describing Lax(B, **Par**) as a category of variable sets, and then, in Section 3, obtain an equivalence with a full subcategory of **Cat**/B, by restricting that of Lax(B, **Rel**) and $\mathbf{Cat}_f/B$. Necessary and sufficient conditions for exponentiability in Lax(B, **Par**) are presented in Section 4. In Section 5, we give a short and simple proof that Pseudo(B, **Span**) is a topos, whenever it is a coreflective subcategory of Lax(B, **Span**), and then show that this occurs precisely when B satisfies the interval glueing property IG defined in [3]. We conclude, in Section 6, by showing that Pseudo(B, **Rel**) is rarely a topos, and present conditions under which it is a cartesian closed coreflective subcategory of Lax(B, **Rel**). We do not consider the analogous question for Pseudo(B, **Par**) since, in this case, there are pseudo functors which are not exponentiable in Lax(B, **Par**).

Our interest in Pseudo(B, **Span**) began when Robin Cockett mentioned its equivalence to the full subcategory **UFL**/B of **Cat**/B consisting of functors satisfying the unique factorization lifting condition. In a 1996 talk [8], Lamarche had conjectured that **UFL**/B is a topos. Subsequently, Bunge and Niefield [3] showed that if B satisfies IG, then **UFL**/B is coreflective in **Cat**/B, and used this to show that **UFL**/B is a topos. Then Johnstone [7] showed that it is not a topos when B is a commutative square, and used sheaves to show that **UFL**/B is a topos when B satisfies certain cancellation and fill-in properties (CFI). Shortly thereafter, Bunge and Fiore gave an alternate sheaf-theoretic proof, using IG, and also showed that the conditions IG and CFI are, in fact, equivalent. Our theorem (see 5.5) shows that these conditions hold *whenever* **UFL**/B is coreflective in **Cat**/B, thus providing a converse to the Bunge and Niefield result from [3].

2. Lax functors as variable sets

In this section, we describe Lax(B, **Par**) as a category of relational variable sets on a small category B (in the sense of [9]).

Recall that a *relational variable set* or **Rel**-*set* is a lax functor $X: B \to \mathbf{Rel}$. Thus, a **Rel**-set X consists of a set X_b, for every object b, and a relation $X_\beta: X_b \nrightarrow X_{b'}$, for every morphism $\beta: b \to b'$, satisfying $\Delta_{X_b} \subseteq X_{id_b}$, for every object b, and $X_{\beta'} \circ X_\beta \subseteq X_{\beta'\beta}$, for every composable pair. Writing $x \to_\beta x'$ for $(x, x') \in X_\beta$, these conditions become

(R1) $x \to_{id_b} x$, for all $x \in X_b$.

(R2) $x \to_\beta x', x' \to_{\beta'} x'' \Longrightarrow x \to_{\beta'\beta} x''$.

A morphism $f: X \to Y$ of **Rel**-sets consists of a function $f_b: X_b \to Y_b$, for every object b such that

$$\begin{array}{ccc} X_b & \xrightarrow{f_b} & Y_b \\ {\scriptstyle X_\beta}\downarrow & \subseteq & \downarrow{\scriptstyle Y_\beta} \\ X_{b'} & \xrightarrow[f_{b'}]{} & Y_{b'} \end{array}$$

for every morphism $\beta: b \to b'$, or equivalently, $x \to_\beta x'$ implies $f_b x \to_\beta f_{b'} x'$. Then $\mathrm{Lax}(B, \mathbf{Rel})$ denotes the category of **Rel**-sets and morphisms.

A lax functor $X: B \to \mathbf{Par}$ consists of a set X_b, for every object b, and a partial map $X_\beta: X_b \rightharpoonup X_{b'}$, for every morphism $\beta: b \to b'$, such that $id_{X_b} \leq X_{id_b}$, for object b, and $X_{\beta'} \circ X_\beta \leq X_{\beta'\beta}$, for every composable pair. Since a relation $X_\beta: X_b \nrightarrow X_{b'}$ is a partial map if and only if $X_\beta \circ X_\beta^\circ \subseteq \Delta_{X_{b'}}$, it follows that a **Rel**-set X is a lax functor $X: B \to \mathbf{Par}$ if and only if it satisfies

(P3) $x \to_\beta x_1', x \to_\beta x_2' \Longrightarrow x_1' = x_2'$

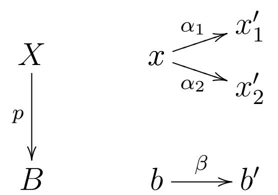
Thus, $\mathrm{Lax}(B, \mathbf{Par})$ is the full subcategory of $\mathrm{Lax}(B, \mathbf{Rel})$ consisting of those **Rel**-sets which satisfy (P3).

3. Lax Functors and Subcategories of $\mathbf{Cat}/B$

In [9], we showed that the well-known equivalence between $\mathrm{Lax}(B, \mathbf{Span})$ and $\mathbf{Cat}/B$ restricts to one between $\mathrm{Lax}(B, \mathbf{Rel})$ and the full subcategory $\mathbf{Cat}_f/B$ of $\mathbf{Cat}/B$ consisting of faithful functors over B. In particular, a faithful functor $p: X \to B$ corresponds to a **Rel**-set, also denoted by X, and defined as follows. For each object b, X_b is the fiber of X over b, i.e., the set of objects x such that $px = b$. Given $\beta: b \to b'$, the relation X_β is defined by $x \to_\beta x'$, if there is a morphism $\alpha: x \to x'$ such that $p\alpha = \beta$. Moreover, the product of X and Y in $\mathrm{Lax}(B, \mathbf{Rel})$ is given by $(X \times Y)_b = X_b \times Y_b$ and

$$(x, y) \to_\beta (x', y') \iff x \to_\beta y \text{ and } x' \to_\beta y'$$

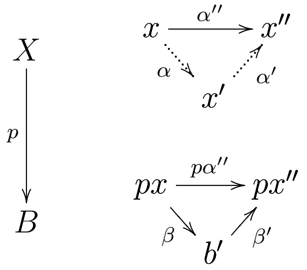
Identifying **Par** with the subcategory of **Rel** consisting of morphisms $R: X \nrightarrow Y$ with $R \circ R^\circ \subseteq \Delta_Y$, it is not difficult to show that $\mathrm{Lax}(B, \mathbf{Par})$ is equivalent to the full subcategory $\mathbf{Cat}_{pf}/B$ of $\mathbf{Cat}/B$ consisting of faithful functors $p: X \to B$ such that given



if $p(\alpha_1) = p(\alpha_2) = \beta$, then $\alpha_1 = \alpha_2$, and $\mathrm{Lax}(B, \mathbf{Par})$ has finite products which agree with those of $\mathrm{Lax}(B, \mathbf{Rel})$.

4. Exponentiability in Categories of Lax Functors

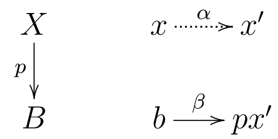
A **Rel**-set X is exponentiable in $\mathrm{Lax}(B, \mathbf{Rel})$ if and only if X preserves composition, i.e., for every composable pair, the containment $X_{\beta'} \circ X_\beta \subseteq X_{\beta'\beta}$ is an equality if and only if the corresponding faithful functor $p: X \to B$ has the weak factorization lifting condition (WFL)



i.e., given α'' and a factorization $p\alpha'' = \beta'\beta$, there exists a factorization $\alpha'' = \alpha'\alpha$ such that $p\alpha = \beta$ and $p\alpha' = \beta'$ [9]. Moreover, $X: B \to \mathbf{Span}$ is exponentiable in $\mathrm{Lax}(B, \mathbf{Span})$ if and only if the corresponding functor $p: X \to B$ satisfies the Giraud-Conduché condition, i.e., WFL plus a connectivity condition on the objects over b' through which α'' factors [5, 4].

To establish a characterization of exponentiable objects in $\mathrm{Lax}(B, \mathbf{Par})$, we use its equivalence with $\mathbf{Cat}_{pf}/B$ as well as the description, given in 2.1, of $X: B \to \mathbf{Par}$ as a **Rel**-set satisfying (P3).

A functor $p: X \to B$ will be called a *partial fibration* if it satisfies the lifting condition



i.e, given x' and β, there exists α such that $p\alpha = \beta$. It is easy to show that a faithful functor is a partial fibration if and only if the associated lax functor $X: B \to \mathbf{Rel}$ satisfies $\Delta_{X_{b'}} \subseteq X_\beta \circ X_\beta^\circ$, i.e., X_β is onto.

4.1. Theorem. *The following are equivalent for a lax functor $X: B \to \mathbf{Par}$ with corresponding functor $p: X \to B$.*

(a) X is exponentiable in $\mathrm{Lax}(B, \mathbf{Par})$.

(b) X is a functor and $X_\beta: X_b \rightharpoonup X_{b'}$ is onto, for all $\beta: b \to b'$.

(c) p is exponentiable in $\mathbf{Cat}_{pf}/B$.

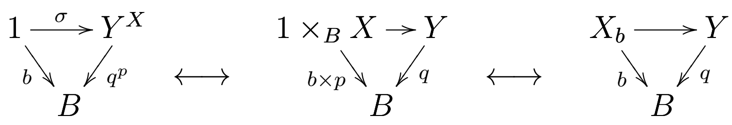
(d) p is a partial fibration satisfying the weak factorization lifting condition WFL.

Proof. We know (a) and (c) are equivalent, since $\mathrm{Lax}(B, \mathbf{Par}) \simeq \mathbf{Cat}_{pf}/B$. The equivalence of (b) and (d) follows from the remarks above, the definition of WFL, and the fact that $X_{id_b} = id_{X_b}$ in $\mathrm{Lax}(B, \mathbf{Par})$, for all b, since $x_1 \to_{id_b} x_2$ implies $x_1 = x_2$ by (P3). Thus, it suffices to show that (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d).

For (b) $\Rightarrow$ (a), suppose X is a functor and X_β is onto, for all $\beta: b \to b'$. Then X is exponentiable in $\mathrm{Lax}(B, \mathbf{Rel})$. Thus, given $Y: B \to \mathbf{Par}$, it suffices to show that the exponential **Rel**-set Y^X is an object of Lax$(B, \mathbf{Par})$, i.e., satisfies (P3), where Y^X is defined as follows [9]. Let $(Y^X)_b$ denote the set of functions $\sigma: X_b \to Y_b$, and define $\sigma \to_\beta \sigma'$ if $\sigma x \to_\beta \sigma' x'$, for all $x \to_\beta x'$ in X.

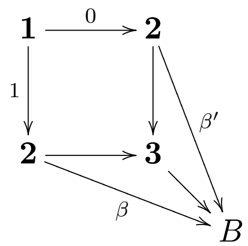
To see that Y^X satisfies (P3), suppose $\sigma \to_\beta \sigma'_1$, $\sigma \to_\beta \sigma'_2$, and $x' \in X_{b'}$. Then $x \to_\beta x'$, for some $x \in X_b$, since X_β is onto, and so $\sigma x \to_\beta \sigma'_1 x'$ and $\sigma x \to_\beta \sigma'_2 x'$. Thus, $\sigma'_1 x' = \sigma'_2 x'$, since Y satisfies (P3), and it follows that $\sigma'_1 = \sigma'_2$. Therefore, X is exponentiable in Lax$(B, \mathbf{Par})$.

For (c) $\Rightarrow$ (d), suppose p is exponentiable in $\mathbf{Cat}_{pf}/B$. Then the elements σ of the fiber $(Y^X)_b$ can be identified with the functions $X_b \to Y_b$, since

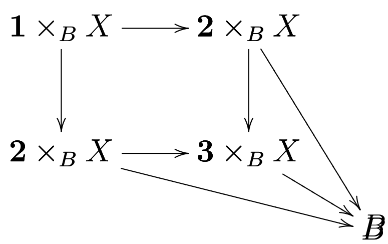


Moreover, every morphism $\sigma \to \sigma'$ over β satisfies $\sigma x \to \sigma' x'$, for all $x \to x'$ over β, since the counit is the evaluation functor $\varepsilon: Y^X \times_B X \to Y$, under this identification.

To see that p satisfies WFL, suppose $\alpha'': x \to x''$ in X and $p\alpha'' = \beta'\beta$, where $\beta: px \to b'$ and $\beta': b' \to px''$. Then the composite $\beta'\beta$ gives rise to a pushout in Cat_{pf}/B of the form



where $\mathbf{2}$ and $\mathbf{3}$ are the categories $0 \to 1$ and $0 \to 1 \to 2$, respectively. Since $- \times p$ preserves pushouts (as it has a right adjoint), it follows that the corresponding diagram



is a pushout in Cat_{pf}/B. The pushout $P \to B$ of this diagram can be constructed as follows. Let X_β and $X_{\beta'}$ denote the subcategories of X obtained by identifying $p \times \beta$ and $p \times \beta'$ with their images in X. Then the objects of P are the union of those of X_β and $X_{\beta'}$, and the morphisms are those of X_β and $X_{\beta'}$ together with pairs $(\alpha, \alpha'): x \to x''$ such that $\alpha: x \to x'$ in X_β and $\alpha': x' \to x''$ in $X_{\beta'}$, subject to an appropriate equivalence relation. Since $\alpha'': x \to x''$ corresponds to a morphism of $\mathbf{3} \times_B X$, and hence one of P over $\beta'\beta$, the desired factorization of α'' follows.

It remains to show that p is a partial fibration. Suppose x' is in X and $\beta\colon b \to px'$ in B. Let Y denote the category

$$\begin{array}{ccc} & & y_2' \\ & & \\ y_1 & \xrightarrow{\gamma} & y_1' \end{array}$$

with $q\colon Y \to B$ given by $q\gamma = \beta$ and $qy_2' = px'$. Let $\sigma \in (Y^X)_b$ denote the constant y_1-valued map, $\sigma_1' \in (Y^X)_{px'}$ the constant y_1'-valued map, and $\sigma_2' \in (Y^X)_{px'}$ the function

$$\sigma_2'\hat{x}' = \begin{cases} y_1' & \text{if } \hat{x}' \in \text{Image}(X_\beta) \\ y_2' & \text{otherwise} \end{cases}$$

Thus, for $i = 1, 2$, we have $\hat{x} \underset{\beta}{\to} \hat{x}'$ implies $\sigma\hat{x} \underset{\beta}{\to} \sigma_i'\hat{x}'$, and so $\sigma \underset{\beta}{\to} \sigma_i'$, and we get a diagram

$$\begin{array}{ccc} Y^X & & \sigma \begin{array}{l} \nearrow \sigma_1' \\ \searrow \sigma_2' \end{array} \\ \Big\downarrow & & \\ B & & b \xrightarrow{\beta} px' \end{array}$$

Since $Y^X \to B$ satisfies (P3), it follows that $\sigma_1 = \sigma_2$. Thus, $X_{b'} = \text{Image}(X_\beta)$, and so there exists $\alpha\colon x \to x'$ such that $p\alpha = \beta$, to complete the proof. ■

4.2. Corollary. *The inclusion* $\text{Lax}(B, \mathbf{Par}) \to \text{Lax}(\mathbf{B}, \mathbf{Rel})$ *preserves exponentiability and exponentials.*

Proof. Since the exponentials Y^X defined in the proof of (b)⇒(a) agree with those of $\text{Lax}(B, \mathbf{Rel})$ given in [9] when X satisfies WFL, the desired result follows. ■

5. Pseudo(B, Span) and UFL/B

Since every pseudo functor $X\colon B \to \mathbf{Span}$ is exponentiable in $\text{Lax}(B, \mathbf{Span})$, one might conjecture that $\text{Pseudo}(B, \mathbf{Span})$ is cartesian closed. As noted in the introduction, this is not the case when B is a commutative square [7] since

$$\text{Pseudo}(B, \mathbf{Span}) \simeq \mathbf{UFL}/B$$

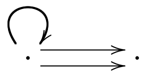
where the latter is the full subcategory of $\mathbf{Cat}/B$ consisting of functors satisfying the unique factorization lifting property UFL, i.e., the condition WFL given in Section 4 plus uniqueness of the lifted factorization. However, $\mathbf{UFL}/B$ is a topos which is coreflective in $\mathbf{Cat}/B$, if B satisfies a condition called the *interval glueing condition* (IG) in [3]. The proof in [3] makes extensive use of (IG), and one in [7] exhibits $\mathbf{UFL}/B$ as a topos of sheaves, using a condition equivalent to (IG). In the following, we show that $\mathbf{UFL}/B$ is a topos, assuming only that it is coreflective in $\mathbf{Cat}/B$. Although this appears to be a more general theorem than that of [3] and [7], we will see that coreflectivity of $\mathbf{UFL}/B$ in $\mathbf{Cat}/B$ is, in fact, equivalent to (IG), thus proving the converse of the result in [3].

5.1. LEMMA. *Suppose* $\mathbf{Y}$ *is a category with finite products and* $\mathbf{X} \to \mathbf{Y}$ *is a product preserving inclusion of a coreflective subcategory. If* X *is an object of* $\mathbf{X}$ *which is exponentiable in* $\mathbf{Y}$*, then* X *is exponentiable in* $\mathbf{X}$.

PROOF. Given $Y \in \mathbf{X}$, then $\widehat{Y^X}$ is the exponential in $\mathbf{X}$, where $\widehat{\ }$ denotes the coreflection, since $\mathbf{X}(W \times X, Y) \cong \mathbf{Y}(W \times X, Y) \cong \mathbf{Y}(W, Y^X) \cong \mathbf{X}(W, \widehat{Y^X})$. ∎

5.2. THEOREM. *If* $\mathbf{UFL}/B$ *is a coreflective subcategory of* $\mathbf{Cat}/B$*, then* $\mathbf{UFL}/B$ *is a topos.*

PROOF. A straightforward calculation shows that the inclusion $\mathbf{UFL}/B \to \mathbf{Cat}/B$ preserves products, and so $\mathbf{UFL}/B$ is cartesian closed by Lemma 5.1. To see that it is a topos, let Ω denote the UFL subobject classifier in $\mathbf{Cat}$ [3]



and consider $\Omega \times B \to B$ via the projection. Then $\mathrm{Sub}_{\mathbf{UFL}/B}(X) \cong \mathrm{Sub}_{\mathbf{UFL}}(X) \cong \mathbf{Cat}(X, \Omega) \cong \mathbf{Cat}/B(X, \Omega \times B) \cong \mathbf{UFL}/B(X, \widehat{\Omega \times B})$, for all $X \to B$ in $\mathbf{UFL}/B$. ∎

5.3. COROLLARY. *If* $\mathrm{Pseudo}(B, \mathbf{Span})$ *is a coreflective subcategory of* $\mathrm{Lax}(B, \mathbf{Span})$*, then* $\mathrm{Pseudo}(B, \mathbf{Span})$ *is a topos.*

PROOF. Since $\mathrm{Pseudo}(B, \mathbf{Span}) \simeq \mathbf{UFL}/B$, the desired result follows. ∎

Given $\beta\colon b \to b'$ in B, consider the category $[\![\beta]\!]$ over B whose objects are factorizations

$$\begin{array}{ccc} b & \xrightarrow{\beta} & b' \\ & {\scriptstyle\beta_1}\searrow \quad \nearrow{\scriptstyle\beta_2} & \\ & \cdot & \end{array}$$

morphisms are commutative diagrams

$$\begin{array}{ccccc} & {\scriptstyle\beta_1}\nearrow & \cdot & \searrow{\scriptstyle\beta_2} & \\ b & & \downarrow{\scriptstyle\gamma} & & b' \\ & {\scriptstyle\bar\beta_1}\searrow & \cdot & \nearrow{\scriptstyle\bar\beta_2} & \end{array}$$

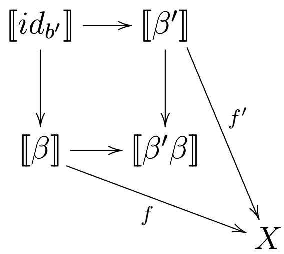
and projection to B takes the factorization $\beta = \beta_2\beta_1$ to the codomain of β_1, or equivalently, the domain of β_2. Then we say B satisfies IG if the induced diagram

$$\begin{array}{ccc} [\![id_{b'}]\!] & \longrightarrow & [\![\beta']\!] \\ \downarrow & & \downarrow \\ [\![\beta]\!] & \longrightarrow & [\![\beta'\beta]\!] \end{array} \qquad (*)$$

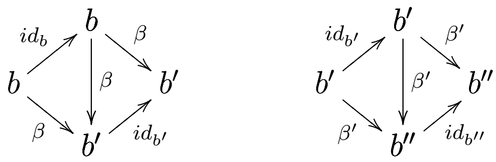
is a pushout in $\mathbf{Cat}$, for all $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$.

5.4. Lemma. *The diagram (*) is a pushout in* $\mathbf{UFL}/B$*, for all* $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$.

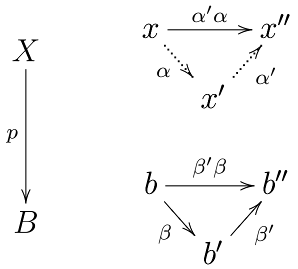
Proof. Suppose $p: X \to B$ is a UFL and we are given a commutative diagram



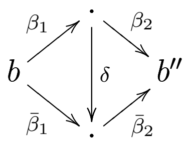
over B. Applying f and f' to the diagrams



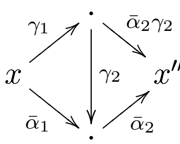
gives morphisms α and α' of X over β and β', respectively, and these morphisms are composable by the commutativity of the diagram. Thus, we get a morphism $\alpha'\alpha$ over $\beta'\beta$



To define $g: [\![\beta'\beta]\!] \to X$ over B, suppose $\beta'\beta = \beta_2\beta_1$ is an object of $[\![\beta'\beta]\!]$, and take $g(\beta_2\beta_1)$ to be the codomain of α_1, where $\alpha'\alpha = \alpha_2\alpha_1$ is the unique lifting of the factorization $\beta'\beta = \beta_2\beta_1$. Given a morphism



of $[\![\beta'\beta]\!]$, let $\alpha'\alpha = \bar{\alpha}_2\bar{\alpha}_1$ be the unique lifting of $\beta'\beta = \bar{\beta}_2\bar{\beta}_1$ and $\bar{\alpha}_1 = \gamma_2\gamma_1$ be the unique lifting of $\bar{\beta}_1 = \delta\beta_1$. Thus, we get a diagram



Then $\alpha_1 = \gamma_1$ and $\alpha_2 = \bar{\alpha}_2\gamma_2$, by uniqueness of the factorization $\alpha'\alpha = \alpha_2\alpha_1$, and we can take γ_2 to be the morphism $g(\delta)\colon g(\beta_2\beta_1) \to g(\bar{\beta}_2\bar{\beta}_1)$. To establish the uniqueness of g, one shows that for any other such morphism $\bar{g}$ the commutativity of the triangles implies that $\bar{g}(\beta'\beta) = \alpha'\alpha$, and so $g = \bar{g}$, by uniqueness of factorizations of $\alpha'\alpha$ over $\beta'\beta$. ∎

5.5. Theorem. **UFL**$/B$ *is coreflective in* **Cat**$/B$ *if and only if* B *satisfies IG.*

Proof. Suppose **UFL**$/B$ is coreflective in **Cat**$/B$ and $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$. Then (*) is a pushout in **UFL**$/B$ by Lemma 5.4, and hence, in **Cat**$/B$, since the inclusion preserves pushouts being a left adjoint. Thus, (*) is a pushout in **Cat**, and it follows that B satisfies IG, as desired. The converse holds by Proposition 3.2 and (the proof of) Lemma 4.3 of [3]. ∎

Applying the equivalence Pseudo$(B, \mathbf{Span}) \simeq$ **UFL**$/B$, gives:

5.6. Corollary. Pseudo$(B, \mathbf{Span})$ *is coreflective in* Lax$(B, \mathbf{Span})$ *if and only if* B *satisfies IG.*

Although IG works well in the proof of Theorem 5.5, the equivalent condition, called CFI in [2], is a good source of examples. This condition consists of two parts, namely, *cancellation*: given a diagram

$$\cdot \xrightarrow{f} \cdot \underset{h}{\overset{g}{\rightrightarrows}} \cdot \xrightarrow{k} \cdot$$

in B such that $gf = hf$ and $kg = kh$, then $g = h$, and *fill in*: given a commutative square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow[k]{} & d \end{array}$$

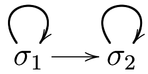
in B, there exists a morphism $b \to c$ or $c \to b$ making the diagram commute. Thus, Theorem 5.5 does not apply to the following two simple examples. A commutative square B does not satisfy the fill-in property, and cancellation fails in the category B with one object and a single non-identity idempotent morphism. Johnstone [7] showed that **UFL**$/B$ (and hence, Pseudo$(B, \mathbf{Span})$) is not cartesian closed in the former case and the following example does so in the latter.

5.7. Example. Let B denote the category

$$\overset{\beta}{\circlearrowright}\, b$$

with one non-identity idempotent morphism β. Then the discrete category $X = \{0\}$ over B is not exponentiable, since the following shows that Y^X does not exist when

$Y = \{y_1, y_2\}$ is the discrete category with two objects. One can show that $Y^X = \{\sigma_1, \sigma_2\}$, where σ_i is the constant y_i-valued map, and there is a (unique) morphism $\sigma_i \to \sigma_j$ over β, for all i, j, since 0 has no endomorphisms over β. Thus, we have a diagram

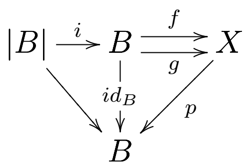


over β, contradicting the uniqueness of the lifting of the factorization $\beta = \beta^2$. Thus, **UFL**$/B$, and hence, Pseudo$(B, \mathbf{Span})$, is not cartesian closed.

6. Pseudo$(B, \mathbf{Rel})$

As in the case of **Span**, every object of Pseudo$(B, \mathbf{Rel})$ is exponentiable in Lax$(B, \mathbf{Rel})$, and so one might conjecture that Pseudo$(B, \mathbf{Rel})$ is cartesian closed. However, we will see that, unlike Pseudo$(B, \mathbf{Span})$, it is not often a topos, since every topos is balanced, i.e., every morphism which is both a monomorphism and an epimorphism is necessarily an isomorphism [6].

6.1. Example. Let B be any category with no non-trivial retractions. If $|B|$ denotes the discrete category with the same objects as B, then the **Rel**-set corresponding to the inclusion $i\colon |B| \to B$ is a pseudo functor. Consider the diagram



Now, i is an epimorphism, since $fi = gi$ implies $f = g$, whenever p is faithful. Since i is clearly a monomorphism which is not an isomorphism in $\mathbf{Cat}_f/B$, it follows that Pseudo$(B, \mathbf{Rel})$ is not a topos.

6.2. Theorem. *If* Pseudo$(B, \mathbf{Rel})$ *is a coreflective subcategory of* Lax$(B, \mathbf{Rel})$*, then* Pseudo$(B, \mathbf{Rel})$ *is cartesian closed.*

Proof. Since the inclusion Pseudo$(B, \mathbf{Rel}) \to$ Lax$(B, \mathbf{Rel})$ preserves products, we can apply Lemma 5.1 to obtain the desired result. ∎

Unfortunately, it is not possible to show that coreflectivity is equivalent to B satisfying condition IG, as it is in the case of **Span**, though we will see that IG is sufficient for a certain class of categories. In fact, IG is not necessary as only one of the non-cartesian closed examples carries over from Section 5. In particular, since every subobject of 1 in Pseudo$(B, \mathbf{Rel})$ corresponds to a UFL subobject in $\mathbf{Cat}/B$, Johnstone's example [7] applies, and so Pseudo$(B, \mathbf{Rel})$ is not cartesian closed when B is a commutative square. However, the following corollary shows this is not the case for Example 5.7.

6.3. COROLLARY. *If B is the category*

$$\overset{\beta}{\circlearrowright} \atop b$$

with one non-identity idempotent morphism β, then $\mathrm{Pseudo}(B, \mathbf{Rel})$ *is coreflective in* $\mathrm{Lax}(B, \mathbf{Rel})$*, and hence, is cartesian closed.*

PROOF. Let $I_b = \left\{ \dfrac{a}{2^n} \mid n \geq 1, 0 \leq a \leq 2^n \right\}$ with

$$\frac{a}{2^n} \to_\beta \frac{a'}{2^{n'}} \iff \frac{a}{2^n} < \frac{a'}{2^{n'}}$$

and let M denote the pseudo functor corresponding to the category over B with morphisms over β given by

$$0 \longrightarrow \overset{\circlearrowright}{m} \longrightarrow 1$$

Then, given a pseudo functor $X: B \to \mathbf{Rel}$, one can show that $x \to_\beta x'$ in X if and only if there is a morphism $f: I \to X$ or $f: M \to X$ such that $f(0) = x$ and $f(1) = x'$. If X is a lax functor, we call this property of an arrow $x \to_\beta x'$, the *pseudo functor property.* Note that this property is hereditary, in the sense that, if $x \to_\beta x'$ satisfies the property, then so do the other morphisms in the image of corresponding f.

Given a lax functor $X: B \to \mathbf{Rel}$, let $\widehat{X}$ denote the $\mathbf{Rel}$-set with the same elements as X and the following relations. Let $\widehat{X}_{id_b} = \Delta_{\widehat{X}_b}$ and $\widehat{X}_\beta$ denote the set of $(x, x') \in X_\beta$ satisfying the pseudo functor property. Then it is not difficult to show that

$$\mathrm{Lax}(B, \mathbf{Rel}) \xrightarrow{\widehat{\ \ }} \mathrm{Pseudo}(B, \mathbf{Rel})$$

is right adjoint to the inclusion, giving the necessary coreflection to apply Theorem 6.2, and it follows that $\mathrm{Pseudo}(B, \mathbf{Rel})$ is cartesian closed. ∎

To apply Theorem 6.2 when B satisfies IG, we first construct a cartesian closed subcategory $\mathbf{Cat}_f/B$, and then establish conditions on B making this category equivalent to $\mathrm{Pseudo}(B, \mathbf{Rel})$, using the following lemma.

6.4. LEMMA. *Suppose B is a category satisfying IG and the only isomorphisms are the identity morphisms. Then $[\![\beta]\!]$ is a totally ordered set, for every morphism $\beta: b \to b'$.*

PROOF. Suppose

$$\begin{array}{ccccc} & & u & & \\ & \overset{\beta}{\nearrow} & & \overset{\beta'}{\searrow} & \\ b & & & & b' \\ & \underset{\bar{\beta}}{\searrow} & & \underset{\bar{\beta}'}{\nearrow} & \\ & & \bar{u} & & \end{array}$$

are objects of $[\![\beta]\!]$. Then, since B satisfies the fill-in property, there is a morphism $u \to \bar{u}$ or $\bar{u} \to u$ of the corresponding factorizations in $[\![\beta]\!]$, and not more than one, since B has no non-identity isomorphisms and satisfies the cancellation property. Thus, $[\![\beta]\!]$ is totally ordered. ∎

6.5. Corollary. *Suppose B satisfies IG, has no non-identity isomorphisms, and $[\![\beta]\!]$ is finite, for all β in B. Then* $\mathrm{Pseudo}(B, \mathbf{Rel})$ *is a coreflective cartesian closed subcategory of* $\mathrm{Lax}(B, \mathbf{Rel})$.

Proof. We will show that $\mathrm{Pseudo}(B, \mathbf{Rel})$ is equivalent to a cartesian closed coreflective subcategory $\mathcal{C}$ of $\mathbf{Cat}_f/B$.

Given a faithful functor $p\colon X \to B$, let $\widehat{X}$ denote the subcategory of X with the same objects and those morphisms $\alpha\colon x \to x'$ over β for which there is a commutative diagram

$$\begin{array}{ccc} [\![\beta]\!] & \xrightarrow{f_\alpha} & X \\ & \searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

such that $f_\alpha(b \overset{id_b}{\to} b \overset{\beta}{\to} b') = x$ and $f_\alpha(b \overset{\beta}{\to} b' \overset{id_{b'}}{\to} b') = x'$. Note that $\widehat{X}$ is closed under composition, since IG says that

$$\begin{array}{ccc} [\![id_{b'}]\!] & \longrightarrow & [\![\beta']\!] \\ \big\downarrow & & \big\downarrow \\ [\![\beta]\!] & \longrightarrow & [\![\beta'\beta]\!] \end{array} \qquad (*)$$

is a pushout in $\mathbf{Cat}$, for every composable pair. Thus, we get a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{i_X} & X \\ & {\scriptstyle \hat{p}}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

in $\mathbf{Cat}_f/B$. Let $\mathcal{C}$ denotes the full subcategory of $\mathbf{Cat}_f/B$ consisting of $p\colon X \to B$ such that $\widehat{X} = X$.

To see that $\mathcal{C}$ is coreflective in $\mathbf{Cat}_f/B$, it suffices to show that $\widehat{\widehat{X}} = \widehat{X}$, for all $p\colon X \to B$ in $\mathbf{Cat}_f/B$. To do so we will show that every $\alpha\colon x \to x'$ of $\widehat{X}$ is in $\widehat{\widehat{X}}$ by showing that the image of $f_\alpha\colon [\![\beta]\!] \to X$ over β is contained in $\widehat{X}$. Given a morphism

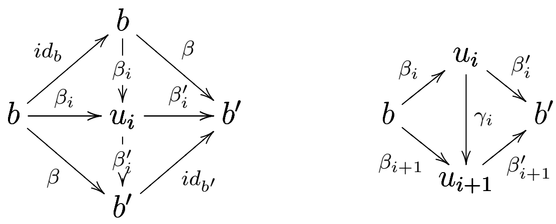
$$\begin{array}{ccccc} & & u & & \\ & {\scriptstyle \beta_1}\nearrow & & \searrow{\scriptstyle \beta_2} & \\ b & & \big\downarrow{\scriptstyle \gamma} & & b' \\ & {\scriptstyle \bar{\beta}_1}\searrow & & \nearrow{\scriptstyle \bar{\beta}_2} & \\ & & \bar{u} & & \end{array}$$

of $[\![\beta]\!]$, there is a functor $i\colon [\![\gamma]\!] \to [\![\beta]\!]$ given by precomposition with β_1 and postcomposition with $\bar{\beta}_2$, and using $[\![\gamma]\!] \overset{i}{\to} [\![\beta]\!] \overset{f_\alpha}{\to} X$, it is not difficult to show that $f_\alpha(\gamma) \in \widehat{X}$.

Now, every object of $\mathcal{C}$ is exponentiable in $\mathbf{Cat}_f/B$, i.e., $\hat{p}$ satisfies WFL, since given $\alpha''\colon x \to x''$ in $\widehat{X}$ such that $p\alpha'' = \beta'\beta$, the morphism $f_{\alpha''}\colon [\![\beta'\beta]\!] \to X$ induces a factorization of α'' via the diagram (*). Since the inclusion preserves products and every object of $\mathcal{C}$ is exponentiable in $\mathbf{Cat}_f/B$ being a WFL, applying Lemma 5.1, we see that $\mathcal{C}$ is cartesian closed.

It remains to show that Pseudo$(B, \mathbf{Rel})$ is equivalent to $\mathcal{C}$, i.e., the objects of $\mathcal{C}$ are precisely the faithful WFL with discrete fibers. We already know that $\hat{p}$ satisfies the first two conditions, and the fibers of $\hat{p}$ are discrete, since $[\![id_b]\!]$ has one element (as the cancellation property implies that every element corresponds to an isomorphism). Thus it suffices to show that every faithful WFL $p: X \to B$ with discrete fibers is in $\mathcal{C}$.

Suppose $\alpha: x \to x'$ is in X and $p\alpha = \beta$. To show α is in $\widehat{X}$, we will define $f_\alpha: [\![\beta]\!] \to X$ over B such that $f_\alpha(b \xrightarrow{id_b} b \xrightarrow{\beta} b') = x$ and $f_\alpha(b \xrightarrow{\beta} b' \xrightarrow{id_{b'}} b') = x'$. Applying Lemma 6.4, we see that $[\![\beta]\!]$ is totally ordered. Since $[\![\beta]\!]$ is finite, it sits over $b \xrightarrow{\beta_1} u_1 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_{n-1}} u_n \xrightarrow{\beta'_n} b'$ in B, and we have commutative diagrams



Since $\alpha: x \to x'$ and $p\alpha = \beta = \beta'_1\beta_1$, using the fact that p is a WFL, we get a factorization $x \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha'_1} x'$ of α over $\beta = \beta'_1\beta_1$. Similarly, since $p\alpha'_1 = \beta'_1 = \beta'_2\gamma_1$, we get a factorization $x \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha'_2} x'$ of α over $\beta = \beta'_2\gamma_2\beta_1$. Continuing, we get $x \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} x_n \xrightarrow{\alpha_n} x'$ of α over $\beta = \beta'_n\gamma_{n-1} \dots \gamma_1\beta_1$, and hence, the desired morphism $f_\alpha: [\![\beta]\!] \to X$ over B. ■

Note that if B and B' are any equivalent categories, one can show that Pseudo$(B, \mathbf{Rel})$ and Pseudo$(B', \mathbf{Rel})$ are equivalent as well, but the same is not the case for the corresponding categories of lax functors. Thus, it turns out that to show that Pseudo$(B, \mathbf{Rel})$ is cartesian closed (but not necessarily coreflective in Lax$(B, \mathbf{Rel})$), we need only assume that B has no non-trivial automorphisms rather than isomorphisms.

6.6. COROLLARY. *If B satisfies IG, has no non-trivial automorphisms, and $[\![\beta]\!]$ is finite, for all β in B, then* Pseudo$(B, \mathbf{Rel})$ *is cartesian closed.*

PROOF. Since B is equivalent to a skeletal category satisfying the hypotheses of Corollary 6.5, the desired result follows. ■

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